Basic Real Analysis
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Lecture 26
Uniform continuity and connected sets Part 02

Uniform Continuity, equivalent ways in terms of sequences, every continuous function on a compact set is uniformly continuous.

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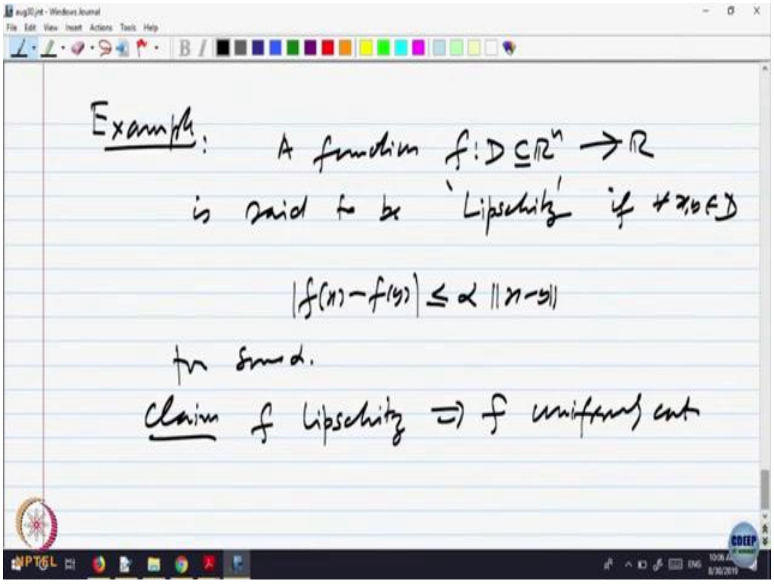

Here, is one more example of continuous functions which you may find is useful. A function, you may come across these things in some other courses so, let me introduce that, a function f from a domain D in Rn to R, is said to be Lipschitz, mathematician who first gave this definition. So, goes by his name is said to be Lipschitz. If, for every x, y belonging to the domain D, if I look at the distance between the images that is less than or equal to some alpha times x minus y for some alpha.

So, we have got the distance between images of any 2 points is bounded by the distance between x and y, for some constant alpha. So, sometimes one says it is Lipschitz of order alpha or does not matter, actually some power sometimes but it is the simplest kind of Lipschitz function you can think of. So, claim f Lipschitz f uniformly continuous.

Because, directly gives a measurement between distance between f x, f y and x and y, so if x and y are close, obviously f x and f y are close, obviously. Or if you like, if a sequence x n, y n goes to 0 then, f of x n minus f of y n will go to 0. Either way you can look at it, and say this is a class of functions which is uniformly continuous. There are many other useful results because the domain is compact. Probably, I should, I think.

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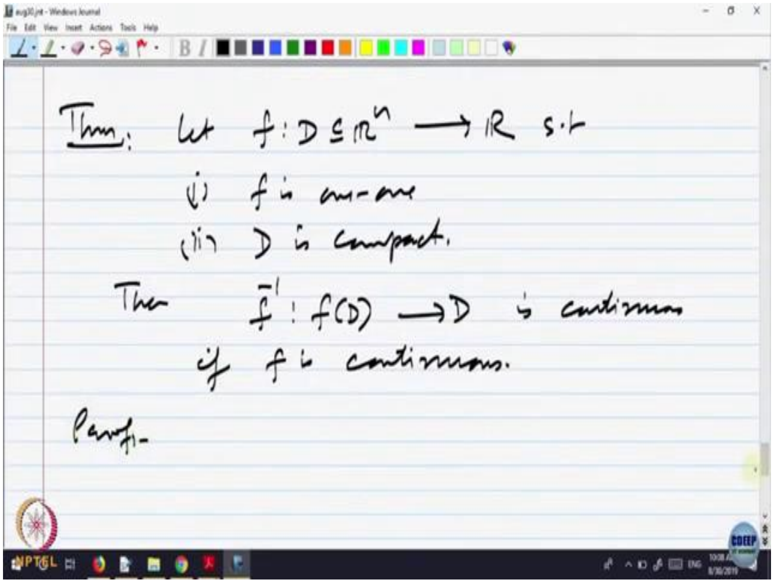

