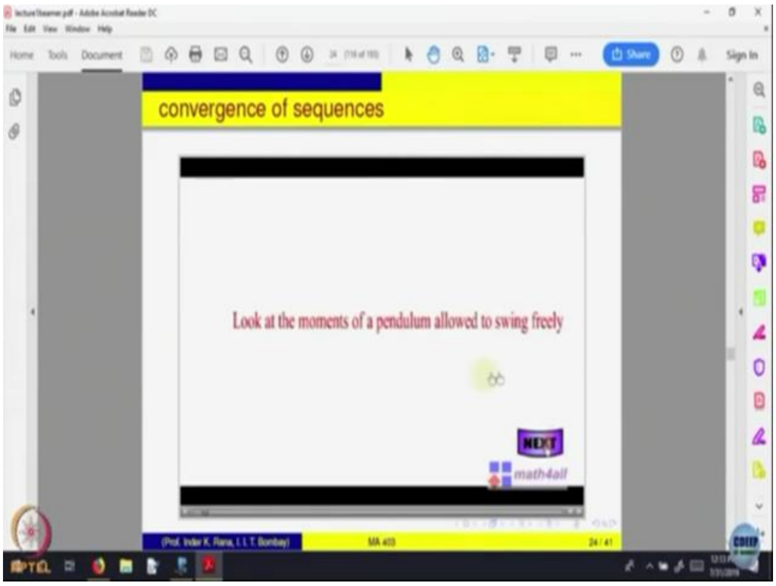
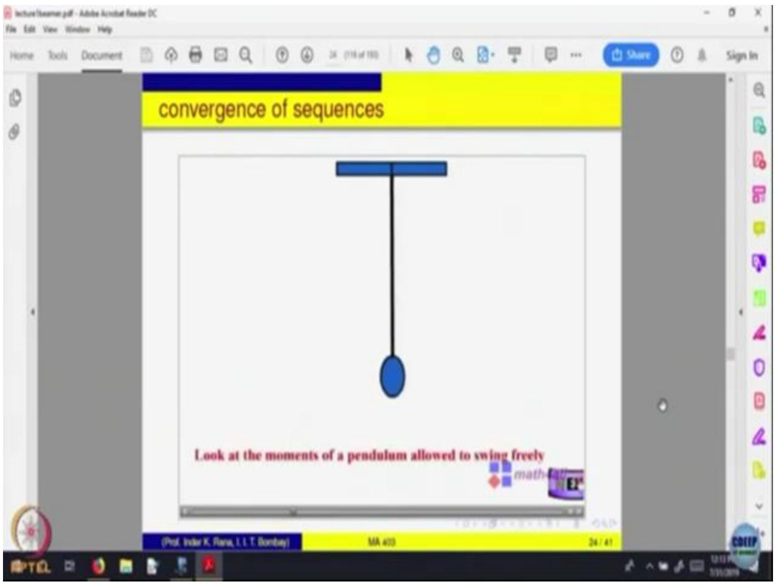
Basically Real Analysis
Professor Inder. K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture No 03
Real Numbers and Sequences-Part III

(Refer Slide Time: 0:22)



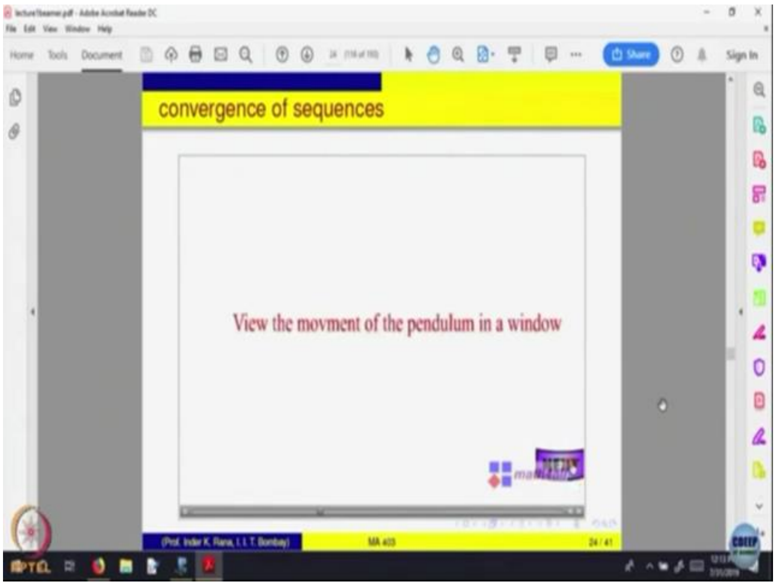
So, now, this is something very intuitive let us just observe what is happening in this. A pendulum is swinging and what do we see when we see the pendulum okay.

