Basic Real Analysis

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Lecture no. 28

Connected Sets and Continuity – Part I

Right, so we were looking at Connected Subsets of Rn and we were trying to prove a theorem, were given illustrations of that theorem.

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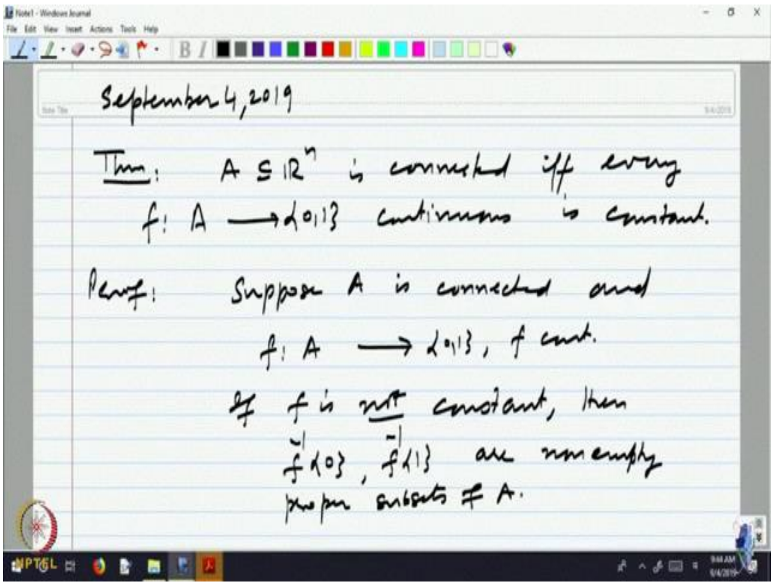

Namely, A contained in Rn is connected, if and only if every function f from A to 2 void set 0, 1, continuous is constant. So, we looked at many consequences of this theorem. So, let us give a proof. Suppose, A is connected and f is a function from A to 0, 1, f continuous. So, it is a function, continuous function on A taking 2 values 0 and 1. We want to show it should be taking only one value, it should be a constant function.

So, if f is not constant, then what will happen? There is at least one point where it takes the value 0, and some other point where it takes the value 1. Then, let us look at the inverse image of 0 and inverse image of 1. So, all the points which go into 0 and all other points which go into 1 are non empty proper subsets of A right.

Because there is a point which goes to 0, there is another point which goes to 1. So, that means, 0 has a pre image, 1 has a pre image and so, they are non empty, and this pre image is of 0 and 1 and they are not whole of A because different points are going to 0 and 1 .

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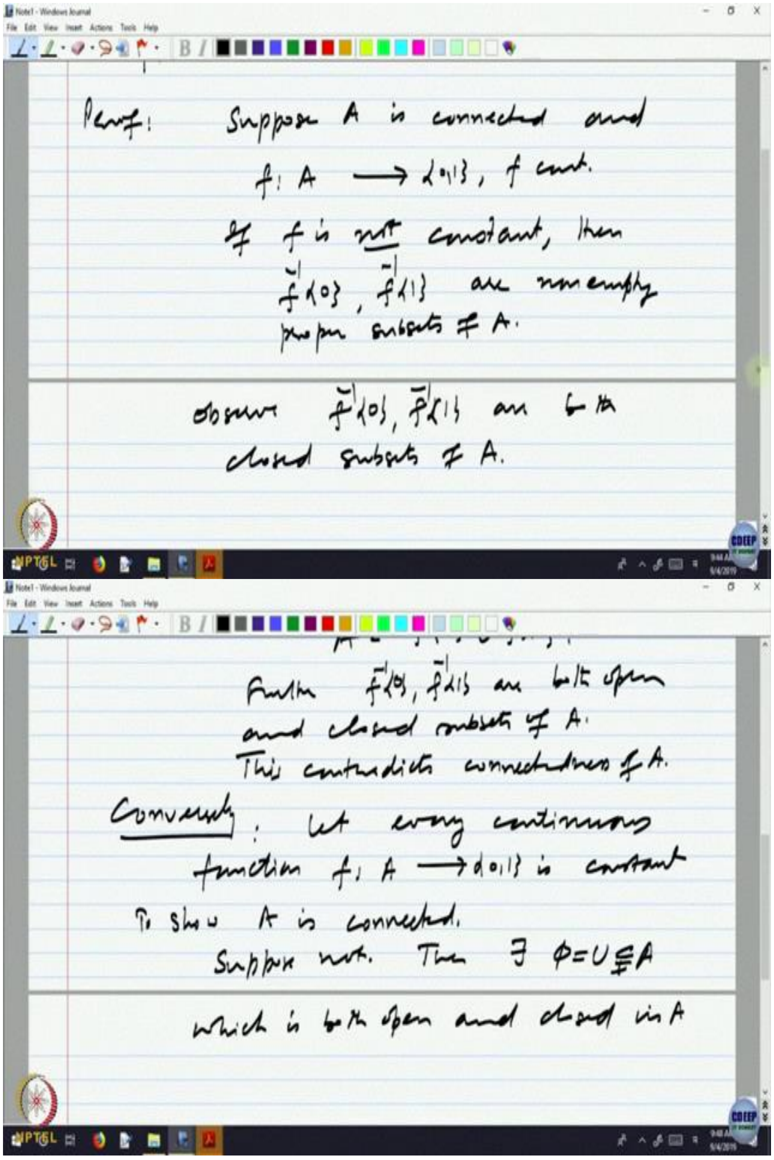

