**Basic Real Analysis**
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**Lecture no. 31**
**Differentiability – Part I**

So, let us start looking at new topic, about Differentiability of functions, of one in several variables. So, we will recall Differentiability for a function of one variable and then go on to functions of several variables. Why one should study Differentiability at all, there are many reasons for that.

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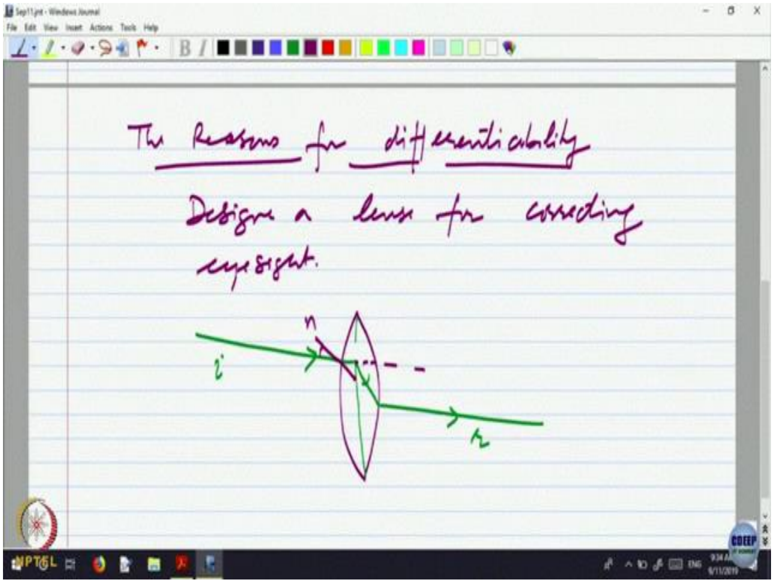

For example, the reasons for study what we call differentiability or the derivative of a function. Let us look at the problem that you want to design a lens for correcting eyesight. So, what kind of lenses are there, there are 2 types. One is convex, other is concave depending on what correction you want. So, let us look at say one of them. So, this is a lens that you want to design. Now, why the lenses are used to correct eye sights, because when the light passes through the lens, it changes its path.

So, let us have a look at it a light ray going inside. So, this is the centre, so it changes its path and then comes out. So, what is the, how much the light rays change their path, how do you measure that. So, there is a question which is dealt normally in physics, it says it depends upon at what angle this is what is called the incident ray and this is called the refracted ray.

So, it depends upon at what angle this ray is hitting that surface and because it is going from one medium air to glass, it changes its path. And then again it comes out, again it changes its path. So, the question is how do you measure, how much this instead of going it straight like this where should it changed. So, how do you measure it. So, in physics after experimentation and all that, they have looked at what is called the angle of incidence, at what angle, the light ray hits the lens. And then, the angle at which it will come out there is called the angle of refraction. So, there is a relation between angle of incidence and the angle of refraction.

So, now the question that mathematically what we want to ask is, if this is a light ray which travels in a straight line hits a curved path, what do you mean by the angle at what angle it hits? How do you measure the angle? So, that is a question, is the question clear to everybody? A straight line, light ray hitting a curved path surface of the lens that is curved. So, what do you mean by the angle of that? We normally, we know that notion of angle only for, between 2 lines, 2 lines intersect what is the angle between them.

So, how do you shift that to a curved surface, so for that, they say that instead of looking at the curved surface, as somebody was trying to point it out, let us draw a normal to this, curved path. So, the angle you measure is how much is the angle between the normal and the light. So, that is angle taken as the angle of incidence. Now, the question comes what is a normal to the surface? How do I draw a normal, so to draw a normal, you should draw a tangent because normally take as the line which is perpendicular to the tangent.

