Basic Real Analysis
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Lecture 35
Differentiability – Part V

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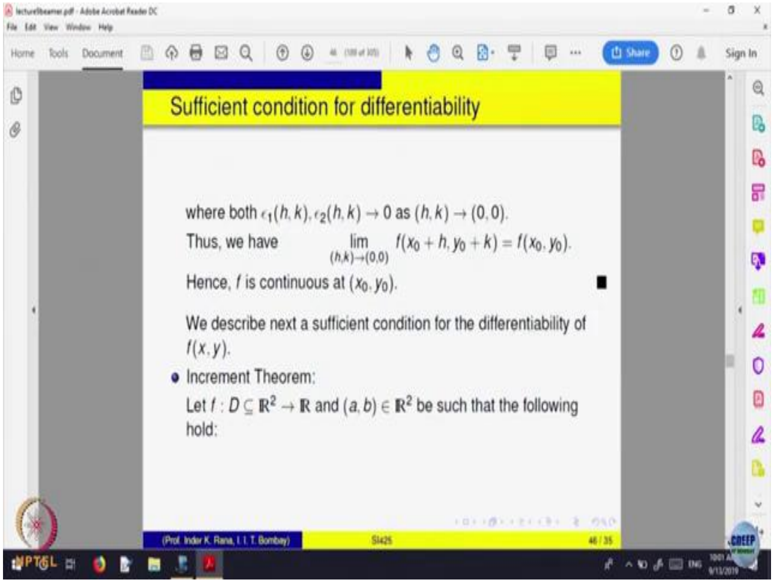

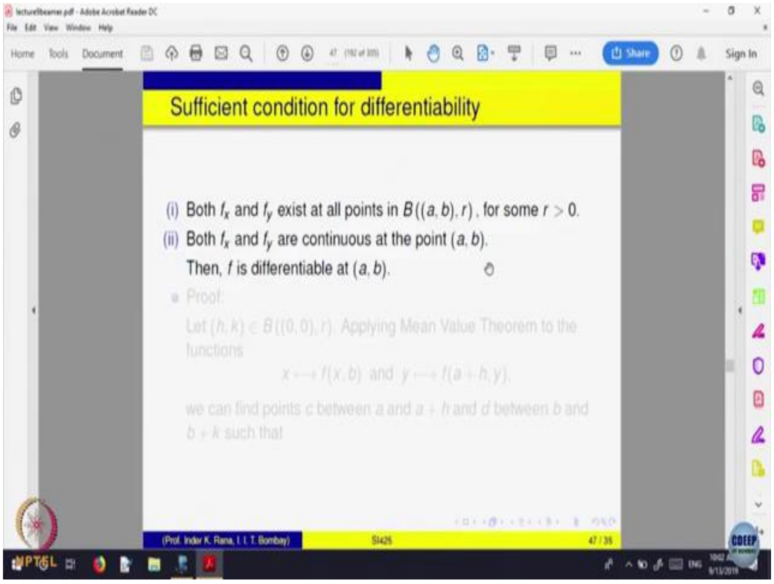

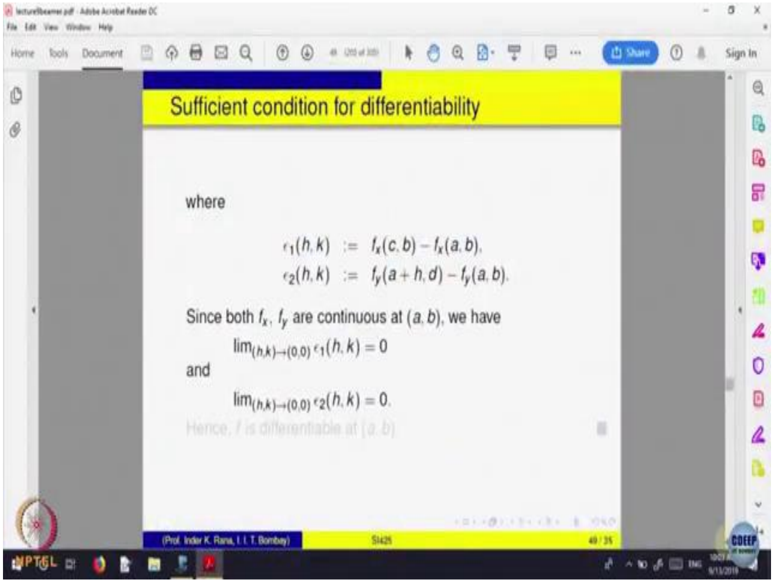

So, I was trying to give you that condition. So anyway, this we have approved that differentiability implies it is continuous, so that is okay because the right-hand side goes to 0. So, here is a theorem which I was referring, which is a sufficient condition for differentiability, which is a very nice one and very useful one.

It is only says that if this is happening, then the function is differentiable, it does not say the converse is always true. So, the earlier condition was if and only if, an equivalent way of defining differentiability, but this condition says suppose F is defined in a domain D such that the following hold, both the partial derivatives exist in a neighbourhood of the point. So, something more is coming not only the partial derivative is exists at that point, we require they exist in a neighbourhood of that point.

Why we need that? Because we want to talk about F, X and F, Y are continuous at the point A, B to partial derivatives as functions of two variables. I want to talk about the continuity of those functions of two variables for it to define the continuity of a function of two variables, the function should be defined in a neighbourhood. Otherwise, you cannot talk about the continuity of that function. So, for that reason the first condition is required because I want to state that F, X and F, Y are both continuous at A, B.

So, they should be first of all defined in a neighbourhood of A, B. So, first condition partial derivatives exists in a neighbourhood and are continuous at that point that is good enough to say that the function is differentiable at the point A, B. So, this is called the increment theorem.