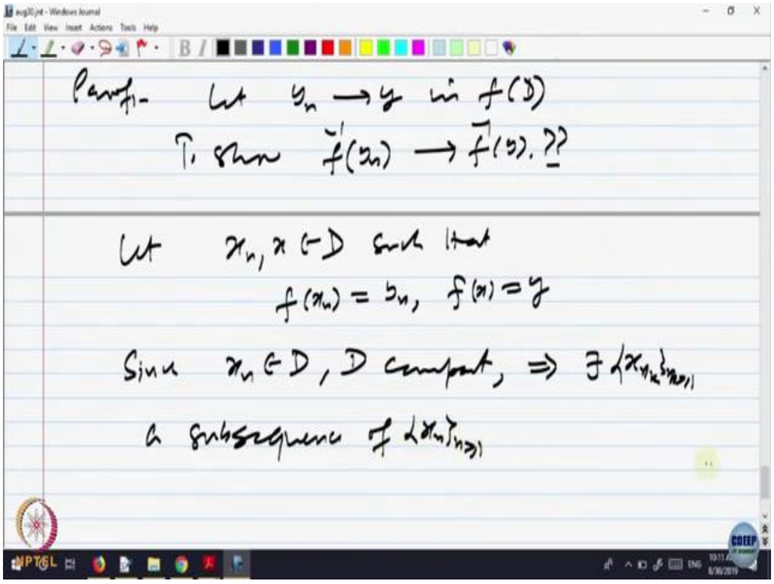

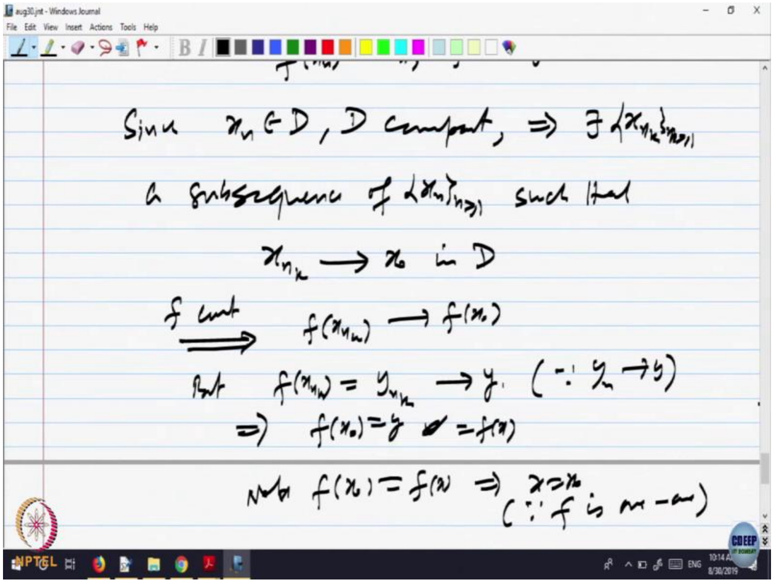

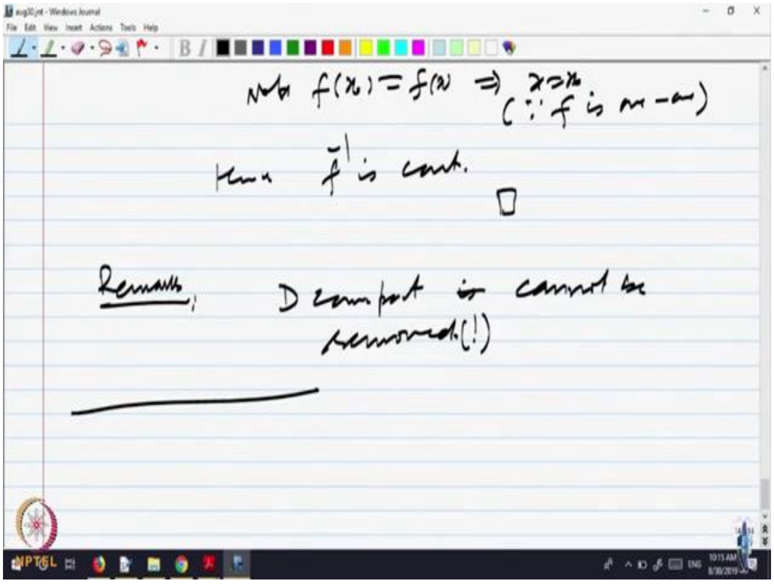

Let me, let me state one more properly and prove if. So, let us say f is from D contained in Rn to R such that f is 1, 1, D is compact, domain is compact and the function is given 1, 1 function. If the function is 1, 1 then on the range there is an inverse function available.