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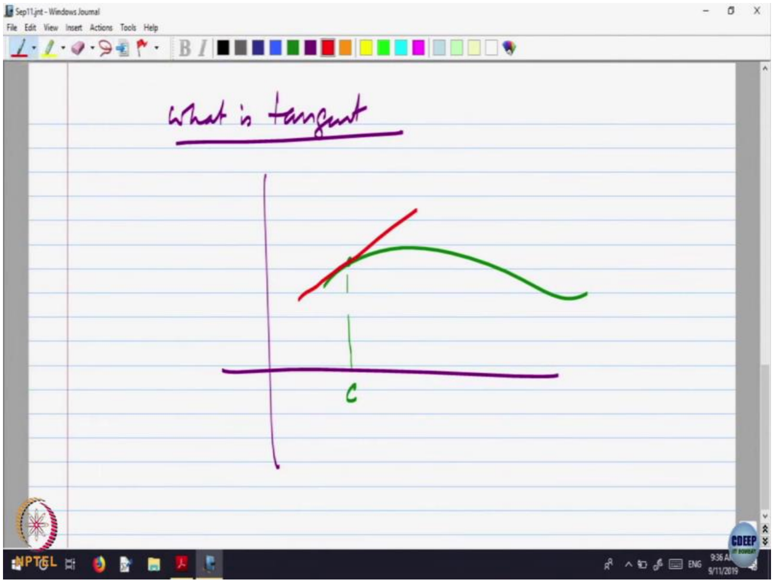


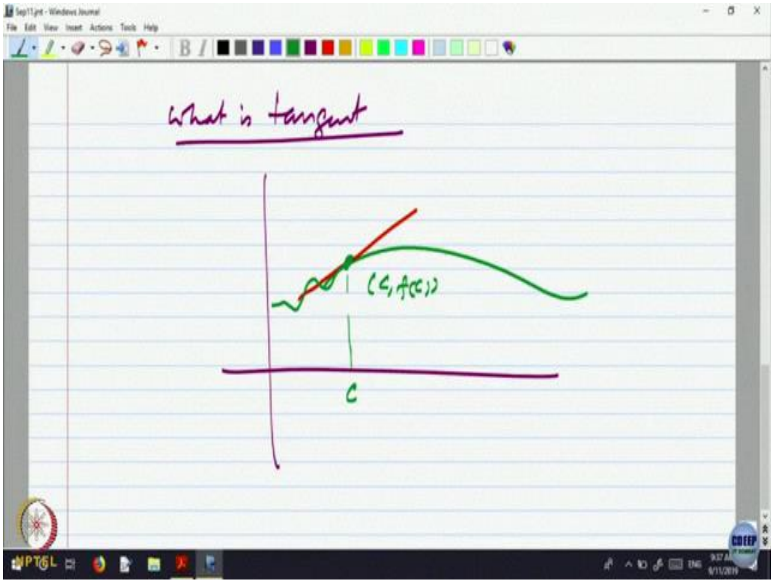


So, what is the meaning of a tangent to a curved surface? So, what is the meaning of, so basically, the comes to, what is, what is tangent? So, let us take the simplest possible case, that we are got a graph of a function. So, this is the graph of a function and at this point, I want to draw a tangent. So, first of all I have to say what is a tangent, what is, before even finding a tangent I have to define what is a tangent.

So, normally in geometry you take tangent as the line which touches that curve, only at one point. But that is English, touches. What is the meaning of touches at one point, you will say it meets the curve only at one point, but that is not. For example, I could have change this, I could have change my graph. So, let us look at the graph, for example instead of going like this, it could go like this. It can cut, it can meet the graph at many points.

So, what is the meaning of touching? How do I capture mathematically what is touching? So, you want to say something, what is the meaning of touching? Why should I take that as a definition of it? Why should I consider that at all? So, the question is, what is the meaning of tangent, what is a tangent? What kind of object it is, first of all?

Tangent is line, to draw a line mathematically what do you need, you need a point? And you know, want to know the slope of that line or you want to have 2 points on that line. Then you know the equation of the line, mathematically. So, when you want, you already know that the point, point is here. So, this point is c comma, f of c. So, we know this line is going to pass through that point. So, that problem is solved, only we want to know what should be the angle of that line. What should be the slope of that line? So, how do I capture the idea that it is that line which touches this, that curve only at one point how do we capture that, idea.