So, not only we want partial derivatives to exist at that point, we want them to exist in a neighbourhood and to be continuous at that point, that is good enough that is sufficient enough to ensure that the function is a differentiable, we will not, the proof is there in the slides but will not go through the proof that is not steadily long. So, will not go through the proof, if you are interested, you can read that.

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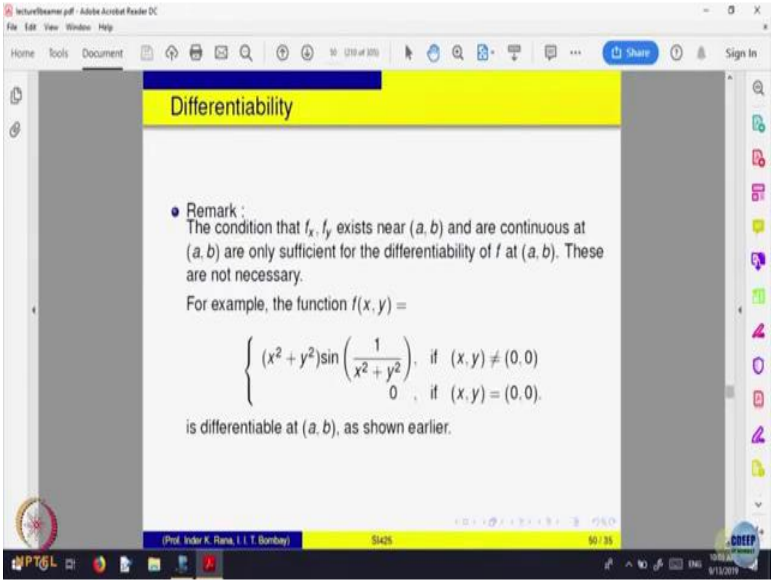

So, there is already difference, okay, here is the same example that we showed, we showed that this example of a function which is differentiable at the point X 0, Y 0, you can calculate the partial derivatives, partial derivative at 0, 0 we saw they exist and are both 0, actually in a neighbourhood of 0, 0 also they exist. For example, what is a partial derivative of this with respect to X at a point X naught equal to 0 and X is not equal to 0 the function is defined by this X square plus Y square into this.

So, I can apply the product rule, I can apply the product rule to calculate the derivative in a neighbourhood that means, Y is not 0. So, this function is as it is where Y is treated as a constraint. So, X square plus Y square sin of 1 over X square plus Y square at a point not equal to 0, 0 that is a definition. So, if I put Y equal to a constant and if you give me derivative with respect to X at a point other than 0, 0.

So, product will applies X square so, it will be two X into sin of this plus X square into derivatives of sin that is cos of we can calculate that derivative. So, if you calculate that, you will see the cos of 1 over something is coming and you can check that function is not continuous. So, derivative at a point in a neighbourhood exists and is not continuous the function is differentiable. So, that this is an example to illustrate that the increment theorem, the derivative exists and are continuous at that point is only a sufficient condition that is not necessary for the function to be differentiated.

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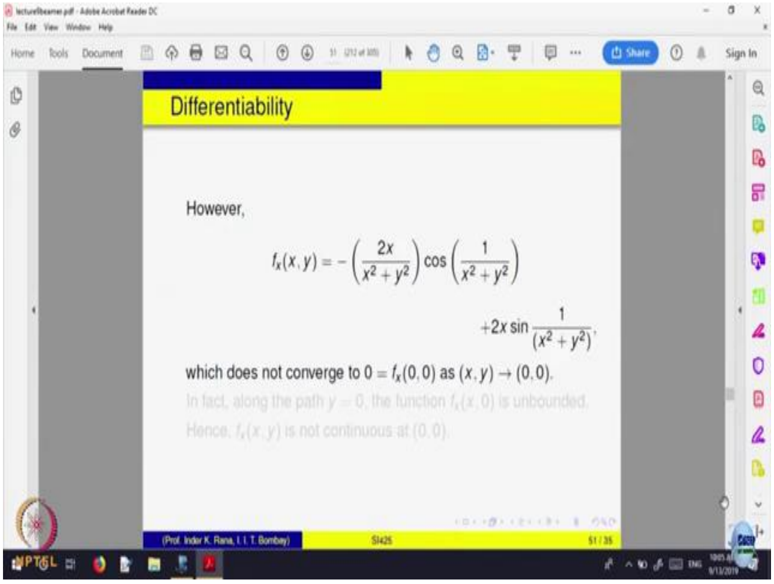

So, one can check, so, this kind of derivative will come and this function is not continuous, you can easily check this function is not continuous at the point 0, 0. So, partial derivatives are not continuous, function is differentiable. So, that increment theorem is only a sufficient condition. If something happens, if the partial derivative exists in a neighbourhood and are continuous at that point, then the function is differentiable the converse need not hold right function can be differentiable without, the partial derivatives being continuous at that point. So, there is an example.

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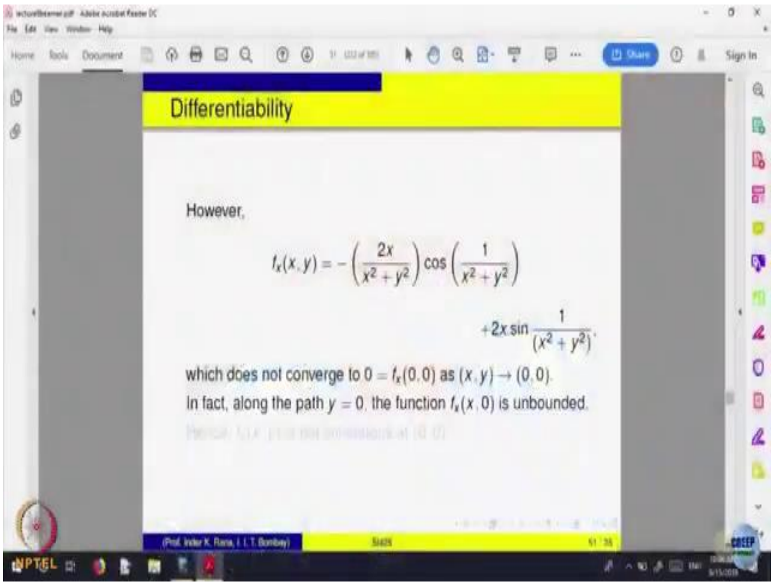

