Basic Real Analysis
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Lecture 32
Differentiability – Part II

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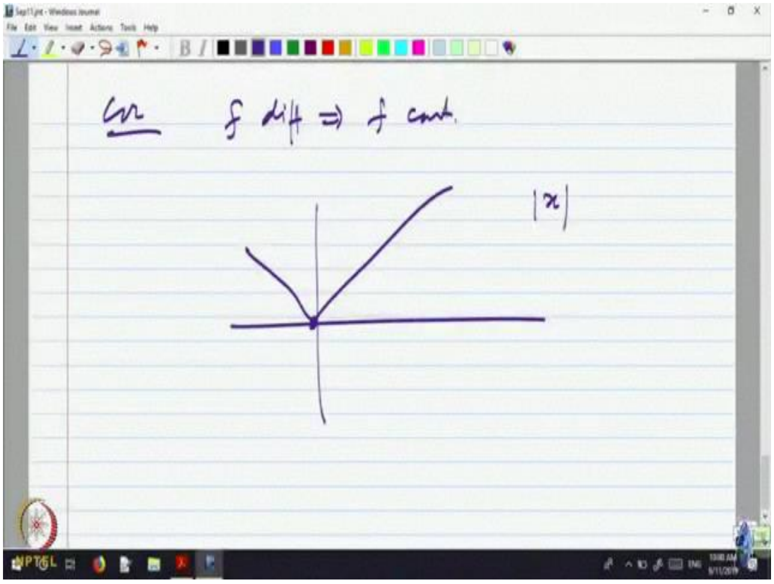

So, let us look at some more properties of okay. So let us say, so we will assume say F is differentiable. I am just revising most of the things which you already know, maybe we slightly giving you a different perspective of the things. Then one would like to know, we I also already mentioned that if a function is monotone, then we know it is continuous excepted countably many points. In fact, I pointed out that there is a deep theorem saying that if F is monotone than it is also differentiable accepted some number of points.

That means, most of the time the graph should be a smooth graph accepted some points which are okay, which have probability zero, forget about that statement anyway. So, F is differentiable at the point C, we would like to know if you know something more about the function can you give me his back some more properties of a function (())(1:37) suppose I know the function is differentiable, that gives me continuity, but I suppose I know something more about the notion of derivative, that how does that derivative of a function look like? Can you give me back some information about the function.

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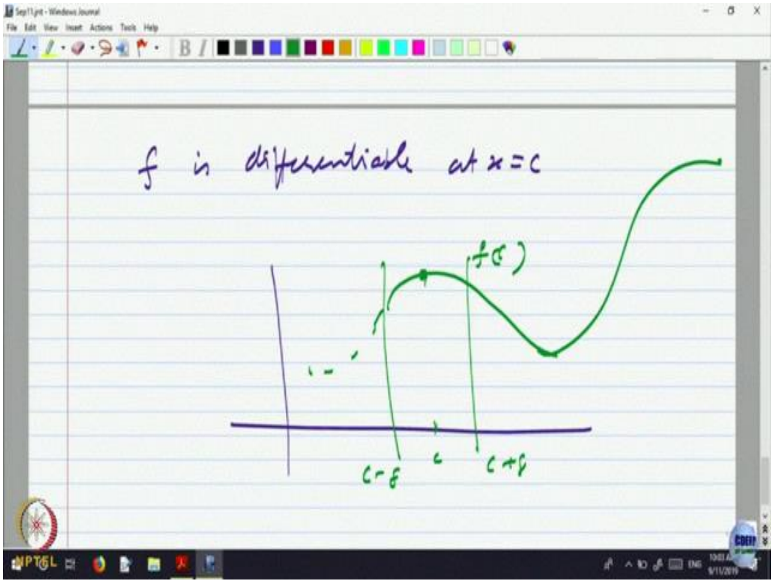

So, for example, let us try to understand it. Let us take the, so this supposing some function and here is the graph of the function for some portion at the point C, graph sort of is going up, but then it starts going down, so at this point F of C what we call as a local maximum. So, what is the meaning of local maximum? It is a maximum for the function, but not everywhere the graph can could go up somewhere.

But in some interval C minus delta to C plus delta I can say there is a interval around the point C such that, there is a neighbourhood of the point C so that the value of the function at the point C is the largest. Then we say the point C is a point of local maximum, we all have gone through this.

Now, if the point of C is a point of local maximum and there is a notion of derivative available at that point, that means I can draw a tangent to the function at that point, then what should this look like? What should the tangent at that point look like? At this point there is a tangent possible, right? So, it looks like the tangent should be horizontal one.

So, geometrically we are guessing, if F is differentiable at a point C, at a point C F has local maximum at C, then we are saying that the derivative at that point should be equal to 0. So, this is geometric observation, you can prove it very easily. So, let us just give a proof which you might have already done in your courses earlier.

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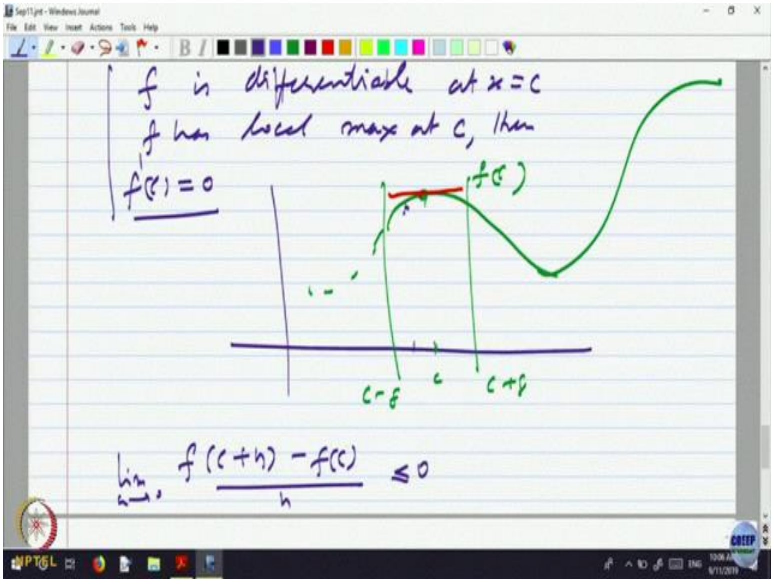

So, I want to calculate the derivative, I already given that it is differentiable. So, what I want to calculate F of C plus H minus F of C divided by H, limit of this H goes to 0, I want to calculate the limit of this as H goes to 0. So, and I want to show it is equal to 0. So, let us analyse when H is positive, if H is positive, what happens to the denominator? H is positive, so our denominator is positive, function is increasing.