The only observation is that this set, so let us observe f inverse of 0 and f inverse 1 are both close subsets of A, they are both close subsets have of A, it is both close subsets. Because if we take a sequence in f inverse of 0, xn converging it to x and f of xn is equal to 0 for every value. So, f of x also is equal to 0.

Or you can look at singleton is a closed set. So, inverse image of close set must be close by definition of continuity. So, our both closed subsets of A and A is equal to f inverse of 0, every point of, f inverse of 1, every point A either goes to 0 or 1. So, inverse image should cover everything. So, what we are saying, so this f inverse of 0. So, you can look at that f inverse of 0 is a subset. What does this complement in A, that is a f inverse of 1.

So, that is closed. So, it is complimentary f inverse of 0, so both are open and closed subsets of A. Because both are f inverse 0 is close, it is complimentary f inverse of 1, which is also closed, so that is both open. And so, and further f inverse of 0, f inverse of 1 are both open and closed subsets of A. That is not possible, contradicting, this contradicts connectedness of A. Because if A is connected it should not have any non empty proper subset which is both open and close.

Let us look at the converse, conversely. Let, let us have the property that every function, every continuous function, every continuous function f, A to 0, 1 is every continuous function is constant function. So, that is given to us, to show A is connected right to that what we are shown.

So, if not again by contradiction, suppose not then there exists a non empty subset U of A proper or non empty subset of A, which is both open and closed in A. So, there is a non empty proper closed, non empty proper subset of U of A, which is both open and closed because it is not connected. So, it should have.

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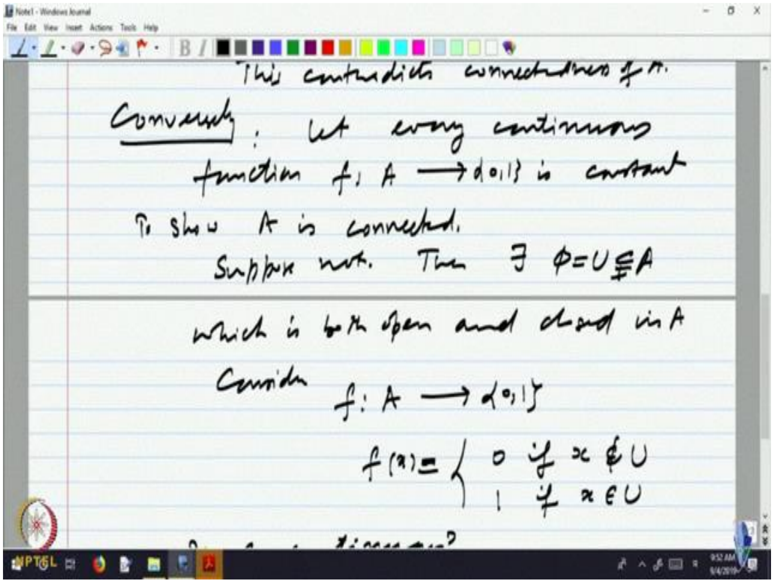