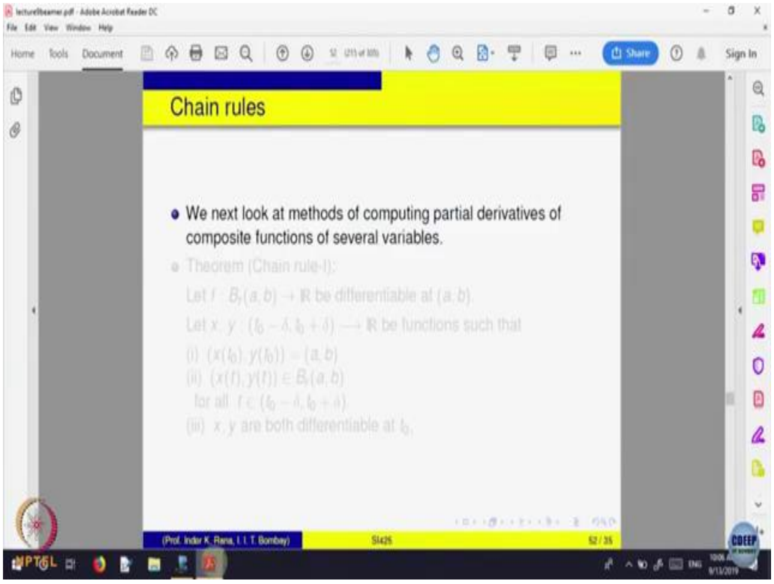

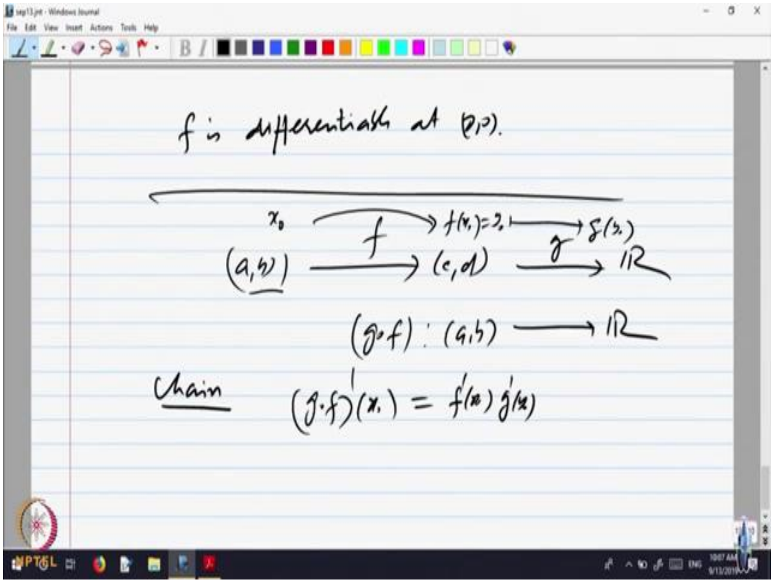

So, good enough now, here is something now, corresponding to one variable chain rule. So, in one variable we had say if you have a function F in C, D and you have a function G in R, say that F composite, this is G composite F is defined. So, that will be a function from A, B to R, you first apply F and then apply G. So, that is a composite function and we had the chain rule. So, what are the chain rule in one variable? You take a point here say X naught, this will go to a point F of X naught call that as Y naught.

G is a function defined there, so that takes this point to G of Y naught and the composite function is defined at the point X naught. So I want to say when is composite function derivative at the point X naught equal to, the composite function is defined on A, B. X naught is a point we want to know what is a derivative in one variable. So, what does the chain rule say? He says it is you need to have the derivative of F at the point X naught, F at the point into G dash of, what is the chain rule? G dash of F of X naught that is Y naught, so there is a increment that comes.

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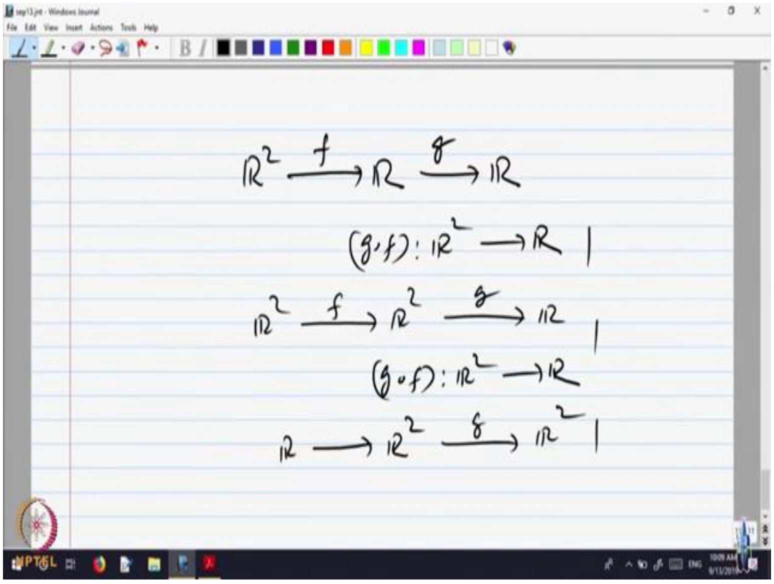

Now in several variables, you can have a function from R 2 to R and then you can have a function from here to here. So, here is a function F, here is a function G. So, what will be composite function? G composite F will be a function from R 2 to R or you can also have a function say F from R 2 to R 2 and then a function G here. So, G compose F will again be a function from R 2 to R.