So, what happens to the numerator? Function is at this point is a local max, sorry. It is a local maximum. That means, if I take any point on the left side here, then the value at this point will be less than the value on the point C. So, this numerator it will be negative, because F of C is the largest value in the neighbourhood. So, the difference will always be negative, denominator if H is positive is positive, so ratio is negative, if H is positive and the limit of that we know it exists.

If you take it limit from the right, I am taking a limit from the right now H positive. So that will be less than or equal to 0 because the ratio is always negative. So, limit must be negative it exists. Is it okay for everybody? Right.

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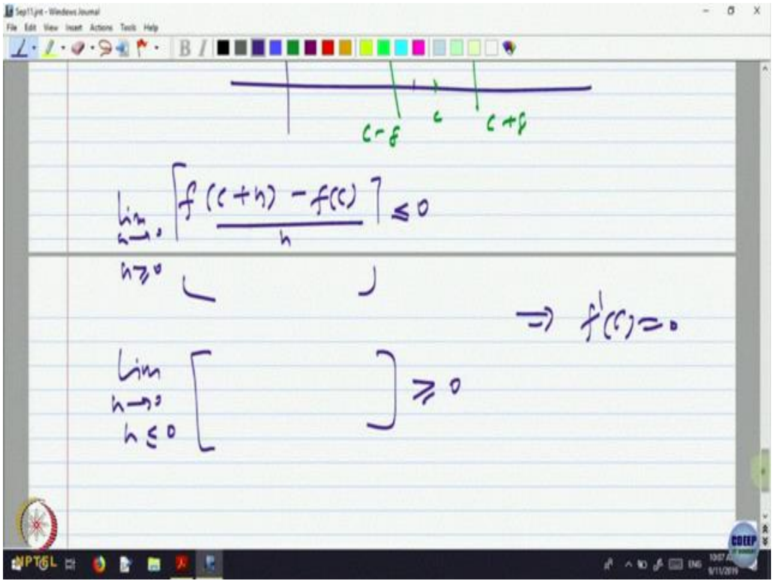

Now, let us look at the same limit. So, here I am taking H bigger equal to 0 of this quantity. Let us take the limit of this quantity from the left side, H is negative now, but the numerator is still positive because the function is local maximum at the point C, numerator is still, what is the numerator? Still negative, denominator negative. So, ratio is always bigger than or equal to 0. So, limit should be bigger than equal to 0.

I am just looking at the function for which you are taking the limit, if H is positive, it is the function values are all negative less than or equal to 0, so limit must be less than or equal to 0 if I look at the left limit, the function is always positive, so limit must be positive. So that implies F dash C must be, so they are left derivative right derivative both are same because the function is differentiable. So, it must be equal to 0. So that is the reason.

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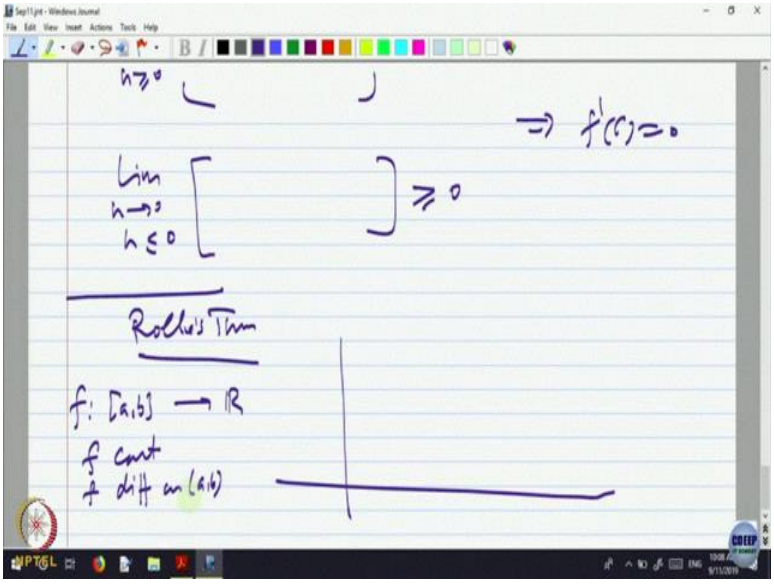

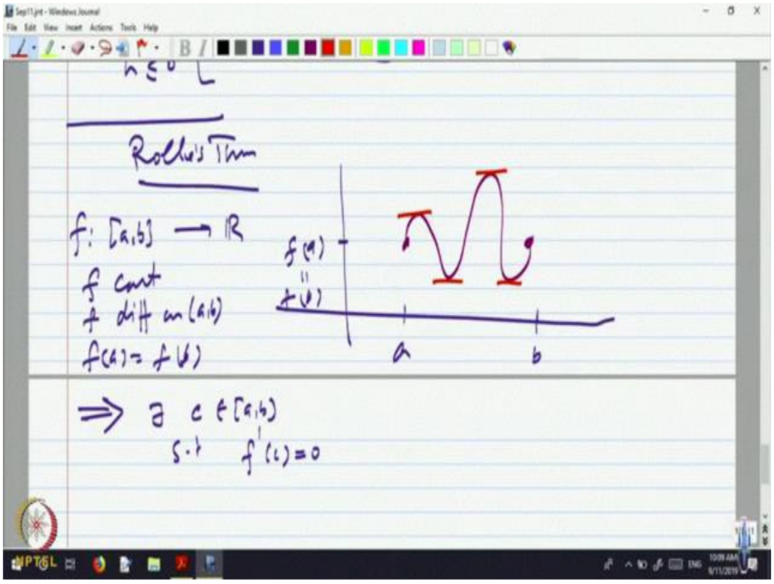

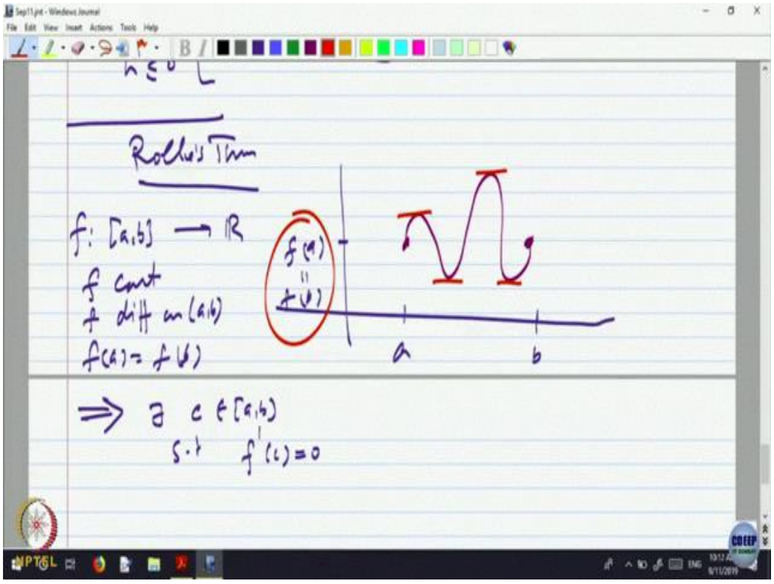

So, this geometric picture, we can prove it very easily by looking at this thing. So this gives a very important theorem namely what is called Rolle's theorem and what is that? So, I think most of you must have gone through it and remember it. So, that says if F is a function on a interval a, b to R closed interval A, B to R, F continuous, F differentiable at least on the open interval a, b.