(Refer Slide Time: 0:39)



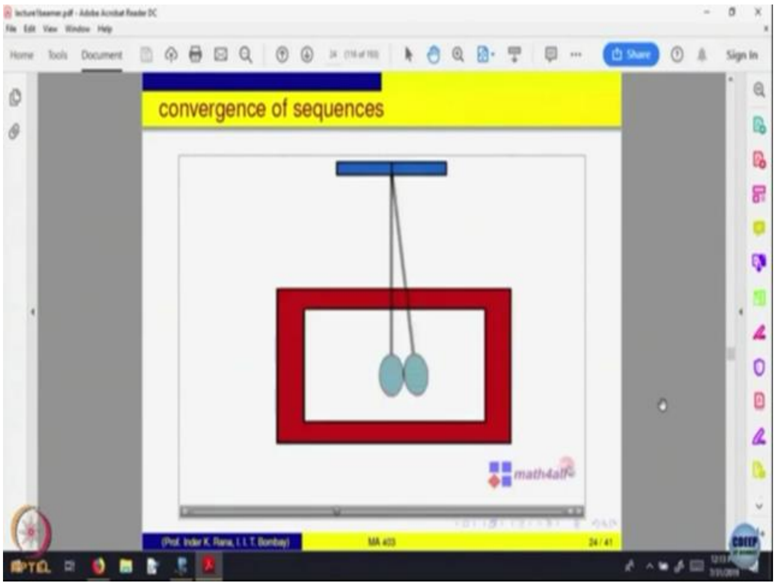
We are able to see the pendulum right swing.

(Refer Slide Time: 0:44)



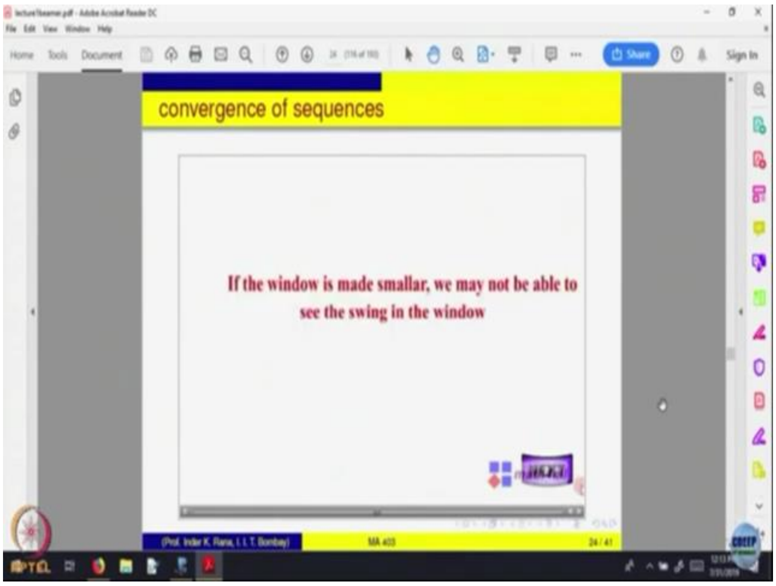
Now let us view this pendulum through a window.

(Refer Slide Time: 0:47)



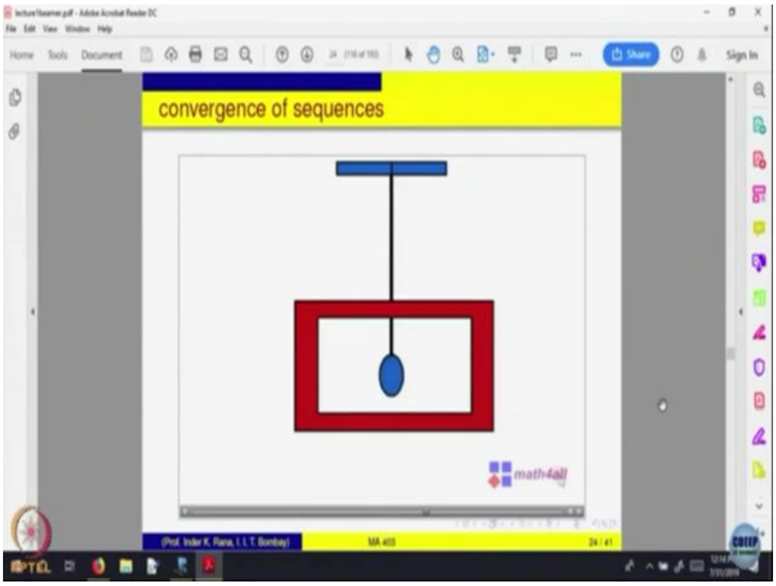
Okay, now I put a window are able to see that right. Window is big enough and you are able to see it.

(Refer Slide Time: 0:55)



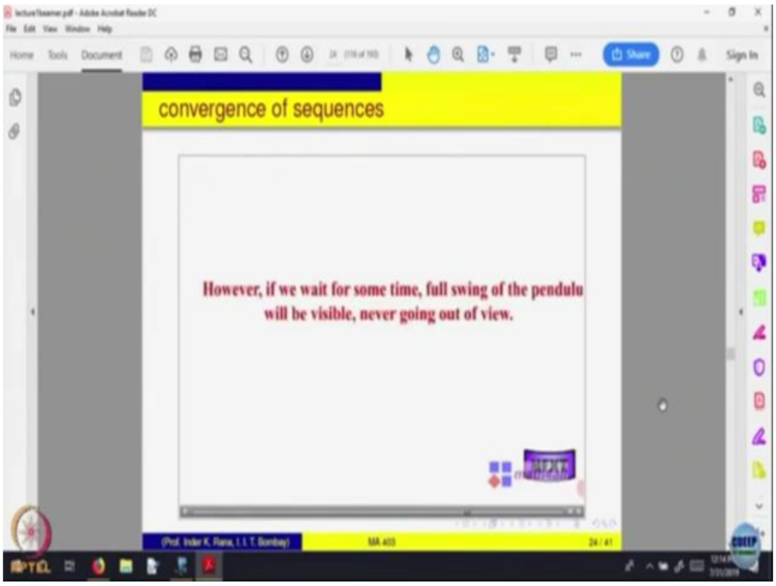
Let us make the window a bit smaller what will happen.