So, there are many possibilities that become apparent. So, you can have a function from here to R 2 itself. So, you can have R 2 or you can have from R 2, there are many possibilities. For example, you can have a function of R to R 2 and G from R 2 to R 2. So, various competitions are possible.

And we want to know, see in the first one you have, F is a function of two variable composed with one variable. So, you can ask, what is the meaning of differentiability of the composite function? So in each one of them, one can ask what is a notion of differentiability. So, for that, there are chain rules.

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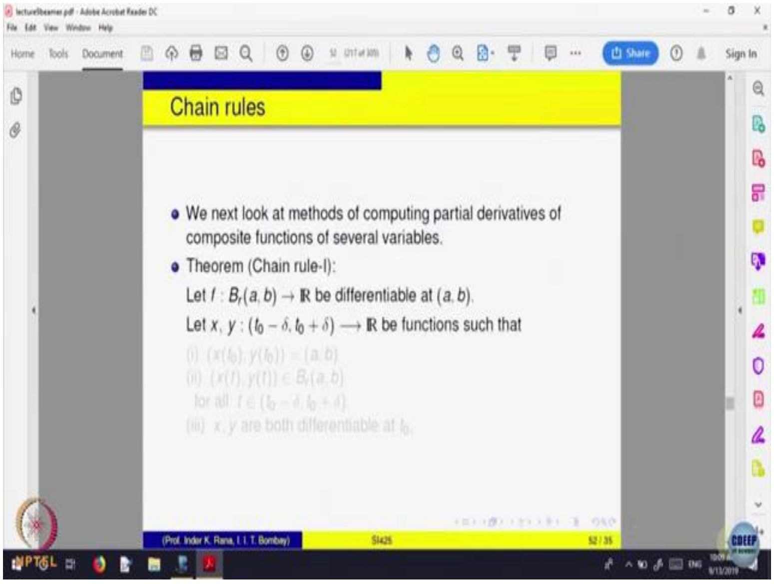

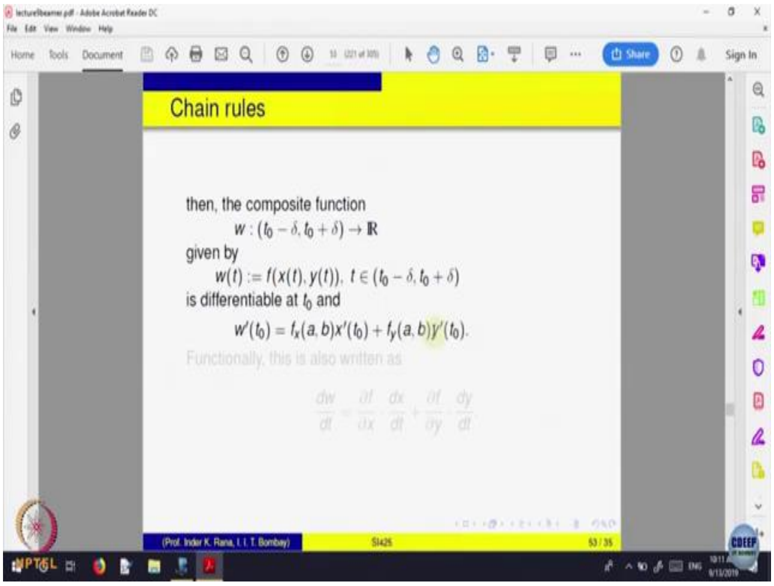

So, will just state some of them, so that you get, so here is so let us look at a function W, which is function of XT, YT that means what? There is a function G, which is a function of F is a function of two variables F of XT comma YT. So, as a function of two variables, but from where does XT, YT come? It depends only on T. So, there is a function of one variable you can call it as G, R to R 2, So T goes to XT comma YT, are you following what I am saying?

So, it can be for example, you are observing the path of a particle in a plane. At time point T, what is the position in the plane? The position of the point in the plane is given by the X coordinate XT and Y coordinate YT. So, a curve in a plane is described by function of one variable taking values in the plane. And that is combined with a function with F. So, F does something to that, so, it is a function.

So, it says that this function will be differentiable at a point T 0 and the derivative is, is a component of F of XT, YT. So, F is a function of two variables. So, there will be increment in the direction X increment in the direction of Y. So, FX partial derivative in the direction of X at that point, chain rule one variable, how does the one variable change X dash of XT naught plus partial derivative of F in the Y direction and the increment the rate of change given by variable.

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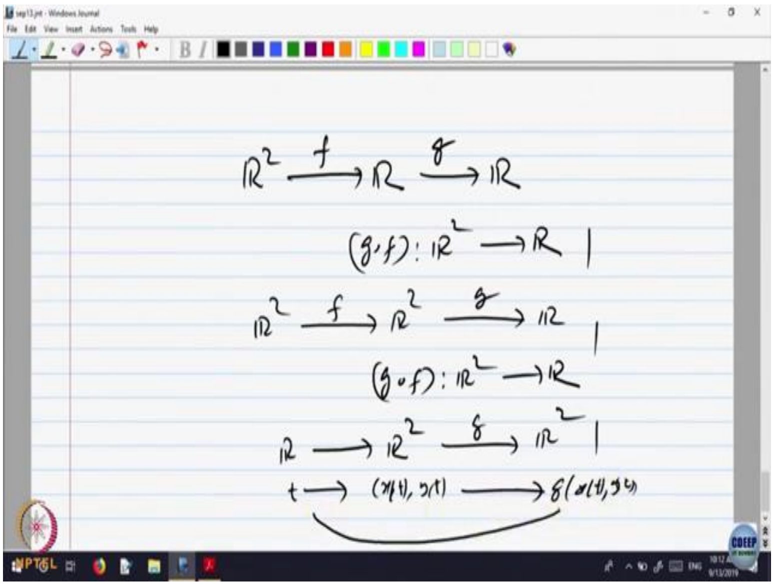