And the third property says that F of a should be equal to F of b, then that implies there is a point c such that, differentiable derivative at the point c must be equal to 0. Geometrically how one guests this result is as follows there is a point a, here is a point b, so, F of a here is F of a and that is equal to F of b so that is a F of b here. So, that is equal to F of b.

So, it is a continuous function. So, let us draw a graph, it starts here, it has to end here. So, what should happen? And you can start going up or down or something. So, let us say it goes like this and comes like this, I should not lift that is one because it is continuous. Now, apparently the graph says there are many points where the slope of the tangent is 0, in this it says here, here and here or you on here.

What are these points? These are the point where there is either a peak or a turf, . So, looks like these are the points where the function has local maxima or minima. So, if I can say, if F is continuous, I can ensure that in the interval a to b, there will be at least one point of local maxima or local minima then I am through by the previous theorem. And that is our theorem that if F is continuous on a closed bounded interval, then it attains its maxima and minima at some points in the interval a, b.

So, F attains maximum minimum in the interval a, b one possibility, maxima equal to minima equal to the endpoints, then what is the graph of the function? Maxima is equal to minima equal to the value at the endpoints, there is a constant function, every point, every place a derivative is equal to 0, so, no problem at all, proof over, if not that means there is at least one point in the interval a, b, where the function takes maxima or minima that is the inside of the interval a, b, if it is a maxima or minima, automatically that point also is local because it is a global thing, is the largest value of the function on the whole of a, b, so locally also it is true. So, the previous theorem applies.

So, basically it is a consequence of the fact that a continuous function or a closed bounded interval attains its maxima and minima and the previous thing that if the function is differentiable then at local maxima or minima derivative must be 0, these two combined together give me (())(11:16). And this theorem you can easily extend this theorem by removing this condition, look at the graph slightly tilted, that means what F of a need not be equal to F of b, then what does the graph look like?

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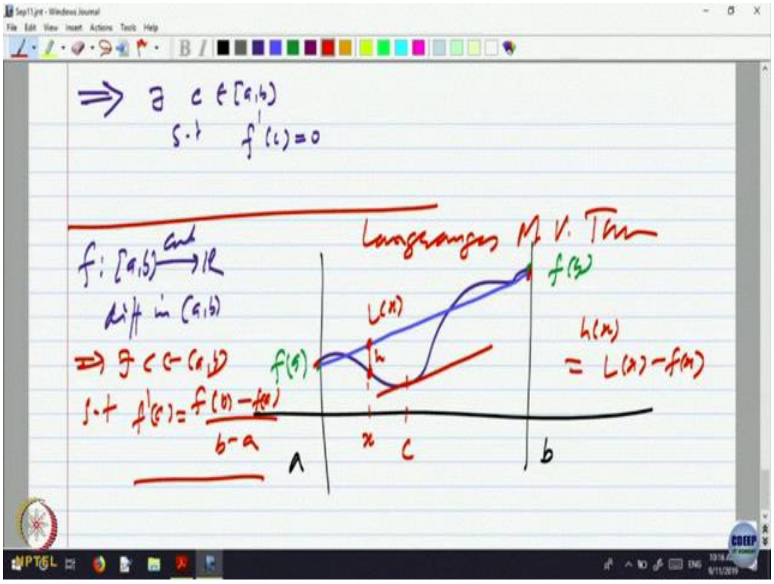

So, let us extend this theorem. So, let us say this is a, this is b and F of a to F of b. So, let us say it is value here and this is a value F of a and that is F of b and the graph is something continuous still. So F on a, b, continuous differentiable in a to b, we do not need endpoints differentiability is clear from the earlier because if inside the local maxima or minima, then derivative at that point is 0.

So, F differentiable in a, b, we are not putting the condition that F of a is equal to F of b, we are omitting, that then what should happen? So, let us look at the line joining this, imagine this line to be moved up and down. Then at some point there is a point where the tangent is parallel to this cord.

So there is a point C because in the Rolle's theorem, what is the slope of the cord? F, b minus F of a, if F of b is equal (())(13:05) that is 0 so, you get 0. Otherwise, is a generalisation of that in general, so it says implies there is a point C belonging to a, b such that F dash of c is equal to F of b minus divided by (())(13:28) that is slope of the cord.

So, this is what is the famous called Lagranges Mean Value theorem, and the proof also is straightforward, we just apply, try to apply Rolle's theorem to a modified function. So, what is a modified function? F of a is not equal to F of b, but if I look at this endpoints at that nothing, values are not equal but it will subtract that, say look at the cord and look at this value and this value because this is a height H. So, this is at any point X, this is the value of the function F of X, if this is the value of the cord,.

So what is that? What happens to this height H as you move the point, if you move this point X towards a or b, what happens to H? At a it is 0, at b it is 0. So, that is a function I should be looking at. So, look at the function H, so called H of X equal to, you can take L of you can call this cord as LX. So LX minus F of X, consider that function, that function has the property that H of X is equal to H of A equal to H of B equal to 0, It is continuous because, the function LX straight-line that is continuous, that is differentiable, the differences also differentiable.

And points the values are equals, so Rolle's theorem applied. So, there is a point where the derivative is equal to 0. So, what are the derivative of H? It is a derivative of the cord that is F, b minus F of a divided by b minus a. And what is derivative of F, F dash? So, F dash minus L dash is equal to 0 and that is precisely this consequence.. So, that is how you proof Lagranges Mean Value Theorem from Rolle's theorem or you can just think of, if you can visualise rotate your axis a bit, so that F of a equal to F of b, it should be true, but we are not going to that kind of argument, we are just looking at a straightforward way of saying. So, that is called Lagranges Mean Value theorem.