(Refer Slide Time: 1:04)



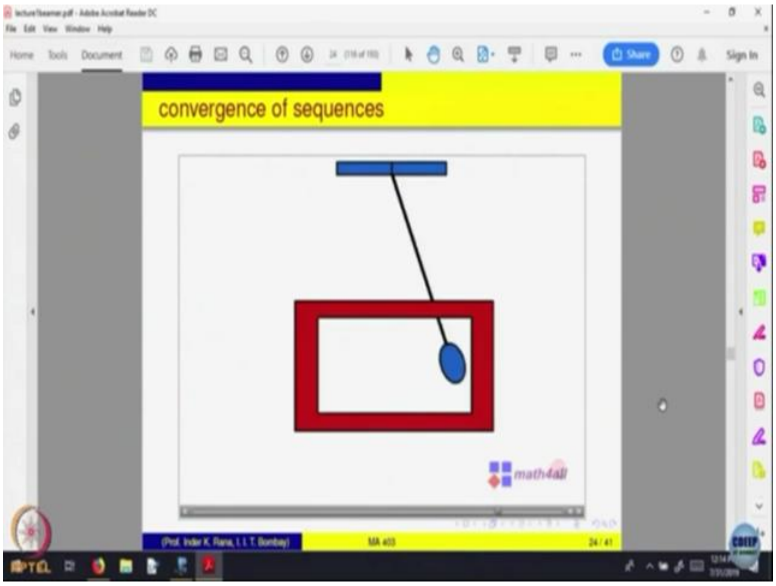
Possibly if a bigger window smaller, I may not be able to see the full swing of the pendulum.

(Refer Slide Time: 1:12)



But what will happen if I wait long enough?

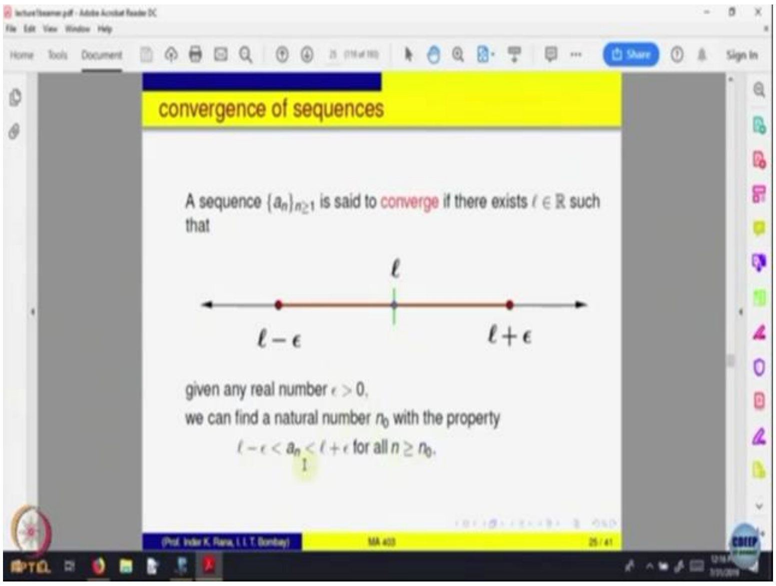
(Refer Slide Time: 1:18)



If I wait, because of gravity physics says that the swing of the pendulum will become smaller. Even if my window is smaller, I will be able to see the full swing. Let us make that window much smaller again. What will happen? Maybe still I cannot see all the swing. But again, I wait more longer enough. Again, I will be able to see. So, what I am saying is I were able to see the position of the pendulum in every window how so ever small the window is.

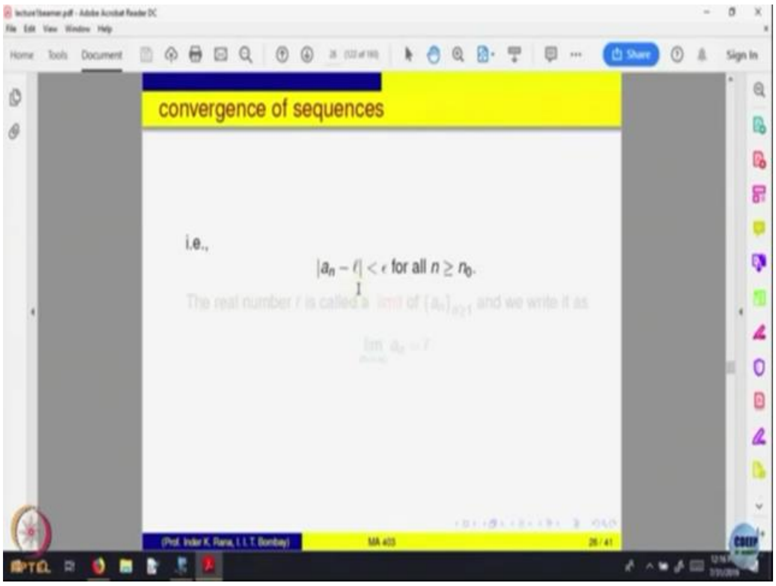
You will say the limiting position of the pendulum is the horizontal one. Is that okay? And that is what is convergence of a sequence position of the pendulum call it as an. So, given a window of some length all the ans are inside. That is convergence of the sequence. So, let us write that as a mathematically saying as follows.