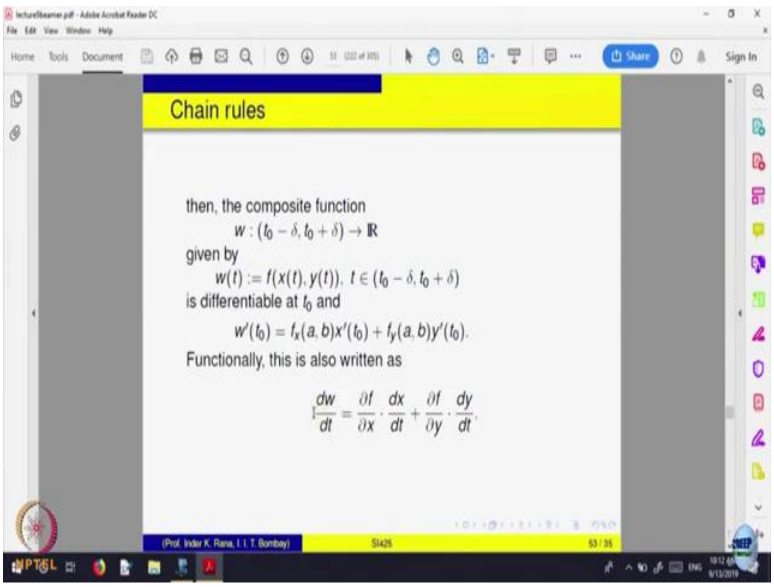

Is this okay, clear to everyone? Let me just draw a picture probably and so, this is like, this is the last one actually T goes to XT and YT. And then, something that goes to G of XT, YT. So, T goes to the composite function. Is intuitively one can write it as W is a function of one variable, the composite is a function of one variable. So, we will talk about the derivative of a function of one variable at a point. So, DW by DT, now W is a composite function F of XT, YT.

So, F in the direction, what is the rate of change? Partial derivative of F with respect to X by chain rule DX by DT. The second variable also to be taken care of, so rate of change of F in the direction of Y, DY by DT. So, that is how you calculate the derivative of function of one variable which is a composite of two functions. Now there are more possibilities for example, this G could itself be a function of two variable T and S, then you will have X comma, X of T comma S, Y of T comma S. So, then instead of DX by DT, there are partial derivative of that within the direction of and so on.

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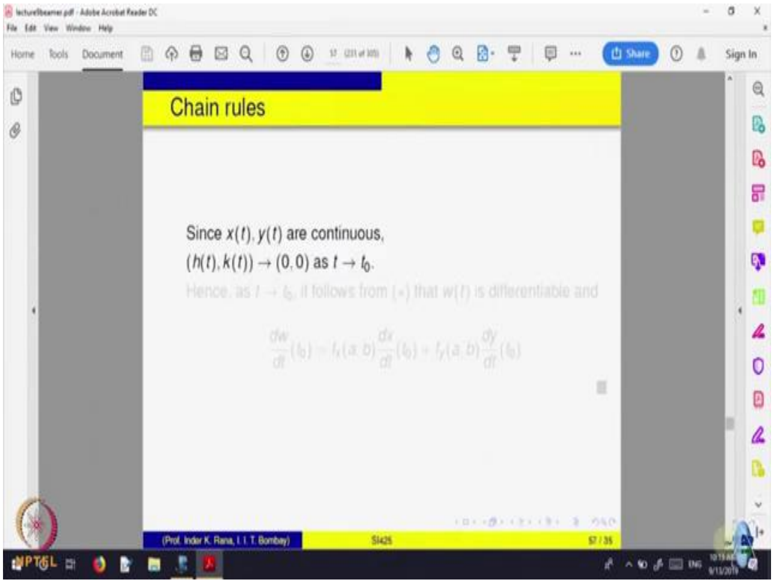