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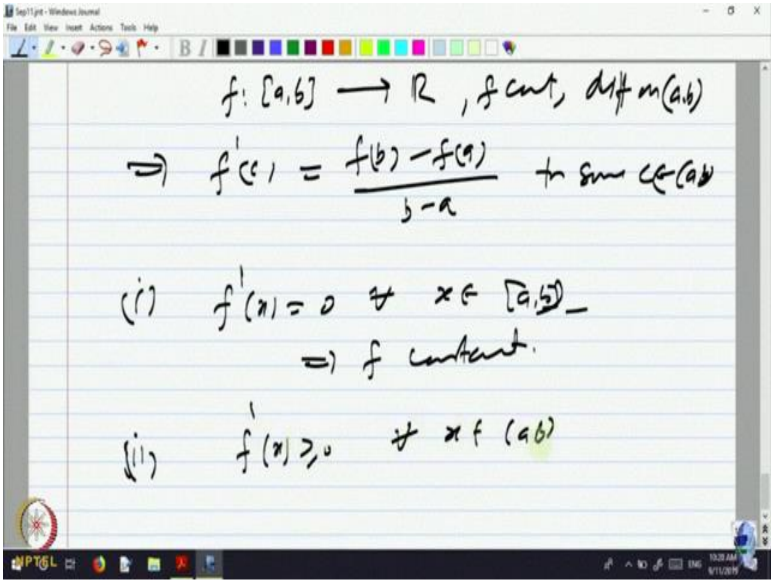

So, this is one of the most important theorems I would say of calculus, it says if F is continuous, F continuous differentiable on at least a to b implies F dash of c is equal to F of b minus F of a divided by b minus a for some c belonging to the open interval a, b. Let us see what are the consequences of this, which you are all gone through so, I will not go through all of them again.

For example, suppose the function is such, now you see how the derivative is giving back you the properties of the function. F is differentiable on the interval a, b. And assume I know something more about the derivative namely, derivative is 0 everywhere. Suppose a derivative is 0 everywhere. If derivative is 0, then what does F dash of c? That is 0 that means F of b must be equal to F of a.

But why B and A, I can apply to any two points in between X1, X2, Lagrange's Mean Value theorem applied to any two points X1, X2 inside a, b and F of X2 minus F of X1 divided by X2 minus X1 must be equal to derivative at some point in between that is 0 anyway. So, it say F of X1 is equal to F of X2. So, if F is differentiable in the interval a, b then it is constant. So there is a consequence of this beautiful theorem, very easy consequence.

So one, F constant implies F differentiable, oh sorry, I should say if derivative is 0, F dash of X equal to 0 for every X belonging to a, b implies F constant. In fact, I can see F is constant, see I will get a point C in between, but you can apply it to what is the value at the point A? If it is 0 inside a, b, it should be 0 at A because it is continuous, function is continuous on a, b.

Lagrange's Mean Value theorem it says in the open interval a, b it is constant, but captivity says that it should be called 0 on the endpoints also. So, it is on the closed interval a, b.. So, this is one of the simplest kind of applications there are more applications of this, what are the other applications of this?

Now, you know that, for example, let me just, I will not go into that we will just state those theorems for you to read. For example, look derivative is 0. Suppose derivative is bigger than or equal to 0. Let us analyse the second case, derivative is bigger than or equal to 0 then what can you say about this ratio? Does it give you something? So, let me, so second that F dash of X bigger then for every X belonging to a, b. So let us say this one.

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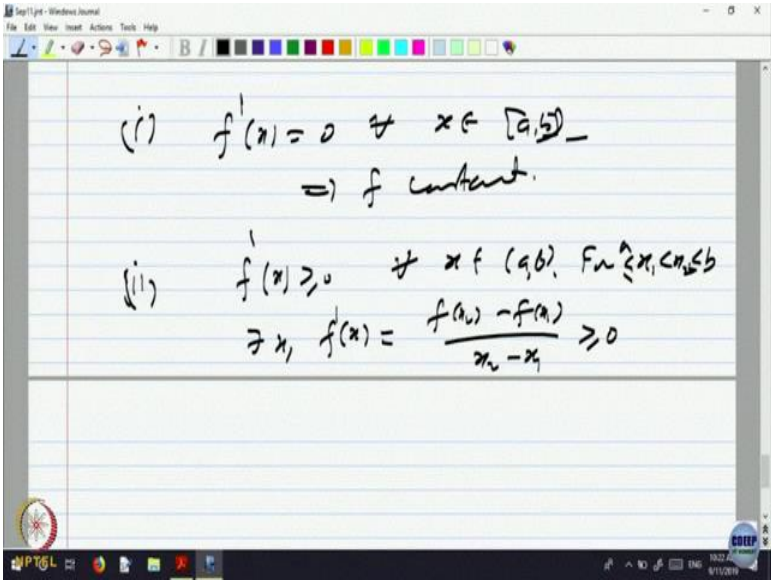

And what does will give me? By Lagrange's Mean Value theorem, F dash of X for any two points X1, X2 in this interval what we will get there is a point X okay, so let me just I am just hurrying through, let me not hurry through. So let me write, so for X1 less than X2 between a and b implies that there exist a point X such that F of X is equal to F of X2 minus F of X1 divided by X2 minus X1, I am applying Lagrange's Mean Value theorem in the interval X1 to X2, which is inside of the interval a, b.

And if this is bigger than or equal to 0, what does that give me? Whenever X2 is bigger than X1 denominator is positive, so, numerator should be positive. So, what does it give you? If X1 is less than X2, then F of X2 is bigger than F of X1, function is monotonically increasing. So, how the nature of the derivative is giving you back bonus points about the function.

All because of Lagrange’s Mean Value theorem, if derivative is bigger than or equal to 0 function is increasing, same proof less than or equal to 0 function is decreasing. So, that is a consequence of Lagrange’s Mean Value theorem. So it is telling you the nature of the function.

So, now, from here you can build up at a point C, I want to analyse whether there is a local maxima or minima for the function at that point C or not, necessary condition derivative must be equal to 0. So, look at the points of the function where the function is differentiable derivative equal to 0, analyse those points whether they are points of local maxima or minima. And also the points where the function may not be differentiable, but still can have local maxima and minima. For example mod X has local minimum at the point 0 it is not differentiable, so that is not a sufficient condition.

So, look at all the candidates namely where the function is not differentiable or function is differentiable and derivative equal to 0, equal to 0. Out of all these points, some points may be local maximum, some points may be local minimum, some points maybe none So, how do we analyse what are the sufficient conditions, the sufficient conditions are if derivative is positive on the left side of that point, derivative exists on the left side in a neighbourhood of that point on the left side.

Derivative is positive, that means what function will be increasing on the left side or because of this theorem. Derivative is to maximum I want decreasing, so derivative is less than or equal to 0 on the right side. So, it will be decreasing, so at level local maximum at that point. So, what are the points F dash should be equal to 0, one. On the left function need not be differentiable, but look at on the left side of that point and on the right side, on the left side if the derivative exists and derivative is bigger than or equal to 0, then the function will be increasing.