(Refer Slide Time: 2:25)



A sequence an said to converge if there is a number l which is a real number such that given any window across the length of the window, so, given any epsilon, look at that window, the length of Windows l minus epsilon to l plus epsilon. So, what should happen? The pendulum should be visible maybe after 10 minutes, so, there is a stage. So, call that stage as n naught such that a n comes inside l minus Epsilon and l plus epsilon and stays inside. So, for all n bigger than n naught some stage or some weight, everything is visible to me, all the terms of the sequence are visible to me. So, mathematically saying that this happens.

(Refer Slide Time: 3:28)



Or you can write it as if you like, you can write it in terms of, there is notion of absolute value of numbers. Does everybody know absolute value of numbers for any number x real number? Absolute value of x is defined as x if x is bigger than 0 and it is defined as minus x if it is less than 0. So, this is essentially geometrically saying it is the distance of a point on the number line from 2.0. Okay, distance is always bigger than or equal to 0.

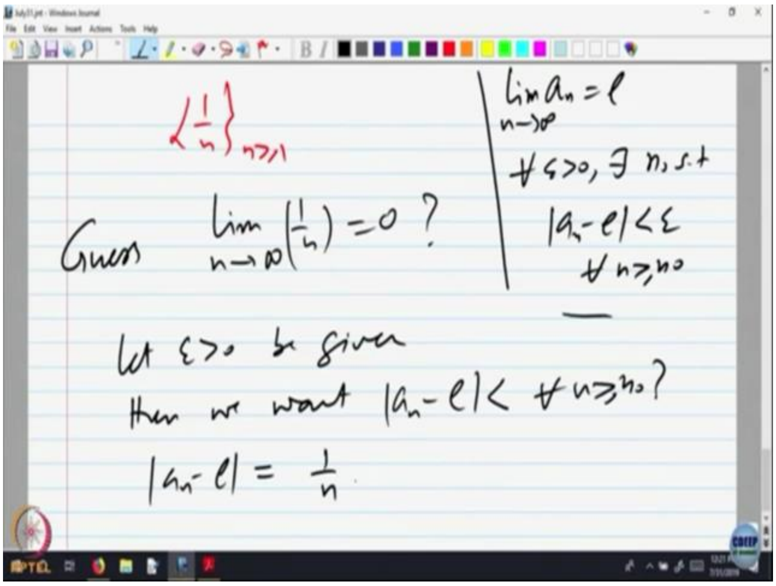
So, so this is a convergence of a sequence of real numbers So, it is quite clear to everybody what is the convergence of a sequence a n. a n is convergent if it comes closer to a value, so I have to say what value so, there exists a number l such that a n comes closer to l what does closer mean? I give you any window. So, length of the window is epsilon. So, given epsilon bigger than 0, look at l minus epsilon n l plus epsilon everything after some stage should fall inside it.

So, a n is inside the window that means absolute value of a n is distance from l. Either this side or that does not matter so, absolute value of that a n minus l distance is that the most absolute from n bigger than n naught. Another way of understanding this would be you see a n keep in mind that area thing a n is an approximation to l. And what does a n minus l indicate?

It is an error you are making actual value is l, you are taking an approximation al, so, the error is a n minus l that error I want small, how small? I will specify how small, so epsilon is a Greek letter for e, okay, that is why one, it has stuck as epsilon. So, given the margin for error, that is a distance for error epsilon I will specify it should come that close. So, a n should come close to l how close? That error should be less than this for all n bigger than n naught after some stage.

So, keep in mind for a sequence convergence depends only on the tail of the sequence from some stage onwards for large n it does not depend on the first few terms. They could be anything because what we are interested in is what happens for n very, very large after some stage, if they all come inside that window for every epsilon, every window then good enough, what first 1 million, 1 trillion does not matter how many times you have to forget, forget them. So, it is a tale of the sequence which is important for analyzing whether it is convergent or not, clear? So, that is what is called convergence, okay.