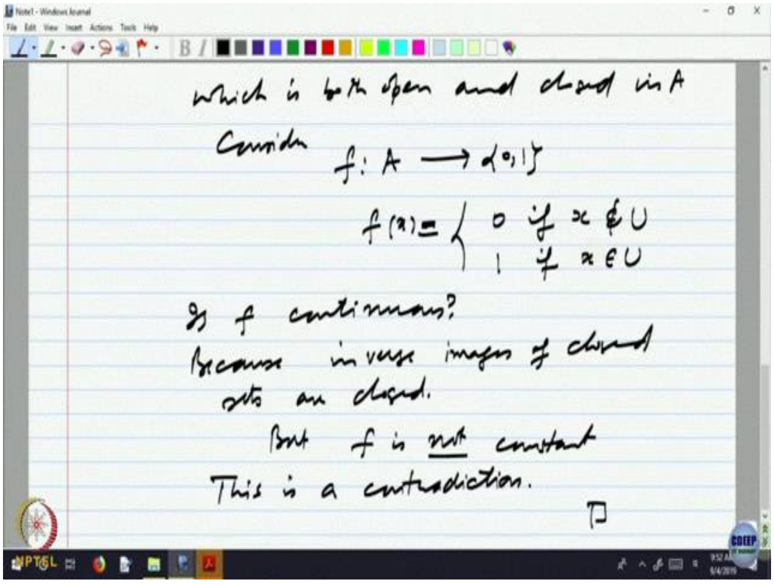

Now here is something, which you will find useful later on also. Consider, let us look at the function f defined on A taking values in 0, 1. So, how I am going to define, f of x is equal to 0, if x does not belong to A and it is 1 if x belongs to A, A is the main. So, let us, the set I want to write is U. We have a set U which is both open and closed, it is proper, it is non empty.

So, define f of x to be equal to 0, if x belongs to U and x does not belong to U and if x belongs to U, then put the value to be equal to 1. So, it is defined everywhere, on A. Is this function continuous? Want to know is this function continuous?

Professor: To check continuity what we have to check? What is F inverse of 0?

Student: (())(8:55)

Let us check inverse of image of every closed set is close. So, what are closed subsets in the range, either empty set right or singleton 0 or singleton 1. What is inverse image of empty set, empty set, there is nothing which is going into anything. So, that is closed.

Professor: What is inverse image of 0?

Student: (())(9:26)

The repressively U, not in U. So, that is U compliment. A is U compliment closed, because we assume U is both open and closed. What is reverse image of 1, that is U which is again close. What is left, is the whole 0 and 1 together, what is the inverse image of 0 and 1 together, of the range that is a domain the whole space is always closed right.

So, inverse image of closed sets are closed. So, this function is continuous because, let us write, inverse images of, are closed. But it is not constant, but this is not constant, but f is not constant. It is not constant function because or U it takes the value, on U takes the value 1, on U complement takes the value 0 and the proper subsets. So, there are points in U, there are points in U complement. So, this is a contradiction.

Because whatever our assumption, our assumption was whenever we have a function right, which is continuous in 01 it should be constant. So, we are produce a function which is not constant, but it is continuous . So, that is a contradiction. So, that proves the theorem, which we saw has a lot of implications in giving examples of connected sets in Rn.