And on the right side if the derivative is less than or equal to 0, then the function will be decreasing. So, you get a sufficient condition F is c is a point, if function should be continuous of course, function is continuous. And on the left the derivative is bigger than or equal to 0, on the right derivative is less than or equal to 0, then the function will have a local maximum and you can go on local minimum similarly.

So all of you have gone through these kind of theorems, in your B.Sc. courses, undergraduate courses. So we will just state those theorems we will not prove those. So they are all consequences of, for example then you can go on to analyse what are called second derivative test, they are called first derivative test, you can go to second derivative test and so on. You can also analyse what are called convexity and concavity a function. So, let me probably I do not know how many of you have gone through but I think it is a good idea to go through some of them.

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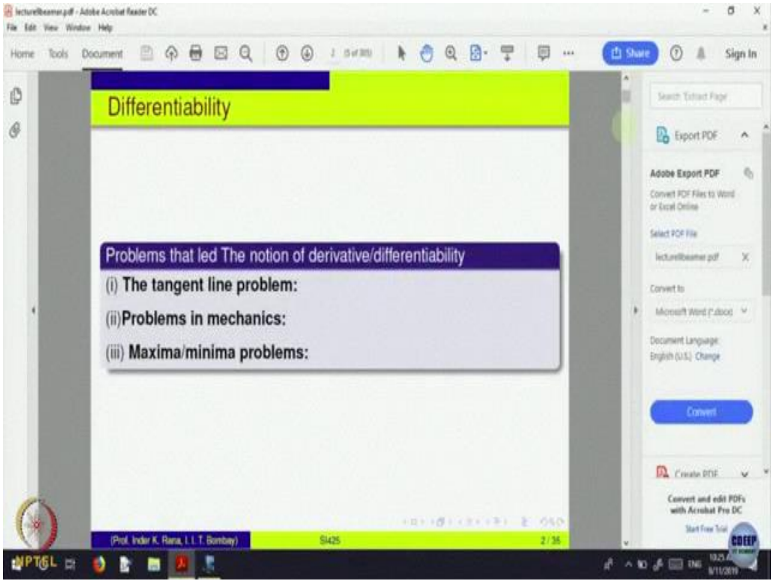

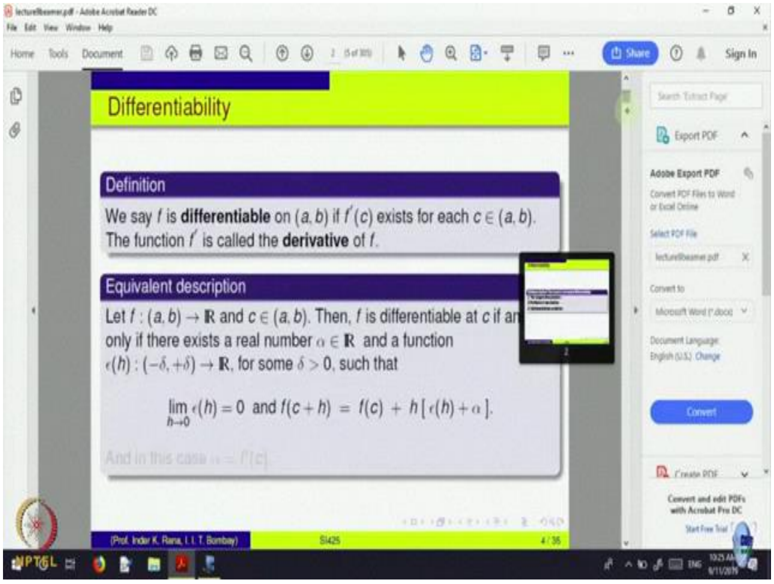

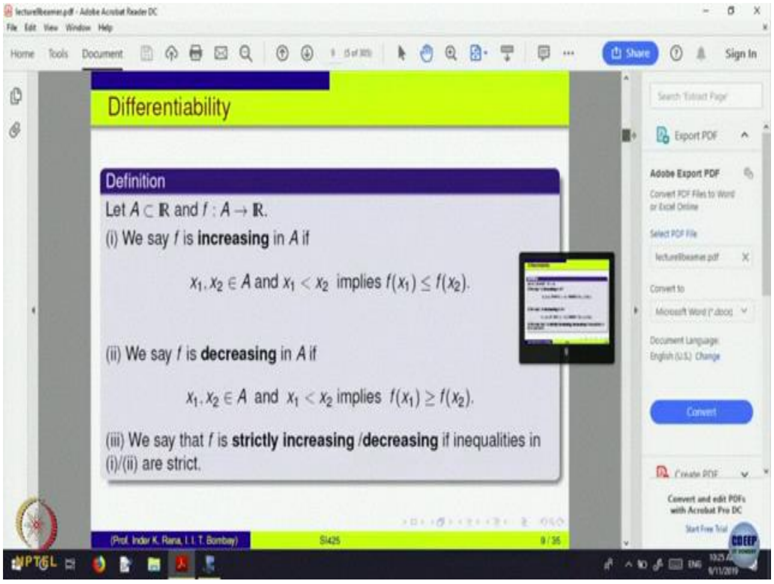

So, let me just show you on the slide. So, maxima, minima that we have seen what motivates one to define. So, algebra for derivatives , increasing, decreasing.

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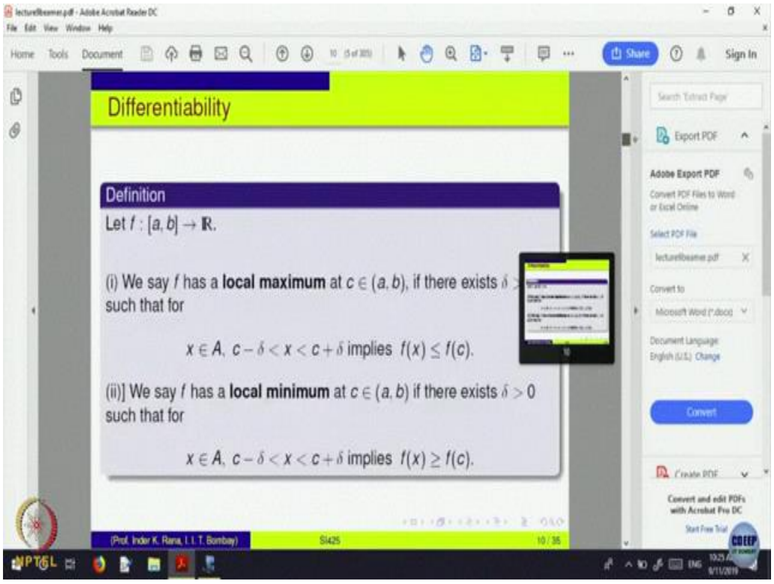