(Refer Slide Time: 6:54)

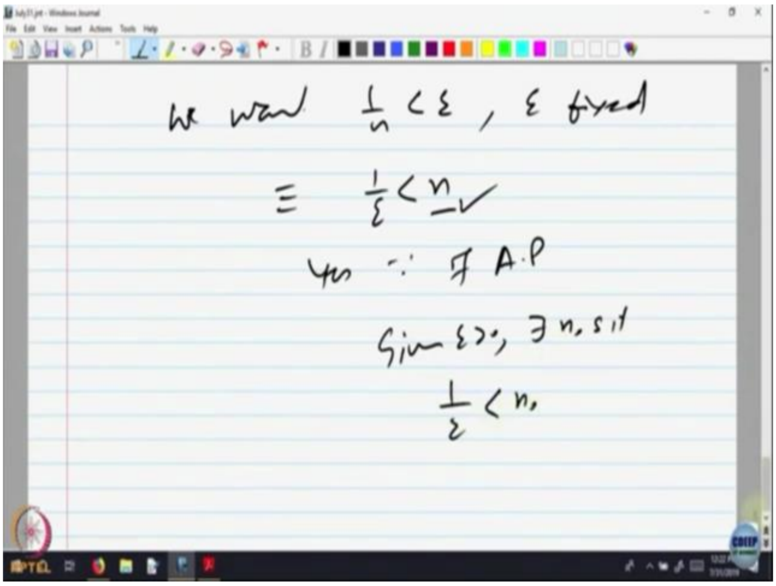


So, this is what, well, let us look at, let us look at some examples I think. So, let us look at the sequence 1 over n so, limit, so we write as limit n going to infinity that is that a n limit n going to infinity equal to l. So, what does the meaning of that? that is only a symbol, that is only a rotational way of writing saying for every absolute bigger than 0 there exists some n naught such that a n minus l is less than epsilon for every n bigger than n naught.

So, let us look at the sequence 1 over n so, whenever you want to analyze whether a sequence is convergent or not you have some way of making a guess of that thing what you think is possible happening, so let us, most of the time you will have to do that and then prove it. Okay. So, make a guess what happens to 1 over n as n is becoming larger and larger, so, n becoming larger meaning 1 over n 1, 1 by 2, 1 by 3, 1 by 4 it is becoming smaller and smaller.

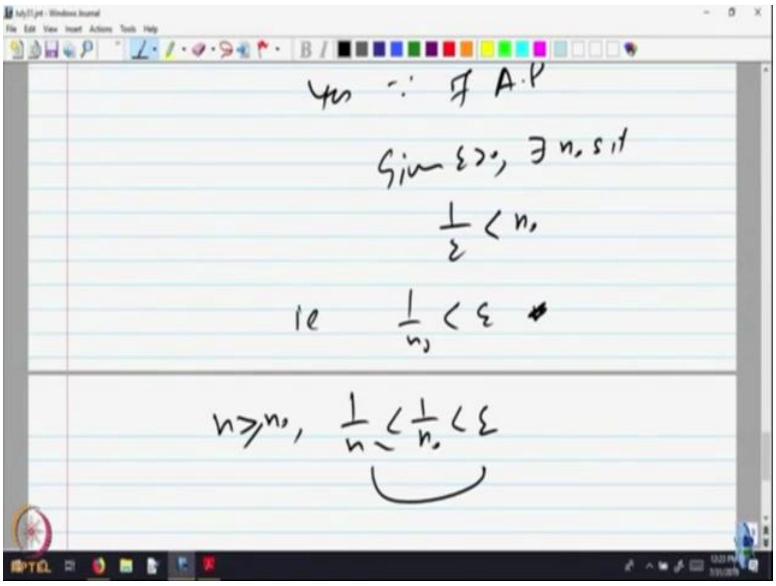
So, guess so, this is my guess limit 1 over n is equal to 0. So, how do I prove it? I had to prove it according to this. So, let epsilon greater than 0 be given then we want n minus l less than epsilon for every n bigger than n naught. So, what is a n minus l here we analyze that the error we want to make it small. So, let us analyze what is the error actually in our case. So, that is 1 over n where n is 0, n is 1 over n, so this is 1 over n so, we want this to be less than.

(Refer Slide Time: 8:55)



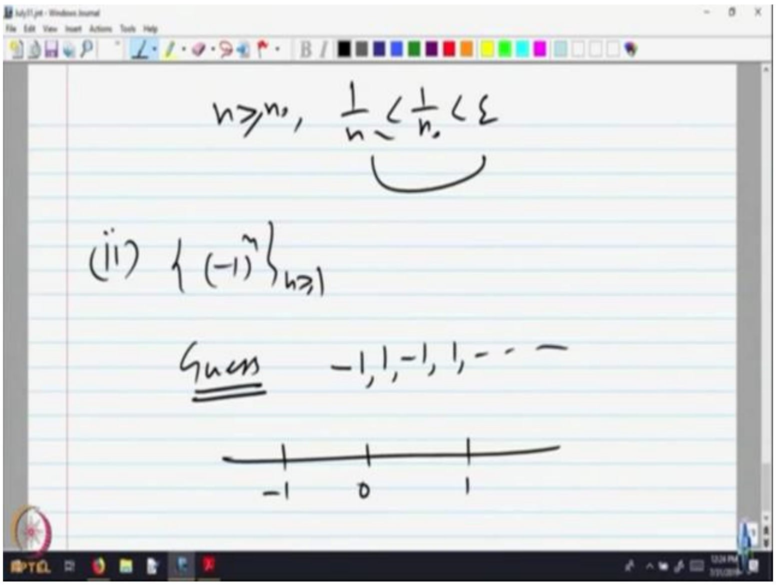
So, we want 1 over n less than epsilon. What is epsilon? Epsilon fixed. That is given to me a margin that is same as saying that 1 over epsilon is less than n. Is it okay? So, that, that means given epsilon I want n such that 1 over n should become less than is it possible? Yes, because of Archimedean property of real numbers given any thing I can go across by using n large enough so, that Archimedean property it says yes, it is possible. So, for that particular epsilon, for given Epsilon begins there is n naught such that 1 over epsilon is less than n naught by Archimedean property.