Every 1, 1 on to function as a inverse from the range to the domain. So, that function is normally called f inverse, from the range f of D to D. So, what is the inverse function, x, the point x goes to y. So, inverse function is you bring y back to x, from the range back to the domain.

So, you can think of as composition of these to be identity map, f composite f inverses is identity map on the set D. The claim is, is continuous if f is continuous. So, what we are saying is, if the domain of a 1, 1 function is continuous then the inverse function automatically is a continuous function, so let us prove that. So proof, so what is to be shown.

So, let yn converge to y in D in f of D. I want to show inverse function is continuous. So, what is the domain of the inverse function that is f of D, the range. So, take a sequence their yn converging to y, show that the image converges to the image. To show, f inverse of yn converges to f inverse of y, so that is to shown.

So, the, and what is given to us, the domain is compact and the function is continuous. So, from the range, somehow, we have to come to the domain because something about the domain only is given. And how can a come back, because yn and yr in the domain, they should be the images of something in the, there in the range.

So, there should be image of something in the domain. So, let us write that, so let xn x belong to D such that f of xn is equal to yn and f of x is equal to y. So, x is a pre image of y and xn is the pre image of x, yn. So now, xn and yn are, xn is in the domain, x is in the domain.

So now, since xn belongs to D and now here comes our compactness, D compact, implies there is a subsequence, a subsequence of xn. See, how compactness always gives you a nice properties, xn such that xnk converges, xnk will converge somewhere, xn is only the subsequence, converges somewhere let us call it as say x naught in D, f continuous will imply what, f xnk converges to f of x naught.

But what is f of xnk, that is ynk because xn is the pre image of yn. And where does yn converge, we are given yn is convergent to y. So that implies what, ynk so that converges to y, because yn converges to y.

So, what we should have, on the other hand it should converge to f of x naught and this y is equal to f of, f x, let me also, let us also write. Actually, this is y, so, what we want to show, we want to show f of x is equal to y. So, what we have gotten, so implies f of x naught is equal to y. This converges this, this is also converges to y, so we have gotten this. Can I say x is equal to x naught?

Professor: Can we say x is equal to x naught?

Student: (())(09:55)

Professor: Why? I know f of x is equal to f of x naught, f of x is equal to x naught, but function is 1, 1, we are not use that fact anyway. Note, f of x naught is equal to f of x because xnk converges to x naught, continuity implies f of xnk will converge to f of x naught, but this also converges to f of x, y is equal to f of x. What is y, y is equal to f of x. So, these 2 are equal, so implies x is equal to x naught because f is 1, 1, f is a 1, 1 function.

So, x must be equal to x naught. So, what we are saying, that y is equal to f of x, that is what we want to show. So, hence, f inverse is continuous. So, it is a continuous function. So, that is one consequence of saying it is. I think, I should make a remark or you should try to remark, D compact is, cannot be removed.

So, what does that mean, In this theorem, we had the condition, D is compact. That condition if the domain is not compact, even if the function is 1, 1 inverse may not be continuous. So, try to construct an example, otherwise in the tutorial class we will discuss it, so what more one should say, I think enough of uniform continuity.

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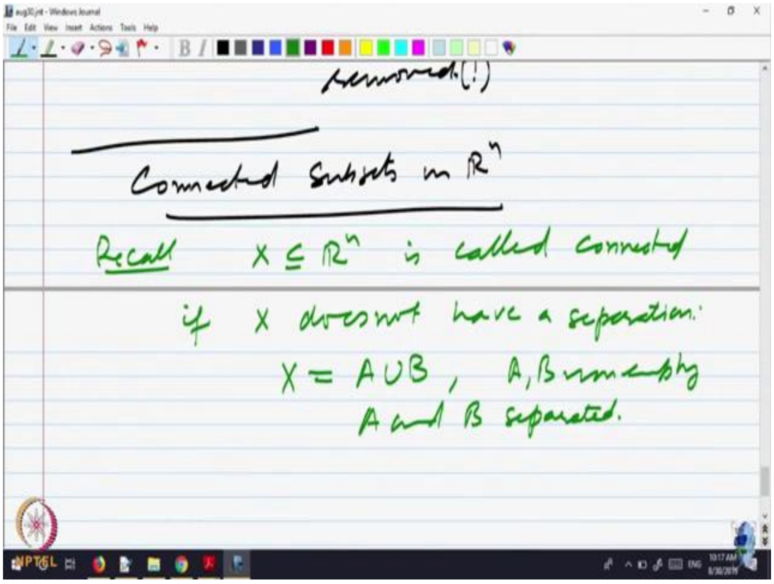