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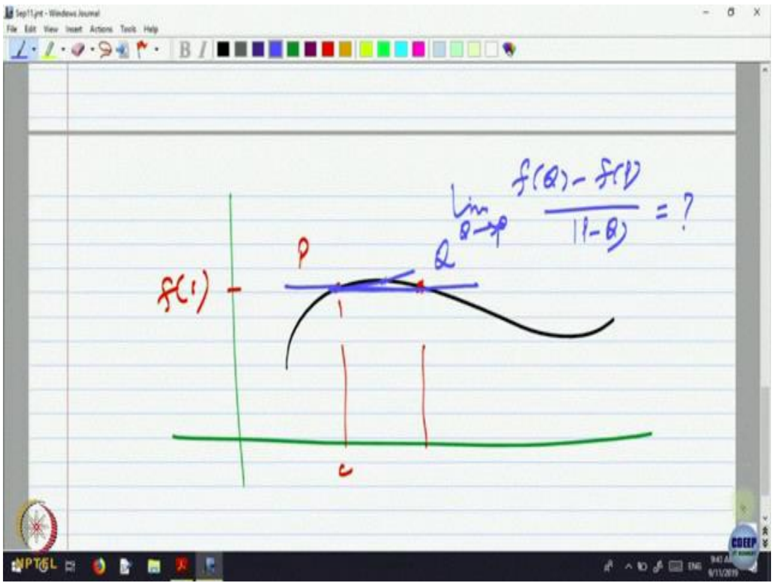

So, to do that, let us just, look at slightly more carefully. So, here is the point c. So, this point is f of c. Now, at that point, we want to say that the line, it is a line which touches the surface only at that point. So, let us call that point as P if you like. Now to capture that idea, let us take a nearby point. And let us take the line joining these 2. Obviously, this line is meeting at 2 points, it is not meeting at one point.

Now, and the possibility is that in between P and Q, the graph of the function may be cutting it many times, maybe thousand times, we do not know. But the idea is, if I make this point closer, so I bring it here and draw this line, then I will be omitting some of the points where it may be intersecting. If I make it more closer, then I may be omitting more points, where it was probably intersecting other than the point P. So, if I may go on bringing it closer and closer, eventually, it will meet, it will omit all other points accept the point P.

So, that is the idea, that is why we want to take, so we want to take this point Q closer to the point P. And why we want to take that because eventually that line will meet only at one point, all in between points will be sort of removed. Now, and that line I would like to call as a tangent. But what I want to know, what is the slope of that. So, slope, for the slope, let us look at the slope of this line P, Q and we are bringing that point closer and closer to P.

So, slope of this line as Q approaches P should give me the definition of slope of the tangent at that point. That is why we take that, f at Q minus f at P divided by the distance between P minus Q and limit Q going to 0 or Q going to the point P. That is a reason, that we take this quantity if it exists, as the slope of the tangent at that point. So, this gives a mathematical way of defining what is called the slope of the tangent at a point, that is how you should understand that.

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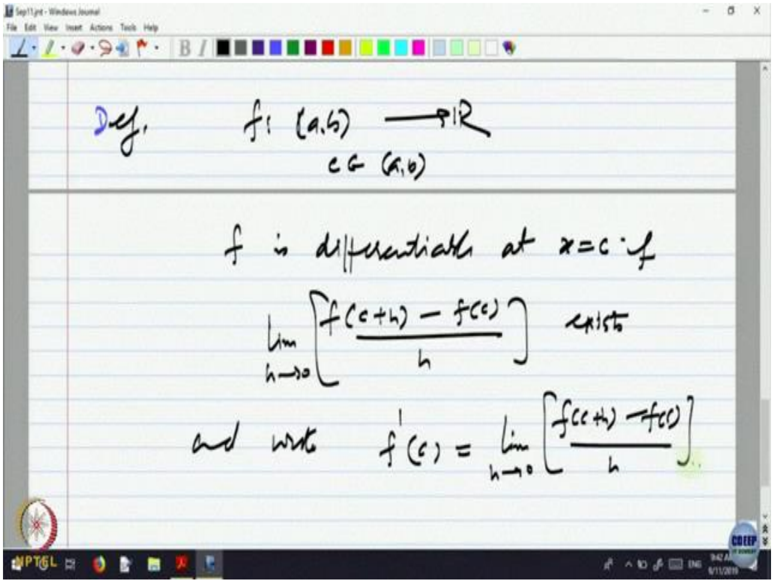