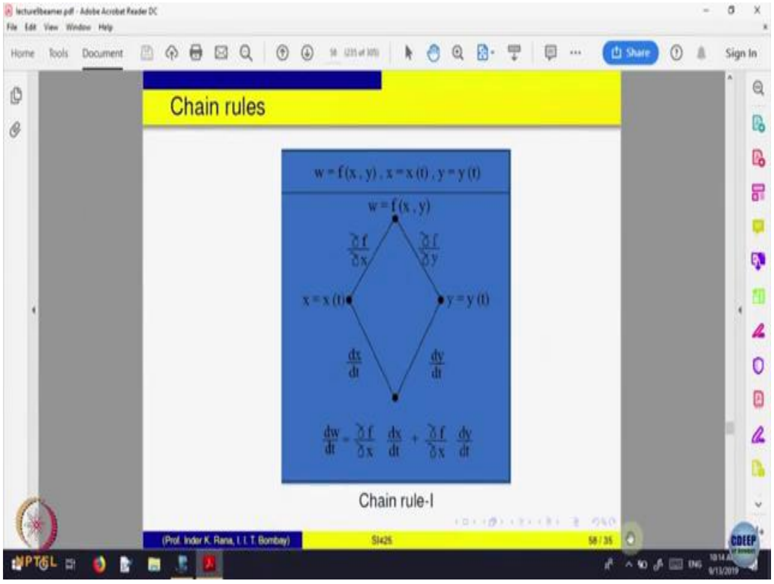

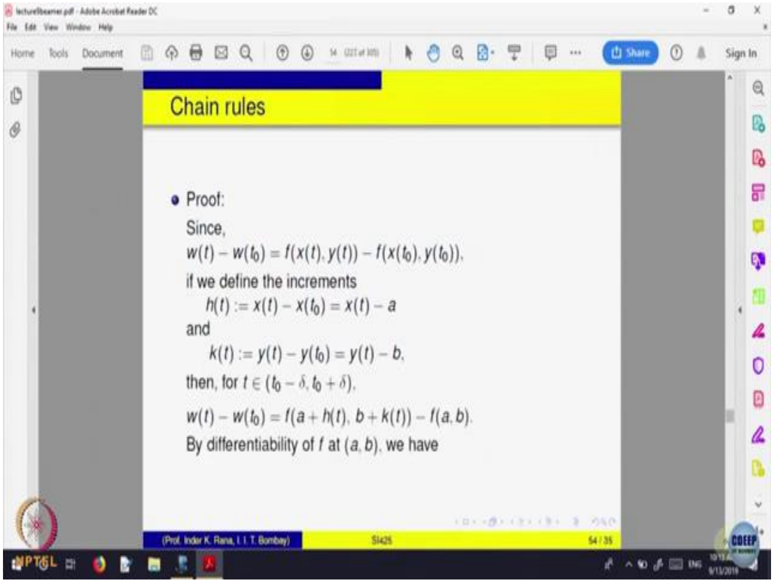

So, let me just show you those also. You just have to, it is not difficult, they are not that straightforward to prove. So, let us go to the proof of this, we will let assume these things. So, this is what pictorial it says W is a function of two variables, X is a function of one variable XT, Y is a function of the variable T, the combined thing is a function of one variable again.

So, we have to go along this plus so a partial derivative of F with respect to X, the increment DX by DT. So, that is along this branch plus above the branch..

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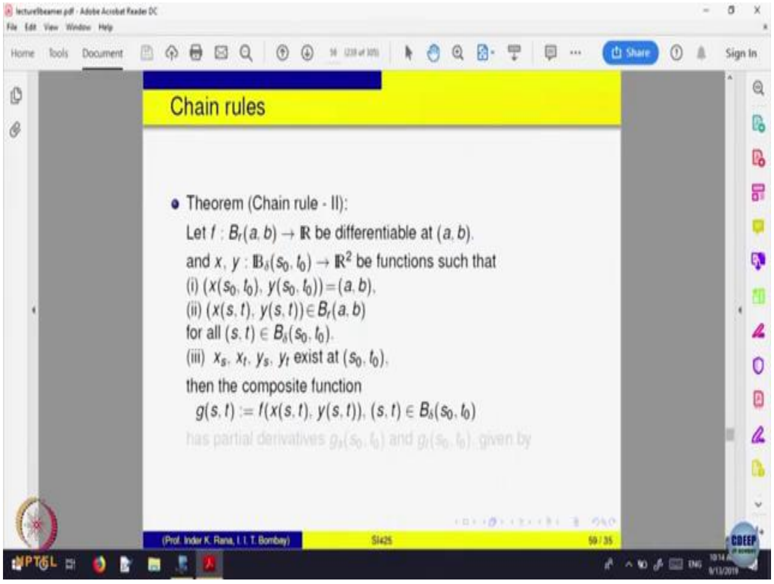