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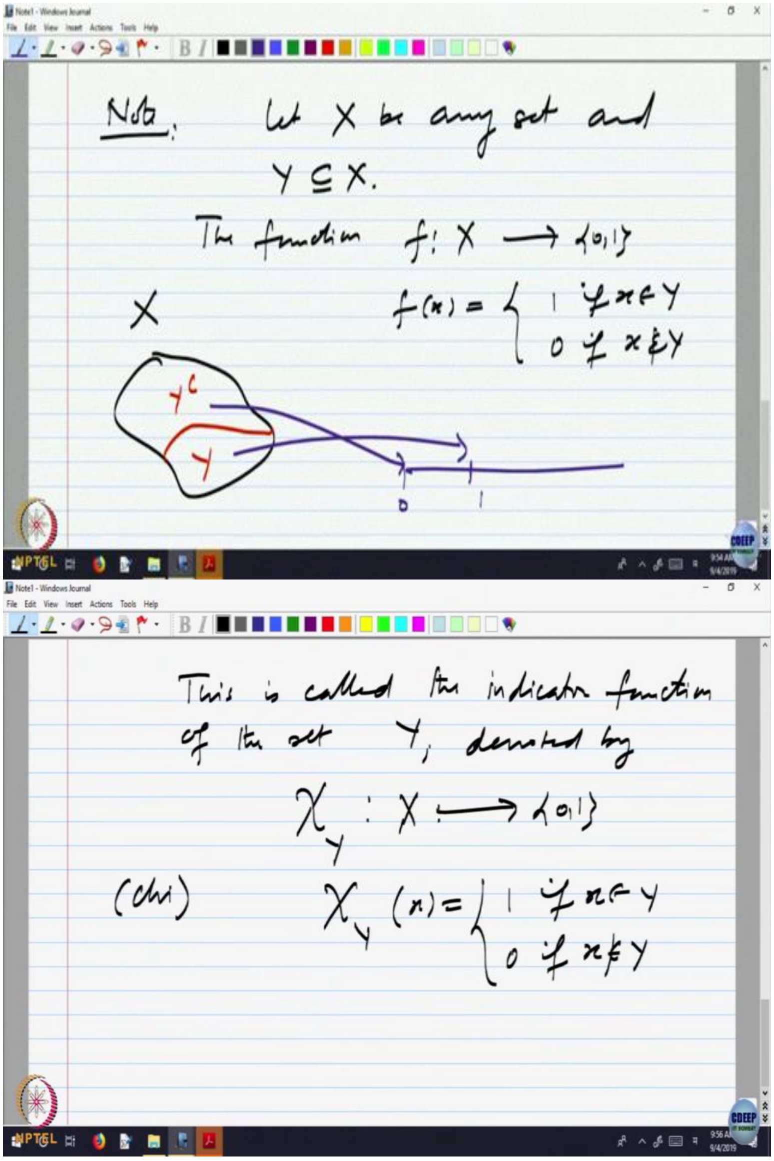

This is a very typical function. So, let me probably put a remark or a note. Let X be any set, any set and let us say Y a subset of X .So, the function, if I define a function f like we defined earlier, on X in 0, 1, f of with values, f of x is equal to 1 if x belongs to Y and 0 if x does not belong to Y. This is the kind of example that we had in our proof right for the set U.

So, what is this function doing, on any set X. This is any set X, look at a subset Y of it, so look at a subset Y of it, so look at a subset Y. So, this is a function defined on the whole space X. So, this is Y compliment. So, what does it do, on Y, it gives the value, the value is, on Y the value is 1 and on Y complement the value is 0.

So, it is kind of thing, if the point lies in the set Y a light goes up. If it is in Y compliment light does not go up, on off, true, false, it takes only 2 values. So, you will find this kind of thing coming in probability theory. So, this function indicates when the point is in the set A. When the point is in A, it takes the value 1, non zero value.

So, this is called, this function is called, this is called the indicator function of the set, of the set Y. So, this is called the indicator function of the set Y, denoted by this is a greek letter called chi. So, this is chi, chi lower A indicating it is the indicator function of A. So, it is defined on the whole space, we are taken not A, we are taken it as Y, defined on the whole space, taking values in 0, 1. The indicator function of Y, at a point x is equal to 1, if x belongs to A and 0 if x does not, why I am writing A again and again, Y and belongs to, does not belong to Y.

On the compliment it is 0, on the set it is Y. This is called the indicator function of, the simplest kind of function. Actually, this function has a beautiful history that historically when the notion of function was not defined properly, there was a mathematician called Dirichlet, who gave this example to indicate that functions can be very simple taking very, only two values. But they need not be given by a formula, you can not have a formula for this abstract.