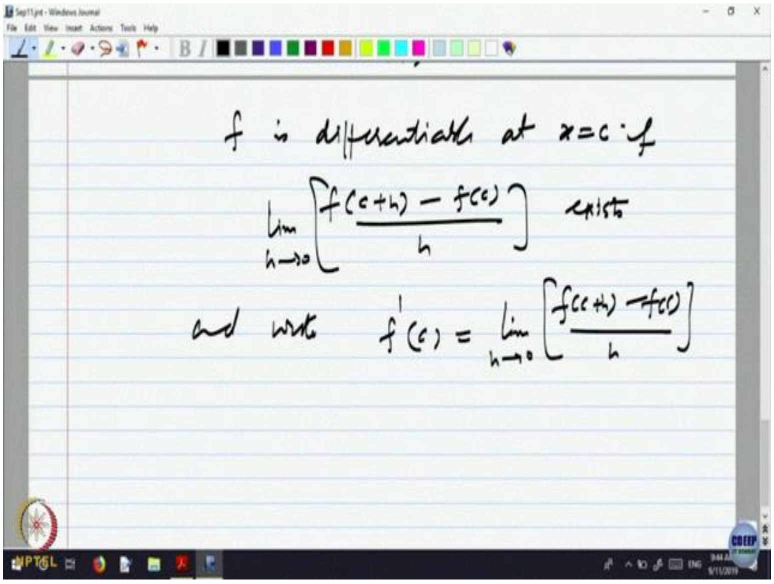


So, let us define it. See, normally in schools we define only for circle. What is the notion of a tangent for a circle, there it meets only at one point, once it meets either it will intersect at 2 points or only at one point. So, that problem does not arise, so they are, so let us define. So, definition is f is a function defined on a intervals a, b to R and c is a point belonging to a, b.

So, we say f is differentiable at the point x is equal to c if. So, what is a nearby point, I take c plus h, a small quantity h. c, h is positive, it goes to the right, h negative it comes to the left, minus f of c divided by h. Is it okay? y2 minus y1, divided by x2 minus x1. So, that is the slope of the chord joining the point at c and c plus h and I should take limit of this as it goes to 0. If this quantity exists, then we say the function is differentiable and write f dash of c equal to this limit. So, this is the limit h goes to 0 f of c plus h minus f of c.

So, all of you have gone through this definition. So, just wanted to, why should I be looking at this kind of thing. The another, reason for this comes from, again from physics basically. The most of these ideas come from practical day to day life. When you are travelling, say in a car and you suddenly look at the speedometer and say I am travelling at 60 kilometres per hour. Now, at this moment, what is the meaning of my speed at this moment is so much? What is the meaning of that? You can calculate mathematically only what is called the average speed. I started 10 o'clock, at 10:10 observed how much is the speed.

So, how you got what is the speed, that is a distance covered divided by the time taken. So, see at from 10 to 10:10, how much distance you have covered, 10 seconds or 10 minutes whatever it is divided by 10. That is an average speed in that interval 10 to time interval of 10 seconds or 10 minutes. Now, if we shorten that, that means, if you go from 10 to 10:05, find out what is average. So, you and if you go on making it smaller and smaller, you will find, you will be getting something on average at that time.

So, that is again f of c plus h is a distance at time point c plus h f of h c is. So, there is a distance covered in time h. So, average, this average c to c plus h and take that average to be limit of that as it goes to 0. So, average at that moment you will call it as your speed. So, in physics that gives you the notion of speed. So, that is, so if you want to look at more realistically, a speed is what is the rate of change of distance, versus time. So, how much time and how much is that distance covered in how much time. So, that is a notion of derivative.

So, will not go into all the properties of the derivative. Namely, if f is differentiable at a point, g is differentiable, algebra of derivatives, all of you have done that. So, we will assume that, we will not spend time on that. So, if f and g are differentiable, f plus g is differentiable, differences differentiable, product rule you know, quotient rule know. Composite role that is a chain rule, function of a function, f composite g if there appropriately defined and g is differentiable at c, f is differentiable at g of c.