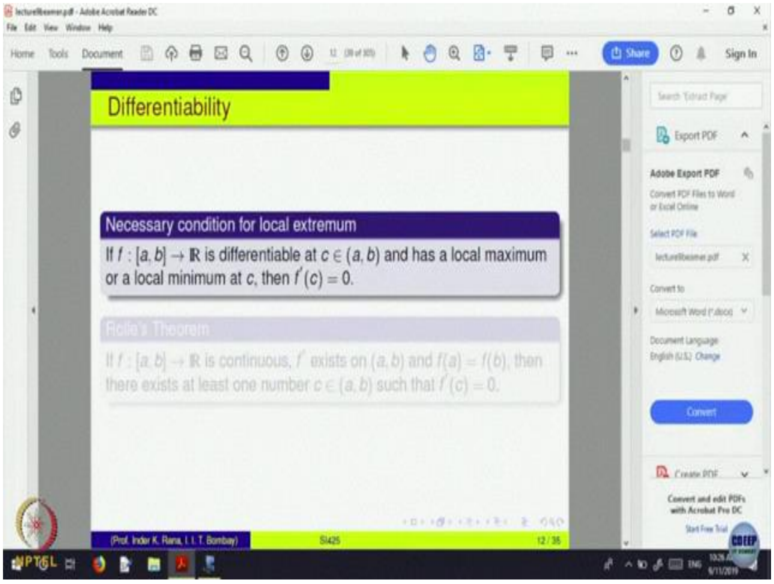

We defined earlier also monotone, so local maxima minima definition, we define on the left side. In a locally it is a maximum of locally it is a minimum. So, here is necessary condition for local maximum that if has a local maximum and is differentiable then the derivative must be equal to 0. So, this is a necessary condition, keep in mind is a necessary condition and how necessary conditions are used.

At points where the function is differentiable, but derivative is not 0 cannot be the points of local maxima, minima or the function is not differentiable that is a point of continuity. So, possible candidates for local maxima minima as a consequence of this are the points where the function either is not continuous or it is differentiable and derivative is equal to 0. So, that gives you a bag full of points called the critical points, you have to analyse which of them are local maxima or local minima.

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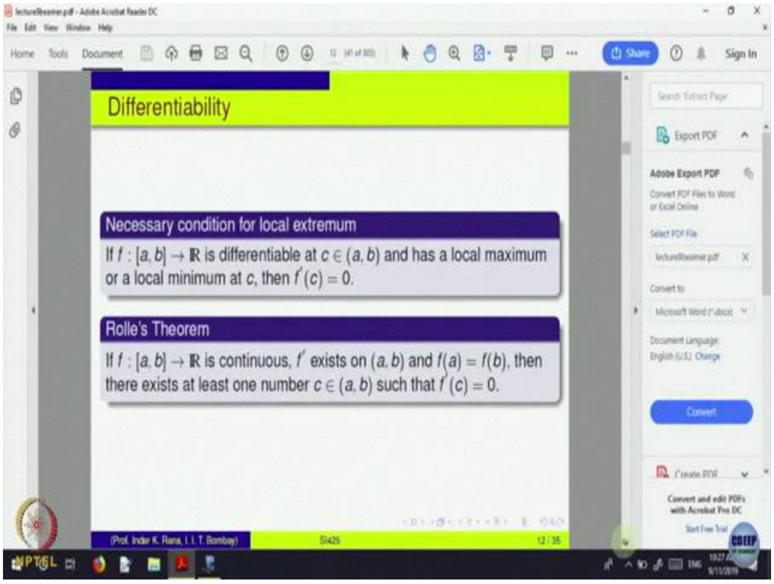

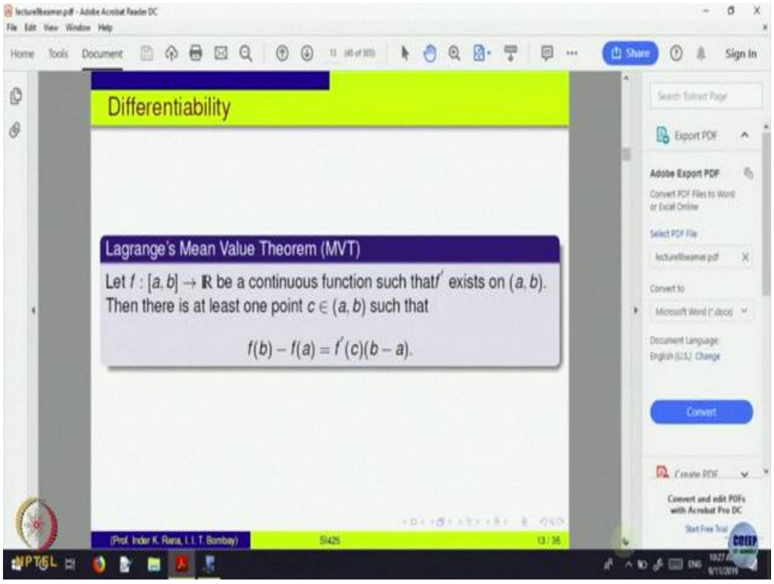

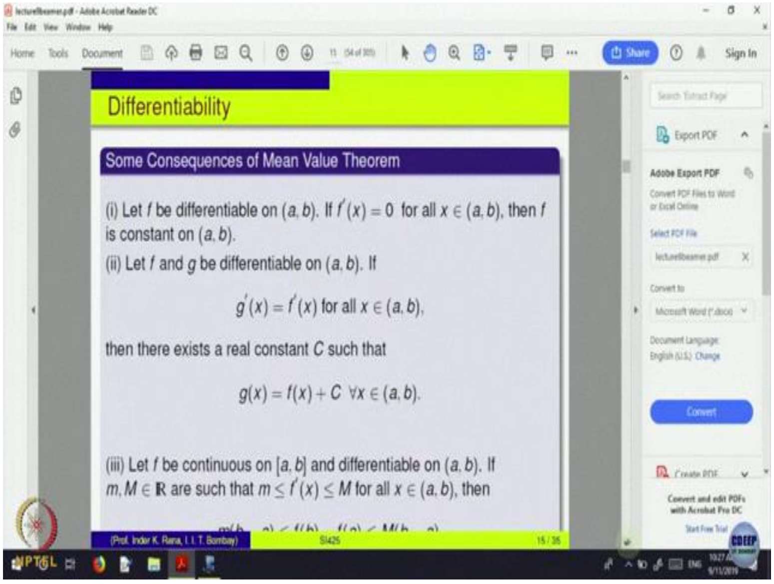

So, Rolle's theorem says okay, continuity in the interval a to b, we said that this is Lagrange's Mean Value Theorem. So the condition of endpoints value equal removed, I am just revising again what I have said, so applications of this, consequences of derivative is 0 the function is constant. So, if the difference of, if two functions have the same derivative then they made differ only by a constant as a consequence of that.