So, till Dirichlet gave that example everybody believe that a function should have a formula or a graph. So, this was the beginning of abstract notion of a function.

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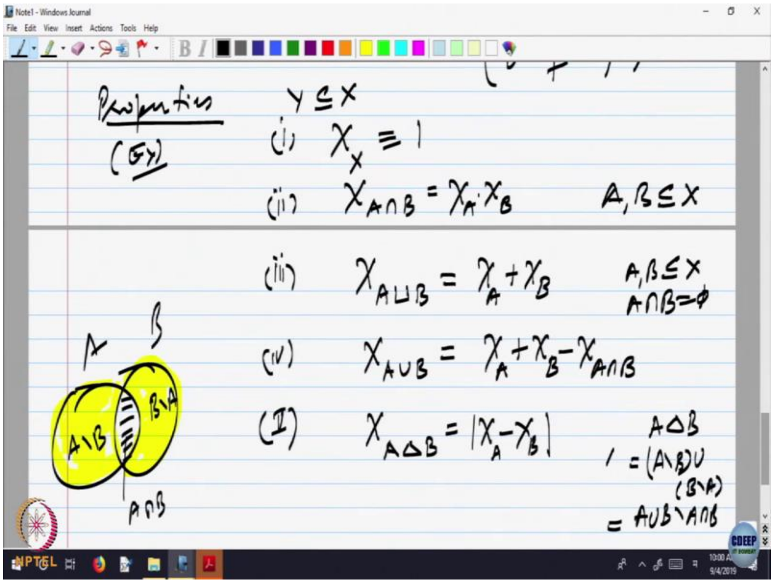

And here, are some properties which, if you have not come across, you should. So, what is, we are what Y contained in X. What is indicator function of X, complement of X is 0, empty set. So, this is constant function 1. Let us look at two sets A and B, subsets of X. What is A intersection B? So, you can, I leave with these are very real simple exercises. So, you try to prove it that is chi of A, multiplied with chi of B.

And third, chi of A union B, this square you kind of union means, A and B are subsets of X and A intersection B is empty, the disjoint sets. Then this is chi of A plus chi of B, very nice properties, such a simple function. And for example, you can also ask what is, in general, when they are not disjoint, this is chi of A plus chi of B. It we will be counting intersection second twice. So, minus indicator function of A intersection B. So, you can think of already a probability theory coming into picture kind of a thing.

And 5, what more, I think this is nice. What is chi of, does everybody know what is a set A delta B, so this is a set A delta B is called the symmetric difference, A minus B union B minus A. So, this set is called the symmetric difference of A and B. From A, this is same as A union B minus A intersection B. So, pictorially if this is A, this is B, and this is A intersection B and what is the symmetric difference, the symmetric difference is precisely. So, this is A minus B and this is B minus A. So, this is A minus B and B minus A and this part is A intersection B. So, symmetric difference.

So, what is this equal to, one can show it is the absolute value of chi of A minus chi of B. So, these are very useful properties of the indicator function, this is functions nothing, no continuity, nothing involved, so you can. So, we looked at connected subsets of Rn, we gave examples of that. So, we have looked at various properties of compact sets, connected sets both in Rn, Rn.

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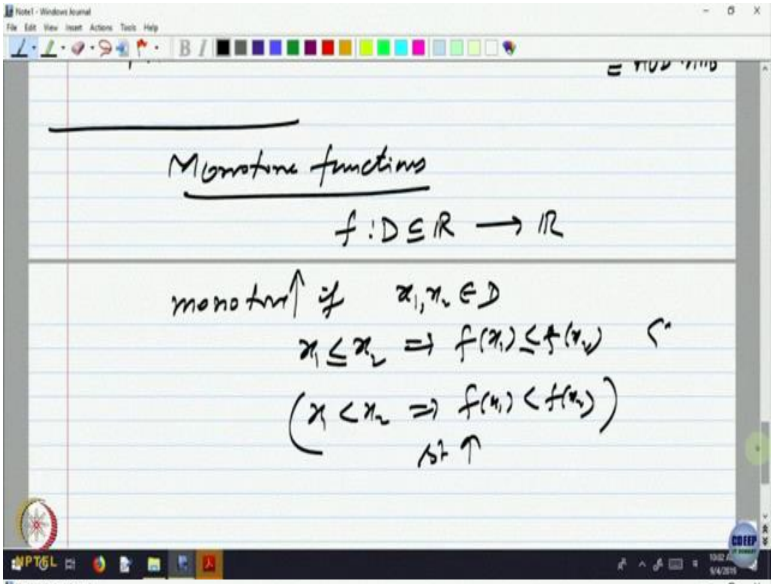