Then f composite with g is differentiable at the point c and the derivative of f composite g is f dash at the point into g dash, the chain rule that comes, that is important aspect of that. So, all those properties we will assume.

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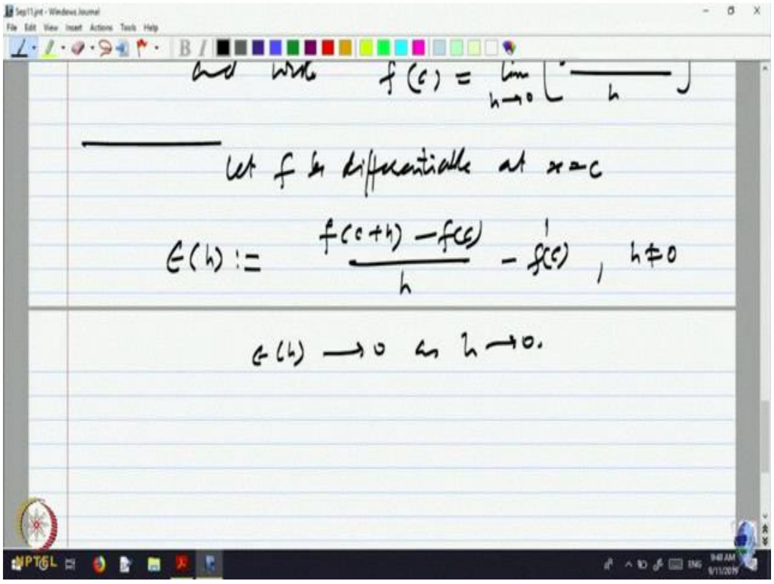