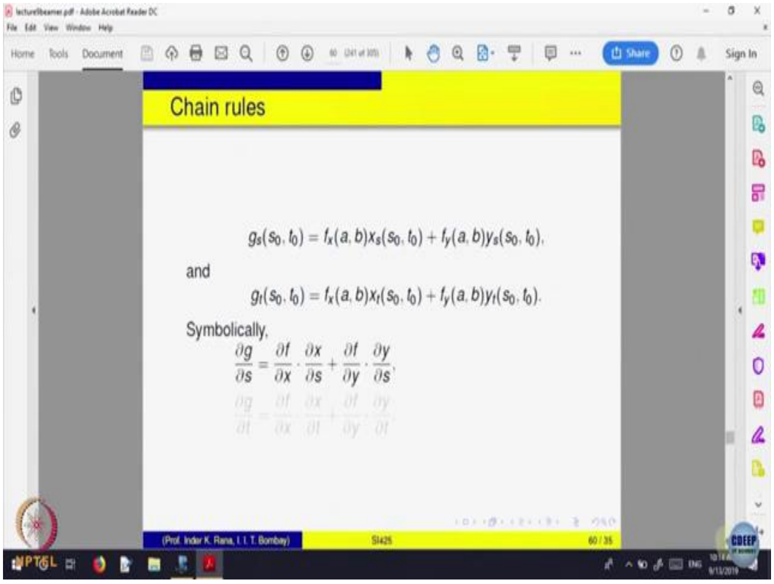

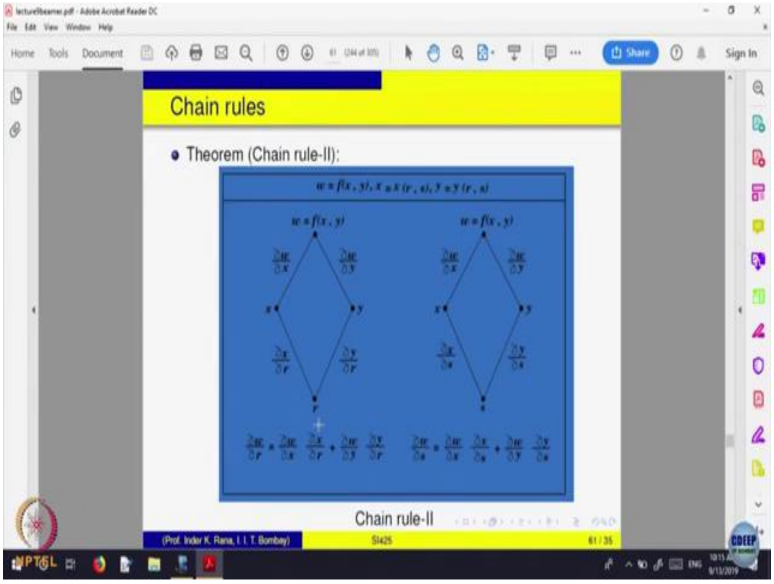

For example, if you have something more, there are many chain rules I think you should just look them and understand them how to, here is another one. W is a function of two variables X and Y but X and Y themselves are functions of two variables, earlier it was XT, YT. Now, X is a function of two variable R and S. So, it gives you two different pictures, partial, So, the composite function is a function of two variables. So, it will have now partial derivatives, earlier the component function was one variable.

So, now a function of two variables, how do you get the partial derivative now? So, two variables, so, partial derivatives you want to know whether it exist or not, if they exist what will be? Along one branch partial, F is a function of two variables. So, partial derivative of this with respect to X, X is a function of two variables. So, which variable we have looking at, here we are looking at with respect to R. So, partial derivative of X with respect to R.

Similarly, for the Y, R, so, add up that gives you a partial derivative of the composite function with respect to R, you want partial derivative of the composite function with respect to S. So, partial to W with respect to X, that is, partial derivative of X with respect to S it will come. So, that is the second one.

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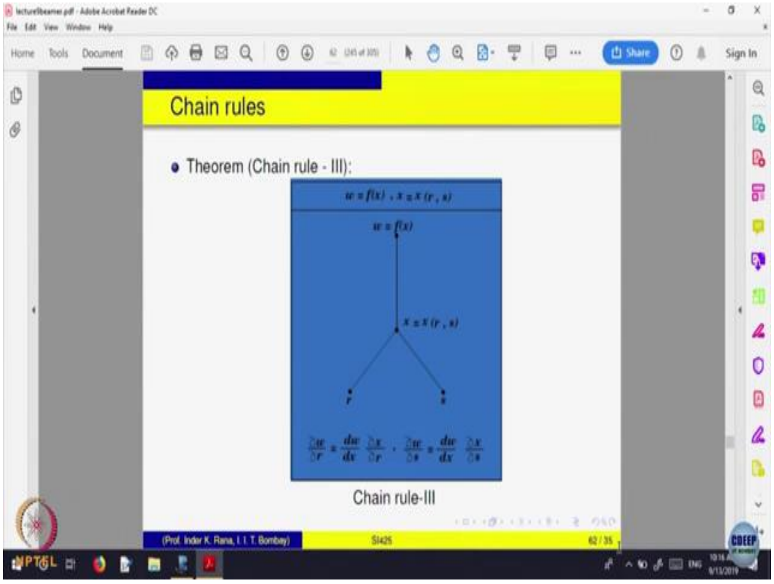

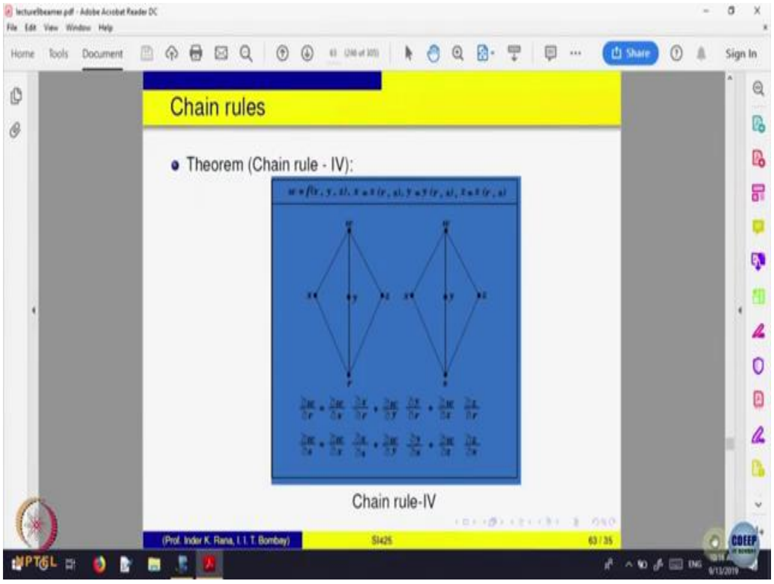

So, depending on what is where how many directions you are going you will write the same rules. So, get used to this chain rules, just go through them once, once you understand you will know what, you can have this kind of thing one variable to two variable to one variable, see one variable and then going to two variable. So, the composite is a function of one variable but taking values in two variables. So, partial derivatives again will come. So, these are various kind of possibilities.

Why they are useful we will see is one application of these kind of things, other possibilities, let me not go through all of them. They are just, see how many branches? How many variables? There is only pictures will appear.

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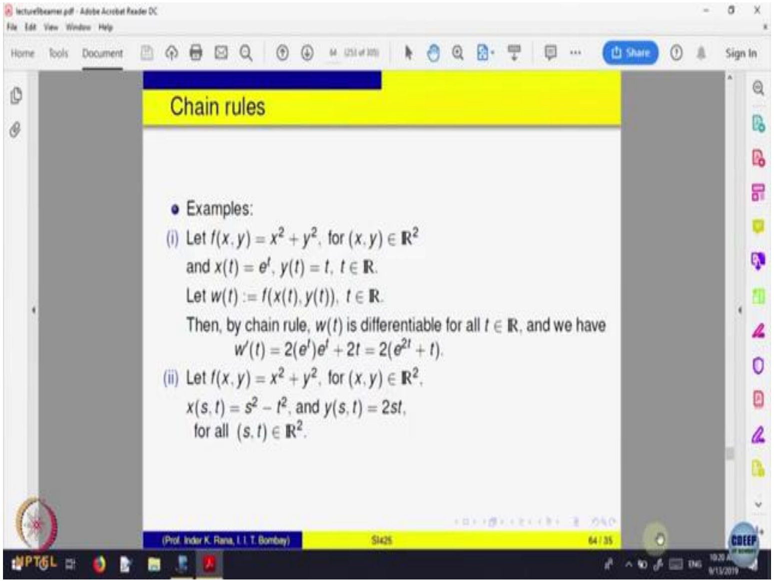