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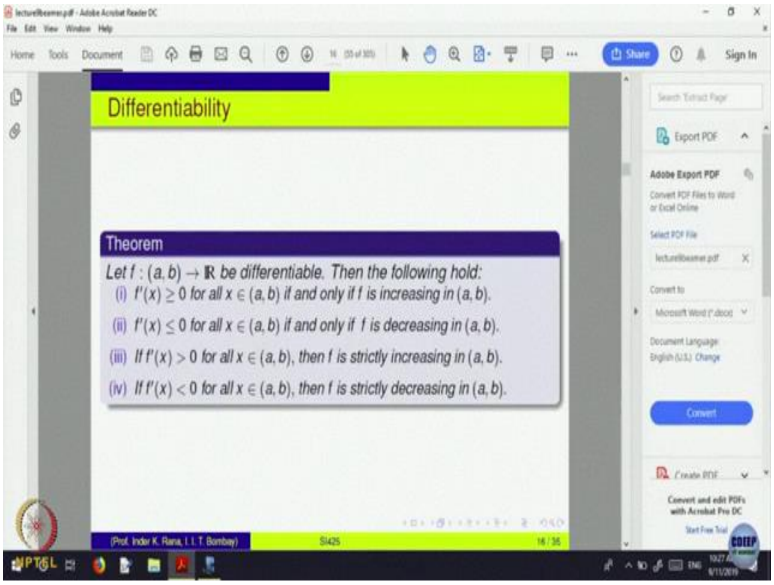

So, this is about increasing, decreasing. If derivative is bigger than or equal to 0 the function is, you can prove other way around also if the function is differentiable. F of X2 minus F of X1 divided by X2 minus X1 will be always the ratio will always be positive. So, limit will be positive, so, other way around is the function is monotonically increasing then the derivative should be bigger than or equal to 0. So, this is a if and only if theorem.

Similarly, decreasing strictly bigger than it is only one way, if F is strictly bigger than 0, then F is strictly increasing because X2 minus X1 will be strictly bigger than 0 and equal to derivate. So, that is so, that is only one way, so, that question one is to keep.

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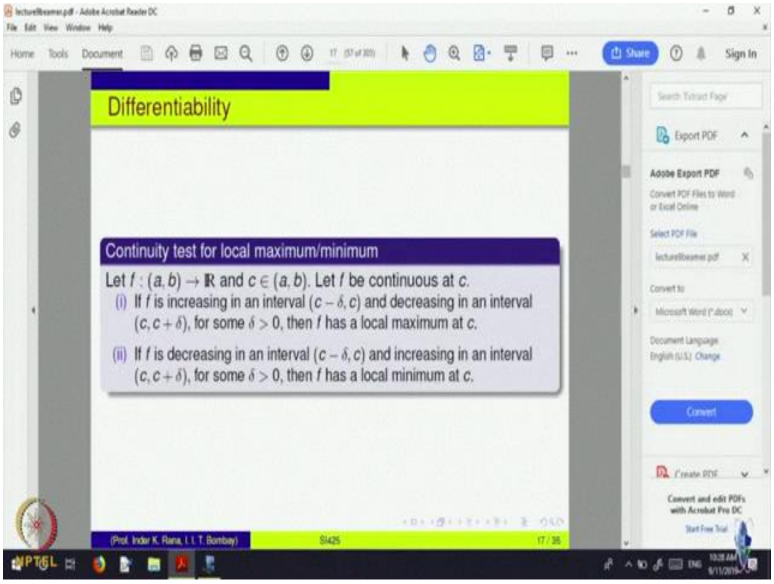

So continuity tests for local maxima minima as I said function is continuous at the point. On the left, I want the values to be less than or equal to one way of saying that in terms of derivative is, derivative is bigger than or equal to, because on the left it is increasing on the right it is decreasing. So, that is one way of just increasing, decreasing straight away in the way, well comparing the values.

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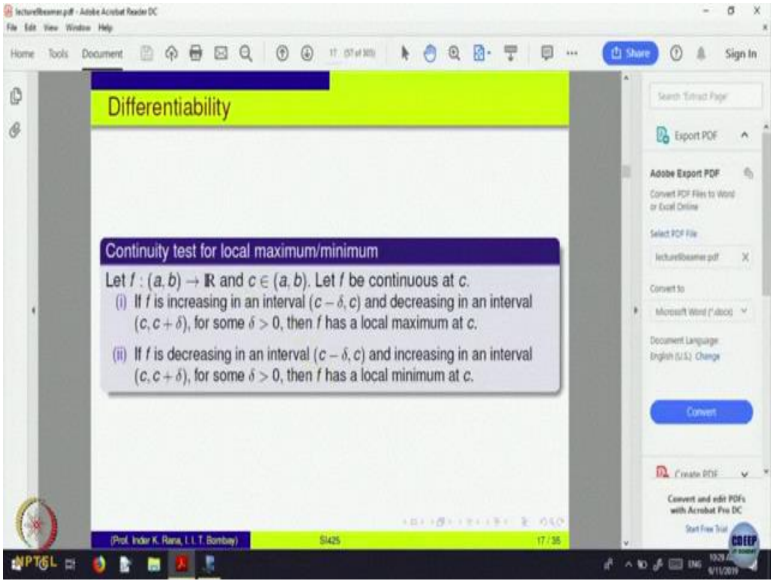