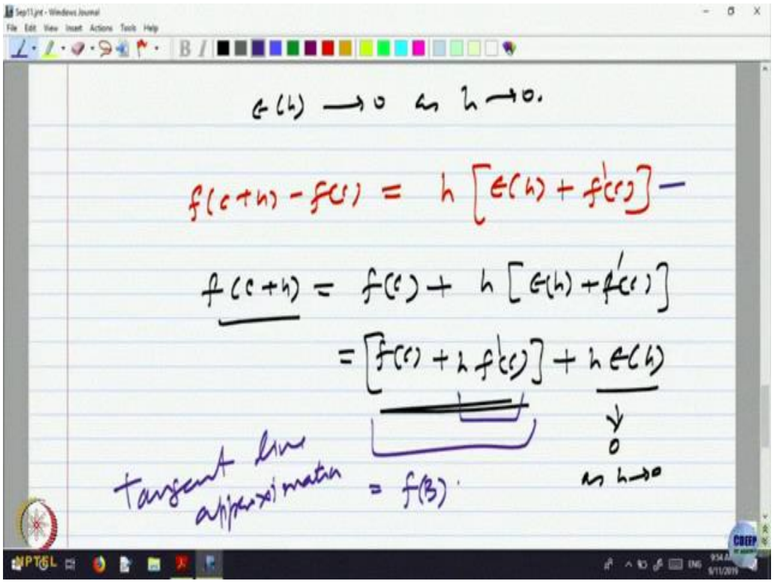


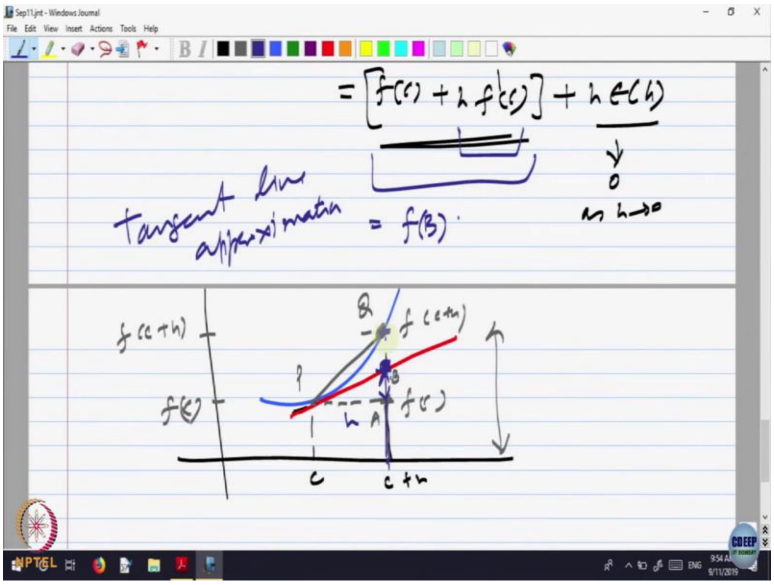

Let me stay at one of the, let me give another interpretation of the derivative which is useful. So, what is this quantity? So, let us say, let f be differentiable at x is equal to c. Then what is this quantity f of c plus h minus f of c divided by h, what does this represent, what does this represents, geometrically this represents the slope of the chord joining the point at c and c plus h. So, and this differs, so this is not the slope of the tangent, it is approximation to the slope of the tangent, you can look at that way. What is the actual value of the slope, f dash of c. So, that is the difference between the 2.

So, let us call it as a, this depends upon h, of course, the point c, so, let us call it, c is fixed. So, define h not equal to 0. So, what I am looking at, is a error that I am making in finding the slope of the tangent at a point. So, I am looking at the slope of the tangent, minus the slope of the secant that is called the chord, is called the secant. So, f differential means this epsilon h goes to 0 as h goes to 0. Epsilon h is a function. Its limit as h goes to 0 is 0, that is a differentiability of the function at that point.

Let us rewrite this equation, star, in slightly differently. Cross multiplied, so that gives me f of c plus h minus f of c is equal to h times epsilon h minus f dash of c. That equation I have cross multiplied both sides by h. So, let us analyse this slightly further, it says f of c plus h can be written as f of c plus h times epsilon h minus f minus or plus, there should be plus because on the other side, h times f of c. So, I have taken, that is plus, plus f dash of c.

So, let me rewrite this as f of c slightly more h times f dash of c plus h times epsilon h. I read it in that equation. Now, what does this tell me, that beautiful interpretation of this it says that if the function is differentiable at a point c, then you know the derivative. It gives you the value of the function at a nearby point f of c plus h, in terms of the value at that point f of c, how much you are away h times, so plus some error. So, it is not exactly equal to, it is because this thing goes to 0 as h goes to 0. What is this quantity, which geometrically?

Let us try to look at it, slightly more geometrically. So, this is, let me draw the function let us draw it this way this time. So, this is the point c, this is the point c plus h, this is the tangent at that point and that is the chord at that point. So, that is f of c and this is f of c plus h. So, this is, this height is f of c plus h, this is side is f of c and there is a. So, let us look at this, so let us call this as points, this is P that is Q, let us call it A and this point that is B.

So, what is, let us interpret this equation now. f of this value is equal to, what is this value, this is height, this height that is f of c plus h. What is that equal to, f of c this is f of c, this part is f of c plus h times f dash, what is h times f dash, geometrically, what is f dash, that is a slope of the tangent. So, what is h times f dash, how much is, so what is this A, B and what is this B, Q? Can you say what is A, B? A to B, what is this height? You know the slope of that red line that is a tangent, you know this distance is h.

So, h times f dash that is, A, B is precisely h times f dash. So, this is, this quantity. So, this plus this, so what you are doing is, instead of taking the value at the point the height to be Q, you are taking the value to be this one that is height at c to from this point to B. So, this is the error, so this is the error the B to Q is the error that you are making. B to Q is the, that distance is the error, you are making.

So, what you are saying is, instead of taking the value or the function at the point c plus h, you are taking the value this. So, I can write this as f at this point B, if you like. Where is a point B, B is on the tangent line, B is point. So, instead of at the point c plus h, instead of looking at the value at f of c plus h, you look at the value at the tangent line that is a good enough approximation plus some error will be there, which goes to 0 as h goes to 0. So, this is the reason this is called the tangent line approximation.