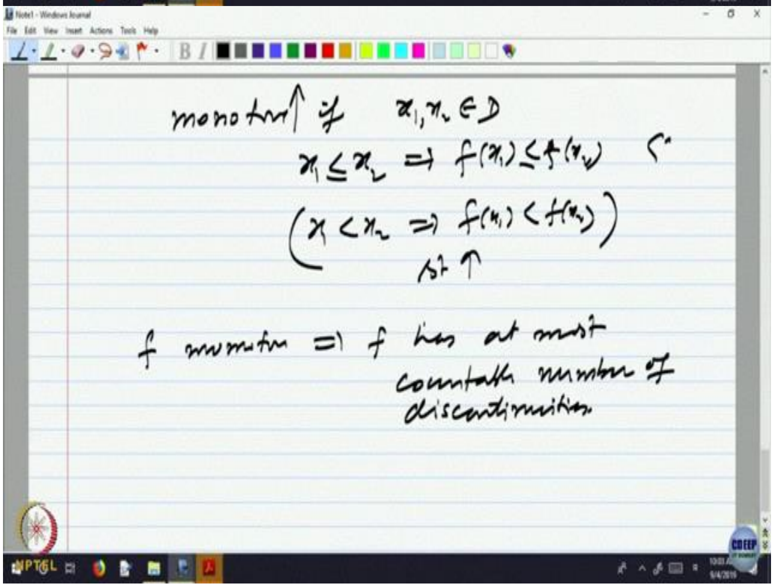

We also, looked at one nice class of functions, probably I should state some of the properties of that. We looked at a function, monotone function, some properties probably I should state more. So, a function f defined on D contained in R is monotone. Remember, if x1, x2, belonging to D, x1 less than or x2 implies f of x1 is less than f of x2. If x1 is strictly less than x2 implies f of x1 is strictly less than f of x2, that was strictly increasing.

So, strictly monotone, this is strictly increasing, increasing. So monotonically increasing let me write arrow up, saying. Monotonically increasing less than equal to strictly and then equality is strict and so on. So, we looked at various properties, we showed that f monotone implies f is continuous or better way of writing that would be the number of discontinuities, f has at most, countable number of discontinuities, that we showed every monotone function is discontinuous. The only possible discontinuities are the jump discontinuities and they are the most countable .

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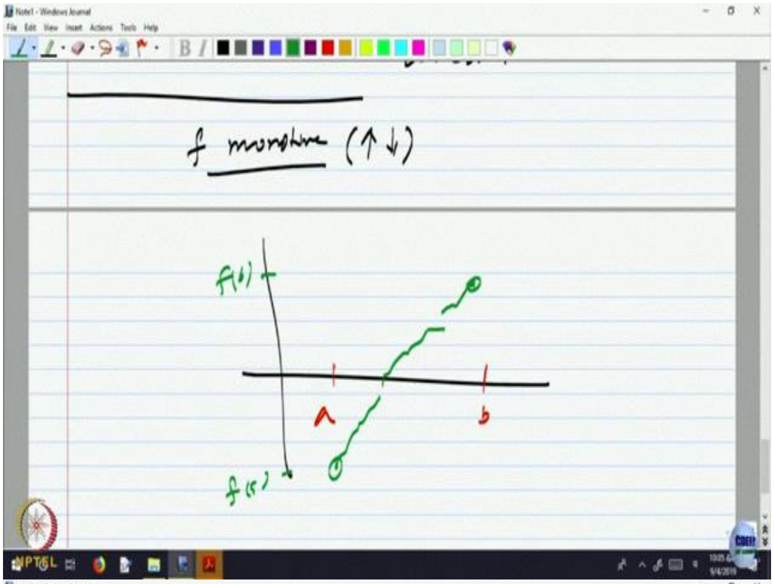