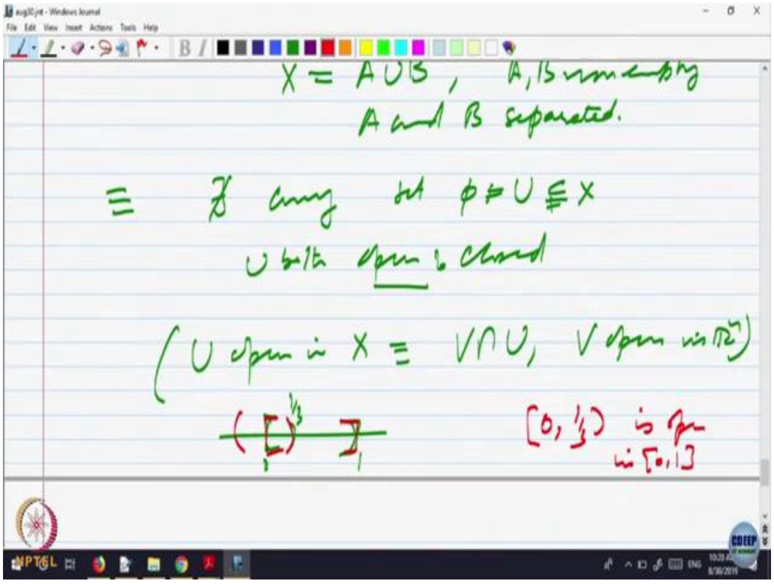

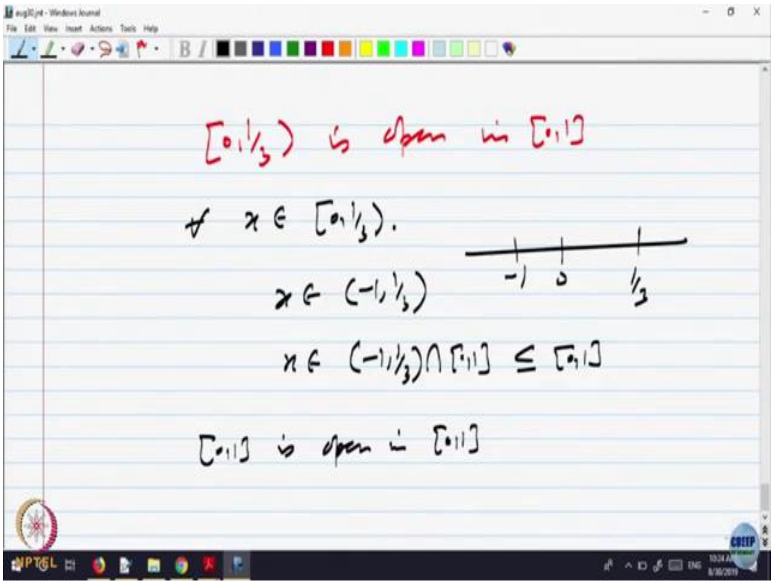

Let me, also come back, to the concept of connected subsets of Rn that I said we will do it later. So, I think this is the right stage to look at connected subsets in, so recall, let us just, because we did it sometime back, so let us recall. A subset x contained in Rn is called connected, I am just recalling. If x does not have a separation, so what are the meaning of saying x does not have a separation, that means, x cannot be written as A union B, A be non empty, A and B separated.

If a set can be written as a disjoint union, it can be partitioned into 2 parts, not only partition, if that is a separation, then we say the set is disconnected. If no such thing is possible, we say it is connected. The typical example was that of an interval, interval is a connected subset of the real line because whenever you try to cut it into 2 parts, 1 part and the other part, they will be disjoint. You can cut it into 2 parts, but they will not be separated, separated means not every point has a neighborhood which does not intersect the other side.

And keep in mind, neighborhood, in Rn, so and we said this is also equivalent to say, that there does not exist any set U which is not empty, which is proper subset of x, U both open and closed.

So, this is equivalent to saying, there does not exist any subset of x which is both open and closed. Because, if it is both open and closed, this and its compliment in x will give us a separation, if not, so it will be separation. But one thing I should make it slightly more clear, which may not be that clear, what is meaning of saying U is open in x, U is a subset of x. So, U open in x is same as saying, it is V intersection U, V open in Rn.