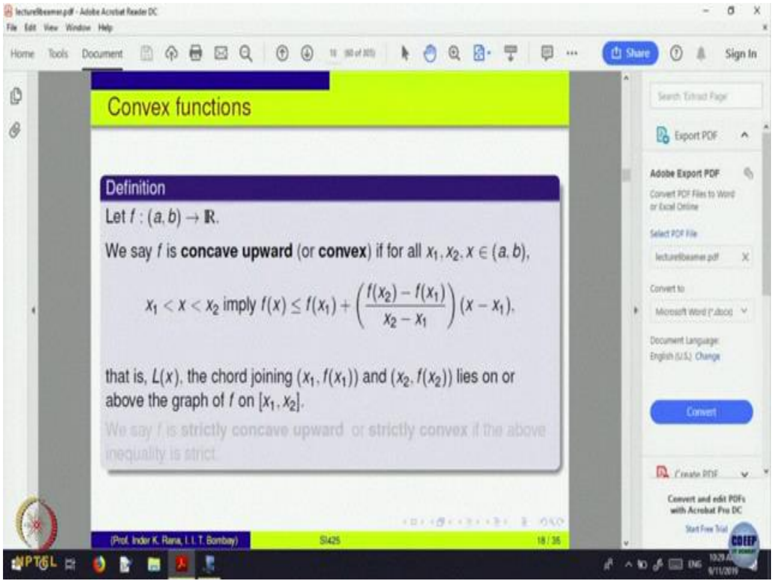

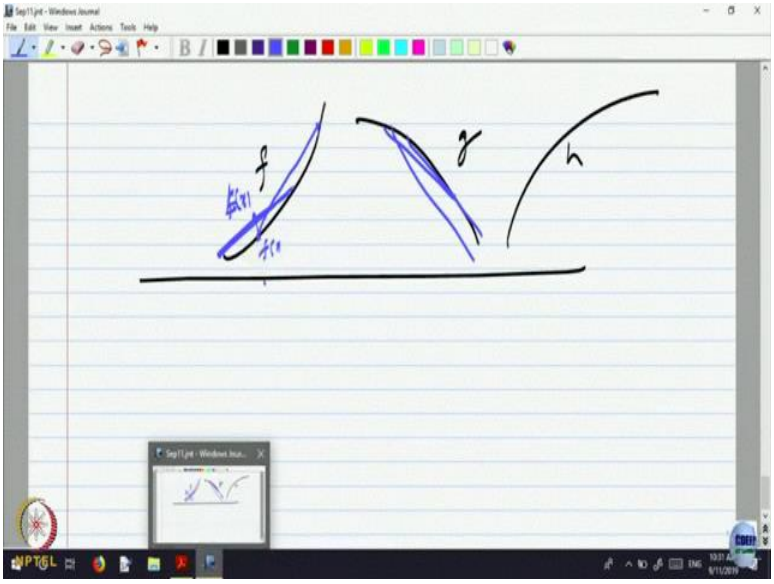

So, that is a continuity test, you can ask the first derivative tests, also okay, we will do it later, maybe. Here is something called concave upward and concave downward functions. So, what is the need for that that is another property, so you can have a function, say which is this function is monotonically increasing, this function is continuous, this function is smooth.

But look at this function, so there is is F, this is G, G is also monotonically increasing or decreasing, it is continuous, it is smooth or you can look at this function, look at this function H compare F and H both are monotonically increasing, both are continuous, both are differentiable but there is a difference between the two. What is that difference? How do I capture that difference between the two.

In one some sense the graph F of F is bending away from axis F of H is bending towards that is all English, what is mathematics? So, mathematically it says for this function F, if I take any two points and join, any two points and join, what happens to the cord? That always stays above the graph of the function. In this, the cord will always stay below the graph of the function.

And I can now make it mathematical, at any point X I know this cord, I know the value at this point, I know the value at the function. So, F of X should be, this is F of X and this is the value at this point is F at on this cord, so, call it L of X. So, the value at the cord should be bigger than or equal to value at the function at that point then it is bending towards. So, you say it is concave down, if you like to call it, concave up. So, is concave up, this is concave down. So, let us formally. So, this is how you capture mathematically the properties.

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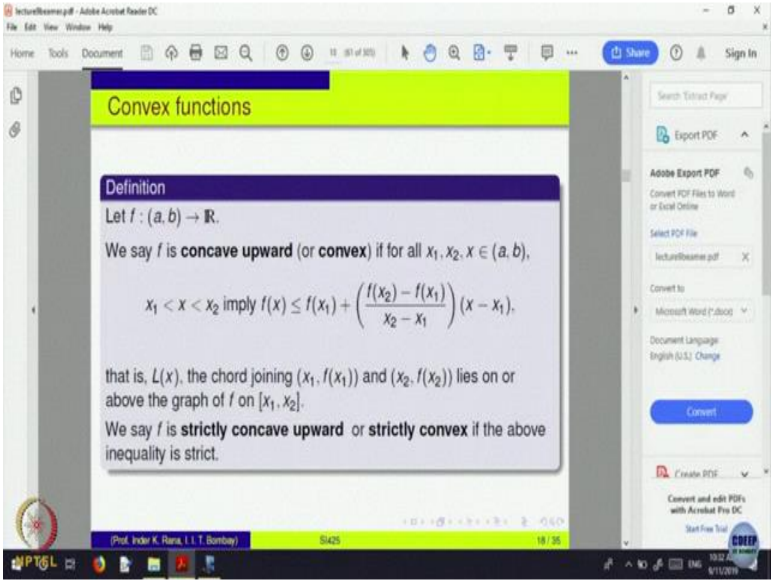

So, this is the look at F of X, if F of X is, what is this right hand side? This is the value of the function at the coordinate X to point X hitting that line, the cord to the equation of a straight line between X1 and X2, the slope, what is the slope of this line? F of X2 minus F of X1. So this is the line joining the points X1 with X2. So F of X is less than or equal to the value on the cord on the line joining. So you call it concave upward or convex functions. We will just keep the definitions will not, from examination point of view, I will state some theorems which will not because the proof are slightly complicated. Similarly, strictly, if this inequality is straight, if there are constant function monotonically increasing will be both convex and concave.

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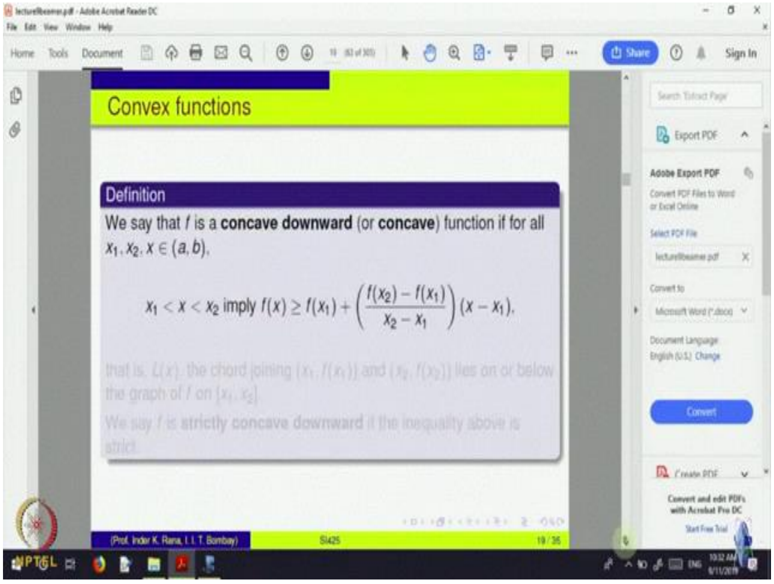

So, if it is bigger other way around, so you say concave down. So, for example, if you look at F of X square, the parabola. So, imagine your graph as a cup, in which a spoon is lying, the spoon never touches the bottom, it only touches rim, that is a parabola, you can think of the parabola, the cord join any two point is always above the graph of the function.

So, that is cup up, so you call it concave up, so concave up is Y equal to X square, other way round you can take the graph of Y equal to minus X square. So that is the other way round, the graph cord is always below, so that is concave down.