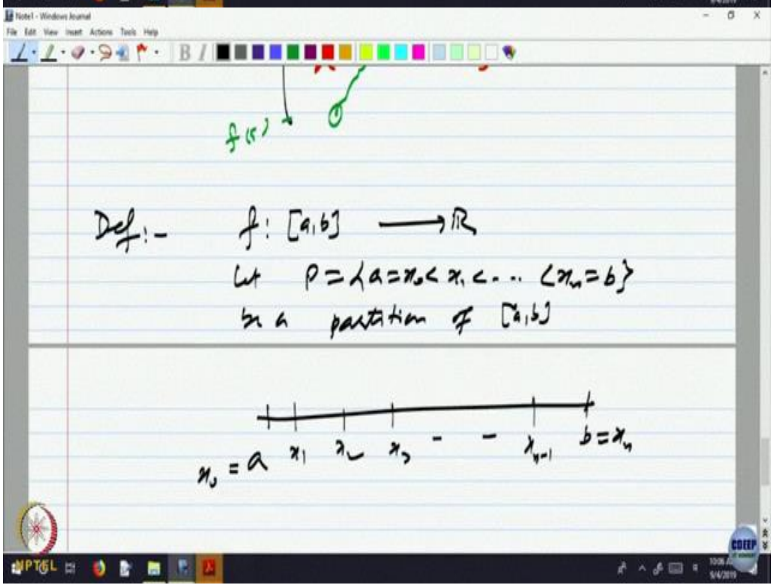

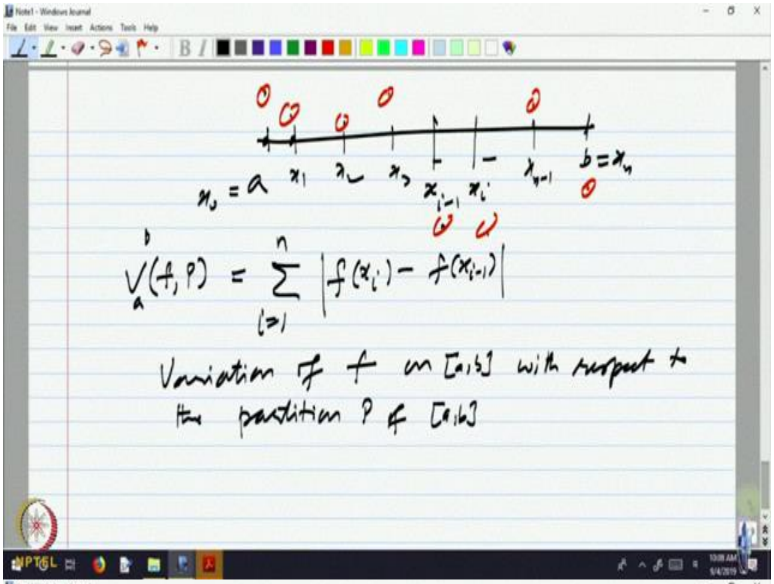

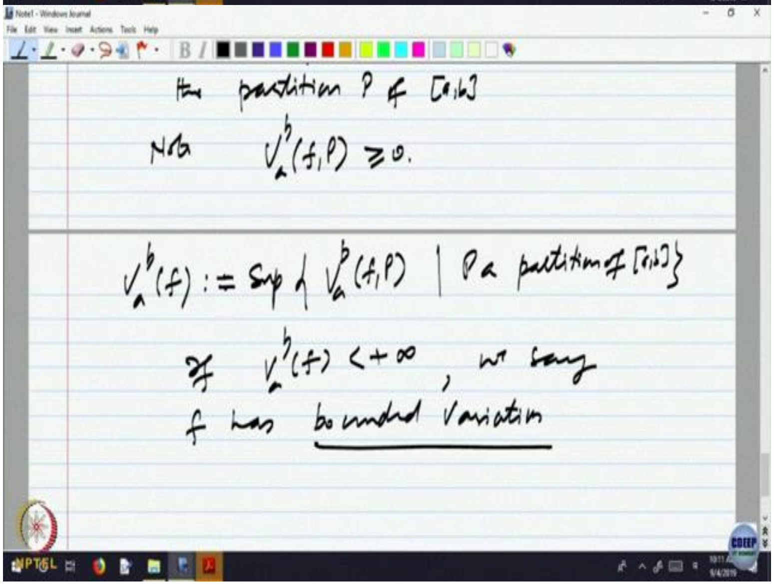

Here, is a interesting thing, which is a very deep theorem, probably I will say it when we come to differentiability. Every monotone function in fact is differentiable.

Professor: Where?

Student: (())(22:08)

Professor: So, there is a notion of what is called sets of length 0. So, probably I will say it later on when we come to differentiability. I will come to it later. I think let us not go into this now. But that is a very interesting theorem, which and a deep theorem, which analyses differentiability of monotone functions also.

So, what I want to say is the following. So, let us look at a monotone function f monotone. What kind of a is a monotone increasing or decreasing. So, here is, let us try to draw a picture of it. So, this is somewhere a and this is b, say it is a monotone function. So, it starts somewhere. This is a value at the point a, say it is monotonically increasing. So, probably it goes up like this and then there is probably there is a discontinuity it goes like this, and maybe it remains constant somewhere and then again starts going up like this.

So, this is a value at the point b, so this is f of a and this is f of b. Now, what we want to analyse is, How much is a monotone function? How much it can vary? How much the values of the function fluctuate in the interval a, b. Some values goes up, some values goes, how much is the fluctuation? We want to analyse that.

So, that kind of thing is measured by something called the variation. So, let us write definition and then we will come back to this kind of a function monotone. So, f is a function defined on a interval a, b to R. So, this is the interval a, b. Let us look at, what is called a partition of the interval a, b. So, let P, so a equal to x0 less than x1 less than xn equal to B.

So, there finite number of points in the interval a, b, such a finite number of point is called a partition of a, b, you are cutting up the interval into parts, be a partition. So, there is, a is equal to x0, x1, x2, x3, xn minus 1 and xn. So, this is a partition.