(Refer Slide Time: 9:56)



So, that is same as saying that 1 over n naught is less than epsilon. So, if n is bigger n naught then what happens to 1 over n? When it is less than 1 over n naught and that is less than Epsilon. So, we gotten a n less then Epsilon. That is how you write the proof that 1 over n as limit which is equal to 0.

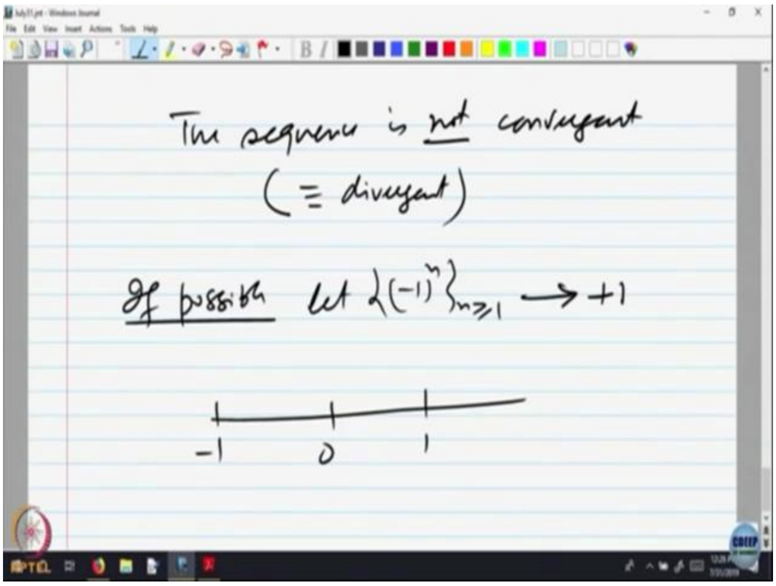
(Refer Slide Time: 10:35)



Let us do one more example. Let us look at that sequence in the example that was minus 1 to the power n, n bigger than or equal to 1. So, I have to make a guess is this convergent or not. So, what are the terms? So, this is one way of look at the few terms of the sequence at least it may give you a hint. So, terms are minus 1, 1, minus 1, 1 and so on. So, if I look at the number line here is 0, here is minus 1, here is 1. So, the sequence is visiting minus 1 and 1 at every time point n and these are only positions where it goes.

So, intuitively, it looks quite clear that it cannot converge to anything other than minus 1 or plus 1. If at all it converges, the possibilities are either minus 1 or plus 1 because these are the only places occupied by it. So, for as a first step, we want to show that, so, guess is it does not converge. So, how do I write a proof of that? So, we want to write a proof.

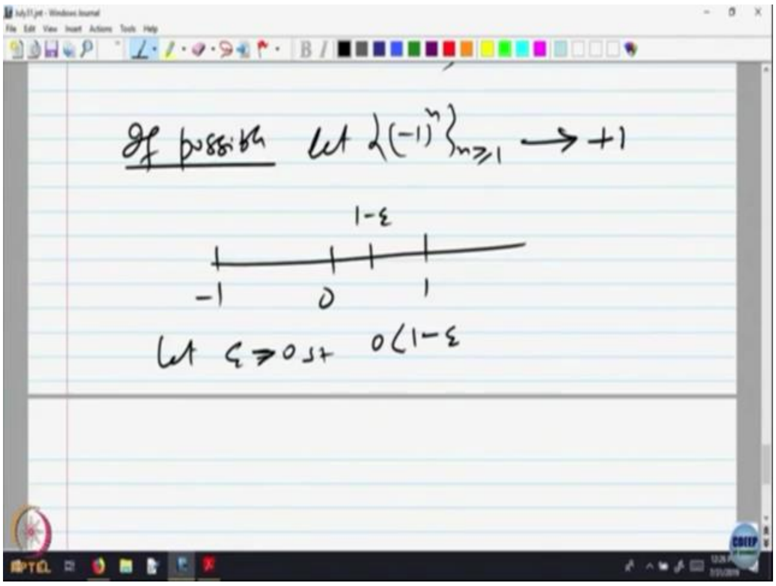
(Refer Slide Time: 12:01)



So, guess the sequence is not convergent that is equivalent to instead of writing not convergent, not convergent letters give it a name, we say it is divergent. Something which is not convergent will be called as a divergent sequence. So, we want to say it is a divergent sequence. So, if possible let this sequence minus 1 to the power n, driving by n bigger than or equal to 1 converge to let us say plus 1 or okay we can do all of them one by one, so here is 0 here is plus 1.