So, this is actually a equation of the tangent line, very precisely. That is a question of the, if you, let the point h vary, that is a equation of the tangent line. So, this is called the tangent line approximation and this is the one which is used most often in many of the calculating machines. For example, your calculator, will give you approximately value of something, that some nearby point. So, that is calculators, this is what is used, normally.

And this is the simplest kind of approximation called tangent line approximation. Instead of looking at the value of the function at the point Q, it could be anything, because the function could be anything, you are looking at the value on a straight line. The simplest possible approximation called the tangent line approximation, that is a geometric interpretation of the derivative.

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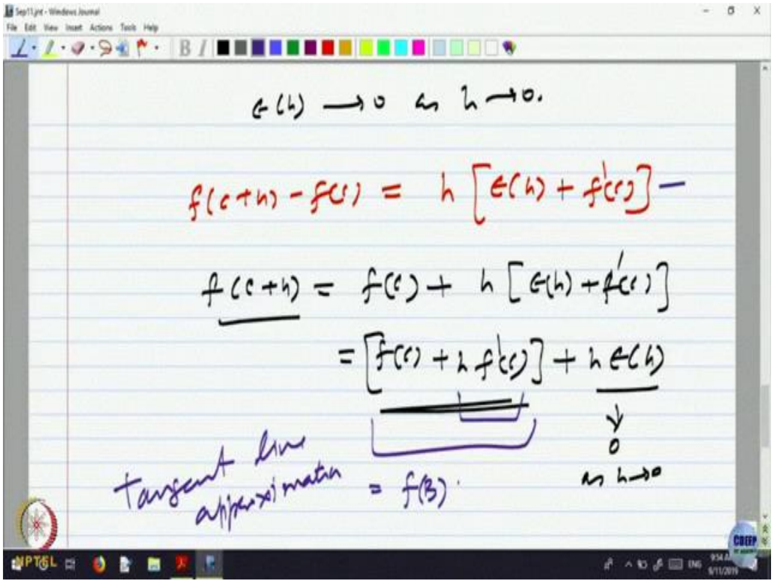


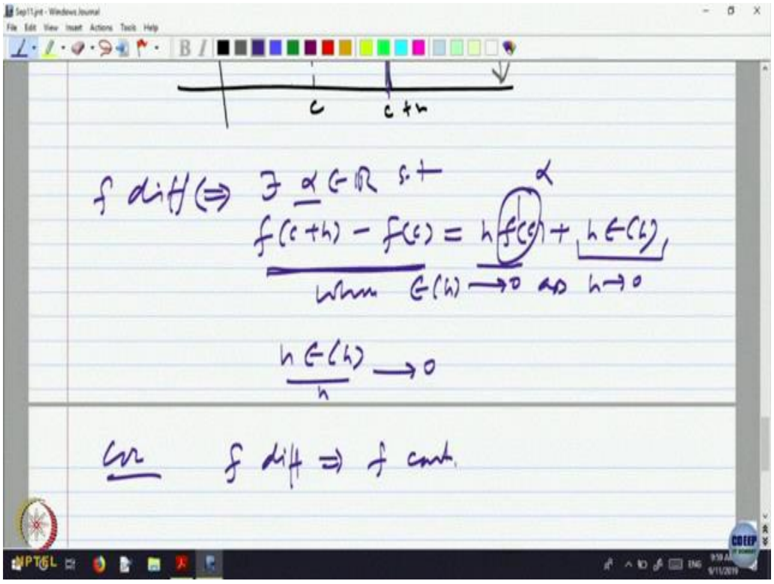


Let us slightly look at a bit further because that will be useful. So, let me look at this equation that star that red one. So, what is that red one, it says f at, differentiability f differentiable implies f of c plus h minus f at c is equal to h times f dash c plus h times epsilon h, where epsilon h goes to 0 as h goes to 0, there is a error. So, I just read it on that equation, I will just read it in that equation, f of c plus h minus f of c is h times f dash plus h times epsilon h.

One thing more you should observe, because normally this is not mentioned in calculus courses. See epsilon h goes to 0, but our error, our error is h times epsilon h, epsilon h was an error in the slope of the tangent and the secant. But in this approximation, the error is this much, there is a tangent line approximation. And now, note that not only epsilon h goes to 0, even if I divide by h that also goes to 0. Because when I divide h cancels, epsilon h that still goes to 0.