So, what we want to do is, we want to look at the value of the function. Say in general, it will be xi minus 1 and xi right. So, look at the value of the function at xi and value as a function at xi minus 1. What is the value at these 2 points? We do not know which is bigger, which is smaller. But if you want to measure, how much is that change in the value, let us look at the absolute value of the difference.

So, that is the change in the value as you go from xi minus 1, to xi, it may be up or down, we do not know. And look at, total of this, we are taking it a general point. So, the value, value at a minus the value the value at this, this and this, probably and this and this, this and this and so on, kind of thing. So, look at this differences and this may be this.

So, this is how much, the function varies at these endpoints, these are variation of the function at this endpoints. So, this is given a name, so this we call it as V, variation from a to b of the function f with respect to the partition P. So, this is called variation of f, is called the variation of f on a, b that is a domain, with respect to the partition, with respect to the partition P of a, b. So, it depends on the partition, different partition it will be different. But, let us observe this is always a non negative quantity, because there is some of absolute values.

So, note Vab f, P is bigger than or equal to 0, it is a non negative quantity. So, if I look at the variation f over P, P a partition. If I look at these numbers, this is a subset of the real line, all are real number which are actually non negative, but we do not know whether it is bounded above or not, it may be bounded above, it may not be. If it is bounded above it will have a least upper bound right.

So, let us right, so call it supremum, let us say the supremum of this, let us write Vab of f to be supremum of of this. So, what are the possibilities, this number which is supremum of these non negative numbers. Maybe, a real number or this set may not be bounded above, if it is not bounded above, we say the variation is infinite. If it is, so if Vab f is, meaning what, this set is bounded above and the supremum exits as says a number we say, f has bounded variation.

So, look at the variation of the function with respect to all partitions. If that is finite, supremum of that is finite, we say the function as finite variation or function is bounded variation, either way.