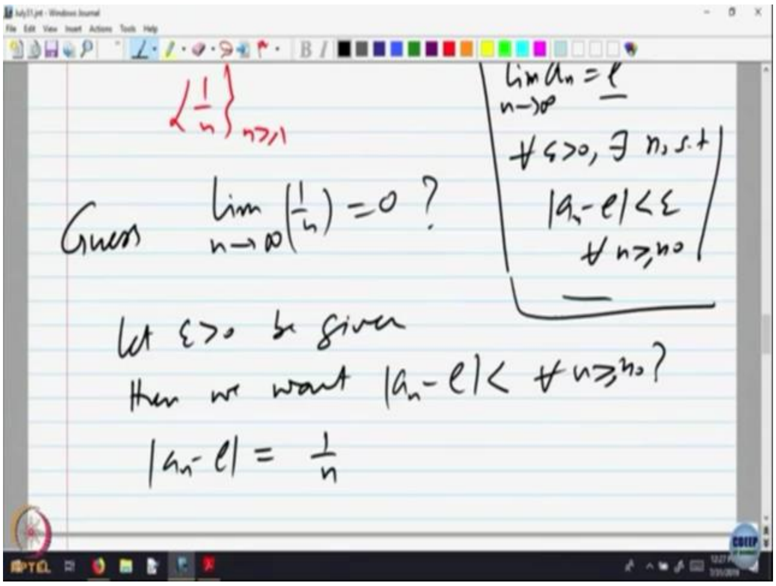
Here is, now, intuitively if the sequence has to converge, it has to come closer to some value and we are saying it is coming closer to the value plus 1 after some stage, but I know that whatever stage I choose, after that positively it is going to come to minus 1 it is going to come away from it.

(Refer Slide Time: 13:25)



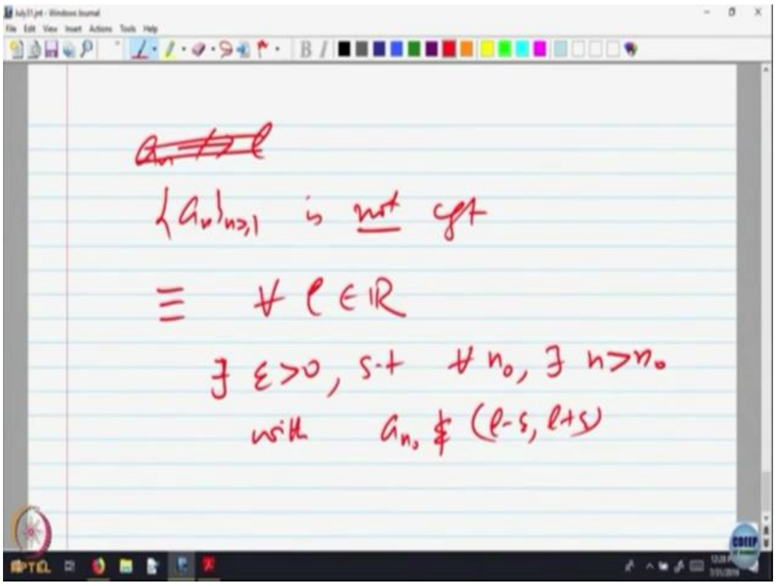
So, that says to me that let us take epsilon equal to, epsilon bigger than 1 such that 1 minus epsilon is bigger than 0. So, let us take here is 1 minus epsilon I want to show it has not converge to plus 1 I had to say that there is a, okay so this is important saying something is not convergent, what does it mean? That is, let me discuss that first.

(Refer Slide Time: 14:00)



If this thing does not happen, my guess is the sequence does not then what I had to do? To prove convergence given epsilon I have to find a stage n naught. To prove it is not convergent, what I have to do? I have to show that whatever l I choose. Whatever l I choose this thing should not happen that means, what? That means, I should be able to, that is, if I want no l to be the limit right that means, I should be able to find at least one epsilon one window. So, that whatever stage I say something goes out of it, everything after that is not visible so, something goes out of it.

(Refer Slide Time: 14:49)