So, the error in tangent time approximation goes to 0 very fast. Even though h is going to 0, so this ratio will, something in the denominator going to 0 means you should blow it up, but it says it does not blow it up, it still goes through 0. So, the rate of convergence for tangent time approximation is very good. It is as good as h itself, divide by h it still goes to, so that is one thing.

And the second observation from here, one should observe. Supposing, I do not know f is differentiable but I can write these difference h times some scalar, plus an error which goes to 0. When I divide by h that precisely says that limit exists and is equal to their scalar. So, this is definition of differentiability. So, what is the definition, if and only there exist some alpha belonging to R such that instead of this quantity, I will write alpha. If this, if we can write f of c plus h minus f of c as h times alpha into something that goes to 0, that error.

Then, when I divide, I will get the alpha should be the derivative, limit exists. So, conversely, this alpha will become the derivative. So, differentiability should also be taken as f is differentiable, if and only this increment can be written as h times that is some scalar plus some error which goes to 0. And this definition becomes more independent of geometry now, it becomes more analytical definition. And that is how we will take it up for functions of several variables. So, that is one way of looking at derivative.

And one thing more, if f is differentiable, what is the limit of the left hand side as h goes to 0, h times f dash that goes to 0, h time something that goes to 0. So, f of c plus h goes to f of c, so that automatically implies f differentiable implies f is continuous. So, as a corollary of this f differentiable implies f continuous.

So, differentiability is in motion which is stronger than continuity. Are their functions which are continuous but not differentiable, yes, we know many of them. Basically, differentiability says there should not be any corner in the graph of the function, there should not be any corner in the, corner is pointed kind of a thing. So, one says loosely, the graph of the function should be smooth, from that is a word smoothness comes from there. Saying function is smooth that means, at every point the function is differentiable.

And differentiability implies continuity also we have seen, so there is no break in the graph of the function. If f is differentiable, on a interval, then there is no break and there is no corner in the graph of the function.

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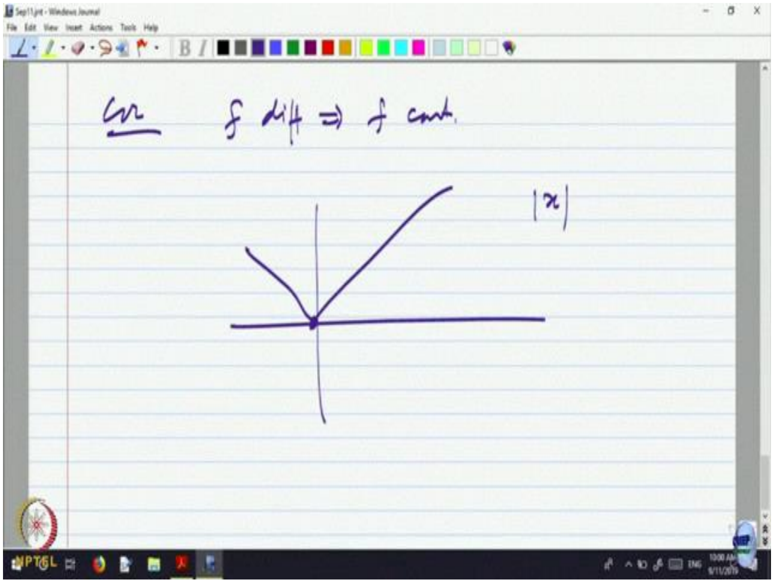

For example, if you have a function mod x. So, there is a function mod x. So, that has a corner. And we know, we can easily prove, you must have gone through this that it is not differentiable at the point x is equal to 0 thus that is a problem with that.

So, that is differentiability in terms of. And so, you can now see that this is giving you another property of the function. What was continuity, function is continuous on a interval that means, there is no break in the graph, differentiability gives you something more not only there is no break, there should not be any corner in the graph of the function, it should be smooth. So, it is giving more properties of the function. So, that is how calculus that actually have been developed historically.