It is part of, the part of an open set in Rn. See, for example, if you take a open, take a set which is open interval, saying this is an open set, I can say it is an open set if I look it as a subset of R, real line whole thing.

But can I say a part of it, say 0 to 1 and let us like it 0 to 1 by 3, can I say 0 to 1 by 3 is open in 0 to 1, yes, but supposing I close it here, I take the closed interval. Now, can I say that 0 to 1 by 3 is an open subset of the closed interval is open in 0 to 1, it is, it is open in 0, 1 closed, but not open in the whole thing.

Because, I can write this as a intersection of bigger open set, intersection 0 to 1. So, that kind of thing you should keep in mind, saying something is open in a subset means, it is intersection of an open set in the whole space with that set. Because, let me say why. So, let us just illustrate this a bit more. I am saying that 0 to 1 by 3 is open in 0 to 1. What does that statement mean? Let us just elaborate it more.

If I want to go back to this and look at in terms of interior points. So, for every x belonging to 0 to 1 by 3, what should I have, I should have a ball around this. So, there exists, so if x belong, then let us look at say minus 1 to 1 by 3 intersection with 0 to 1 is a subset of this point x belongs to this, not this one or let me just write slightly more better.

Let, x belong to this, then x belongs to say minus 1 to 1 by 3, obviously. So, here is 0, here is 1 by 3, here is minus 1. But I can still say x belongs to minus 1 to 1 by 3. Intersection, if I look at its intersection with 0 to 1, which is a subset of 0 to 1.

See, what we are saying is, whatever be the set in R or in Rn, what is the meaning of saying that you are looking at open balls in that set, that set may not have the whole ball as of Rn inside it, but it is the intersection of open ball in Rn with that set, that is what we mean.

So, you can think of it a subset or subspace you can think of. So, that is the main difference between is saying some A subset, and A is a subspace of Rn as far as a distance is concerned. So, what are open sets in a subset, they are intersection of open sets in the bigger, intersected with that set.

So, in that sense, you should think of both open and closed. So, for example, let me just keep that discussion a bit more. For example, if I look at the interval 0, 1, I am saying is open in 0, 1, it is open in 0, 1. Is like whole space is always open itself, whole space is also closed in itself. So, this because, this way of looking at things helps to later on to define many things in higher courses.

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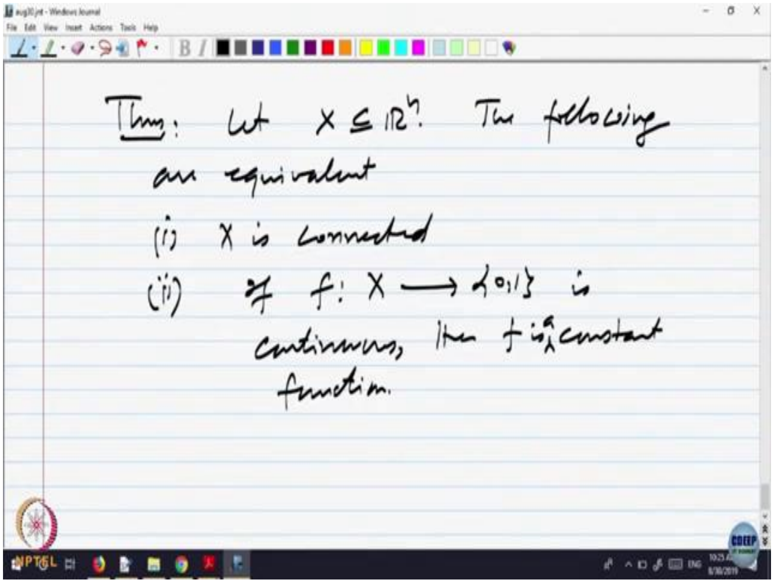

So, let me see what I wanted to do was the following. So, here is a theorem, which is very useful. Let X be a subset of Rn. The following are equivalent, I am going to describe connectedness in terms of functions. 1, X is connected. And second, if f is a function from X to 2 points set 0, 1 is continuous then f is constant, f is, f is a constant function.

So, let us just understand the statement, you are saying if it is, saying a set is connected is equivalent to saying that if we have got a function on X, which is continuous and takes possibly only 2 values 0 and 1, that is not possible if X is connected. It should take only 1 of the values either 0 or 1, it should be a constant function. So, continuous functions on connected sets cannot take 2 different values. Only 1 value is possible, it should be a constant function.