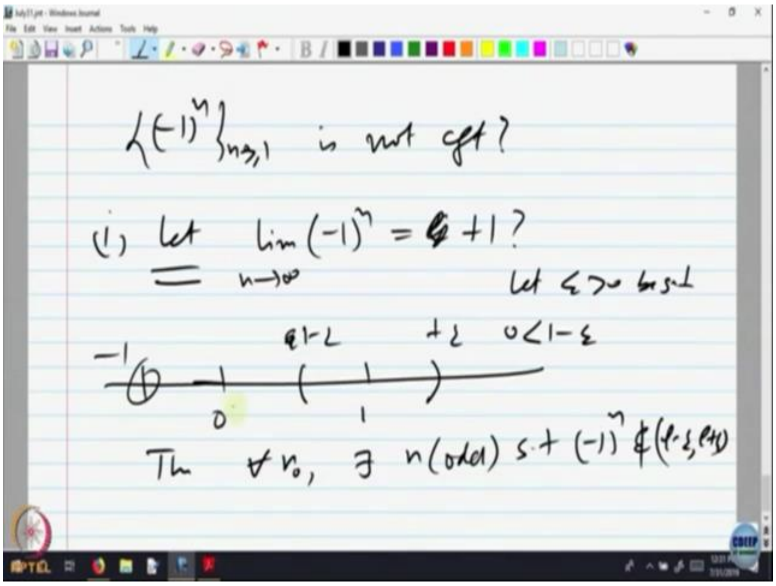


So, let us write that that is important. So, before I write that the, an does not converge to l, an no, I think better way of writing is an is not convergent is equivalent to saying for every l bigger than 0, for every l which is a real number. See limit exists when there exists one l I do not want that to happen that means for every l something should go wrong. What should go wrong? For every epsilon there exists some Epsilon bigger than 0 such that for every stage and not there exists some n bigger than n naught with an naught, naught belonging to l minus epsilon to. Is that okay?

Saying that convergence means there is a l for which something is true. False for every l something is false, what is false? For the, for every l I can find at least one window that means there exists epsilon such that if I look at that window and look at any stage n naught then there is one stage at least after that which goes out of it because convergence says everything must be inside. So, not true meaning whatever stage I give you I can find at least one point after that stage, so, that it goes out of that window.

It is something like in that pendulum. If after some time I give a kick. I keep on giving a kick. So, I can find always a window it will go out, it will not be convergent, something like that should happen, so, not convergence.

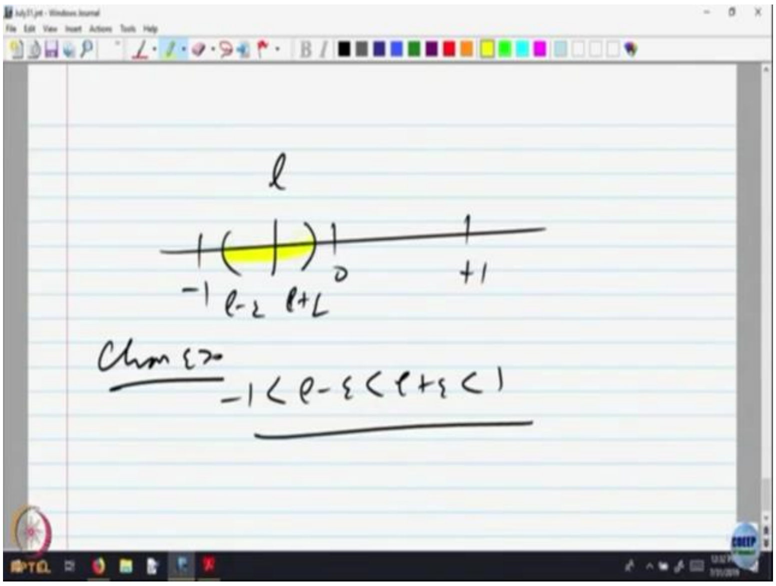
(Refer Slide Time: 16:58)



So, let us look at the example that minus 1 to the power n divided by I am sorry. Not divided by minus 1 to the power n is not convergent. So, case 1 let. So, this is my an let limit minus 1 to the power n and going to infinity be equal to l be equal to say plus 1. So, let us assume, so, if this is plus 1, this is 0, I know the sequence goes to minus 1, whatever stage, if the stage is even then at the odd stage, next odd stage, it will come out, it will go to minus 1. So, positively it will go out of this kind of a window.

So, I have to select there exists an Epsilon, so find an Epsilon, so that this is 1 minus epsilon 1 plus epsilon that is what I said, let epsilon bigger than 0 be such that 1 minus epsilon is bigger than 0 then for every n naught there exists n odd such that minus 1 to the power n does not belong to l minus Epsilon and l plus Epsilon it will go out. So, it will be minus 1 somewhere at the odd stage that is outside. Not only once it goes out, whatever place I give you, after that I take any odd stage it will go out, so, this cannot be, plus 1 cannot be the limit. Similarly minus 1 can be the limit because then I will go to even.

(Refer Slide Time: 18:50)



Can something in between with a limit? This is 0 this is minus 1 this is plus 1, can this l be the limit? No obviously again if I take a window like this. If I take a window like this, then no n is going to be inside it because n is only plus 1 or minus 1. So, for this case I will choose minus 1 less than l minus epsilon less than l plus epsilon less than 1.

So, I choose for this kind of epsilon, so, this is this is going to be my window. So, nothing is going to be inside leave aside something going out, so, this cannot be the limit. So, neither plus 1 can be the limit neither minus 1 can be the limit nor anything other than plus and minus 1 can be the limit that proves the sequence does not converge. So, this is how we will write a proof of something not convergent. Okay.

Is it time to stop? I do not know what is, what time we are supposed to be having lecture. What is the time now? 12:30. So, let us stop here. So, what we have done today is, I try to give you some indication of why rationals are not good enough, why we need reals, what are reals? What are the important properties of reals, which distinguish them from the rationals that is namely one, completeness property and we looked at the sizes rationals are accountable in finite, reals are uncountable in number that is something.

Then we looked at the why we need sequences. And how do we analyze what we are interested in when n becomes large, what happens to the sequence n we defined the notion of convergence of a sequence. Okay, let us stop.