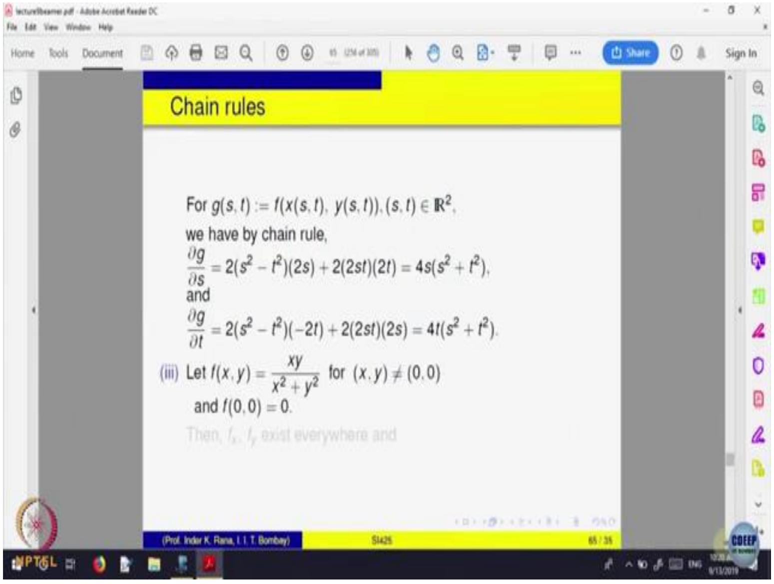

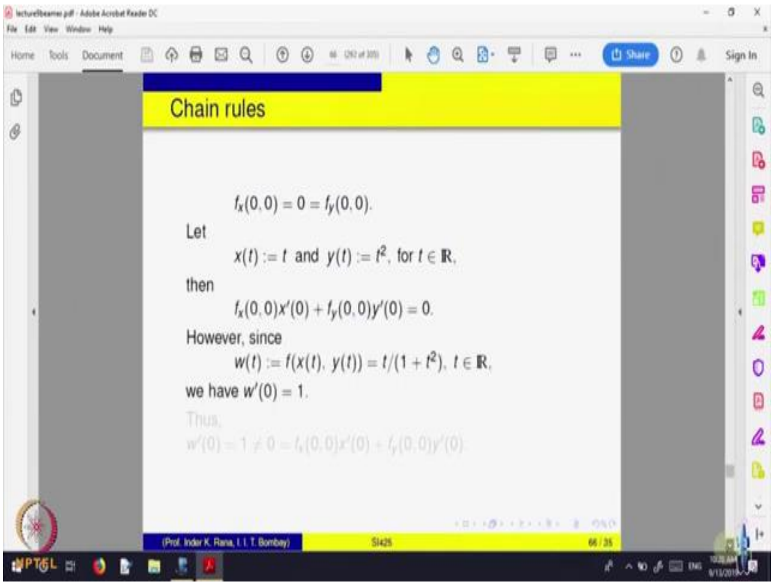

So, for example, let us just look at one example that F is a function of two variables F, X, Y equal to X square plus Y square. Let us see X, this X itself is a function of a variable T, XT is equal to E raise to power T, Y also is a function of T that is equal to T. So, what is the composite function? T goes to XT comma YT, F of XT comma YT. So, it is a function of one variable, composite is a function of T goes to a value in R, T goes to XT comma YT that is R 2.

But F takes XT comma YT to real line. So, is a function of one variable only. So, what will be differentiability, derivative at a point. For one variable the notion of derivative only. So, what is the derivatives? So, composite function if you call it as W, we want partial derivative of W with respect to T at some point say, at any point T. So, what will be the derivative? Partial derivatives of F with respect to X into X dash of T, plus partial derivative of Y, F with respect to Y into Y dash of T.

So, let us write partial derivative of F with respect to X, it is two X. So, first term will be two X into E raise to power T plus two Y derivative is 1 plus 2 Y. So, that is the, you can put the values in terms of T again X and Y, you can put the values in terms of T. So, that is partial, that is a derivative.

For example, another example, let us just. So, that is your competition F, X, Y is same function is same, but now X is a function of two variables S and T and Y is a function of two variables. So, X s, t is this, Y s, t is this. So, what is a composite function W now? It starts with S comma T two variables, maps into X of S comma T comma Y of S comma T. So, this is a function of R 2 to R 2 and then there is a function F of two variables taking into real variable.

So, the composite function is a function on R 2, where the first thing R 2 to R 2 and second one R 2 to R. So, the composite function is a function of two variables taking real values. So, when is a function of two variable composite function, partial derivatives I can find out. So, what are the variables? s and t, composite function with respect to s, how do I find out partial derivative of F with respect to X, so, that is two x into partial derivative of X with respect to s?

Because I was looking at so, into 2 s plus partial derivative of F with respect to Y 2 Y into partial derivative of Y with respect to S. So, that is 2 t, add these two terms you get the partial derivative of the composite function, with respect to s. And similarly with respect to t. So, this is what you will get, it is clear? So that is how you compute, see what functions are coming, and how many are there, so that many additions and multiplications. So I think chain rules are quite clear, more examples that you can read from the slides. (()) (20:08).

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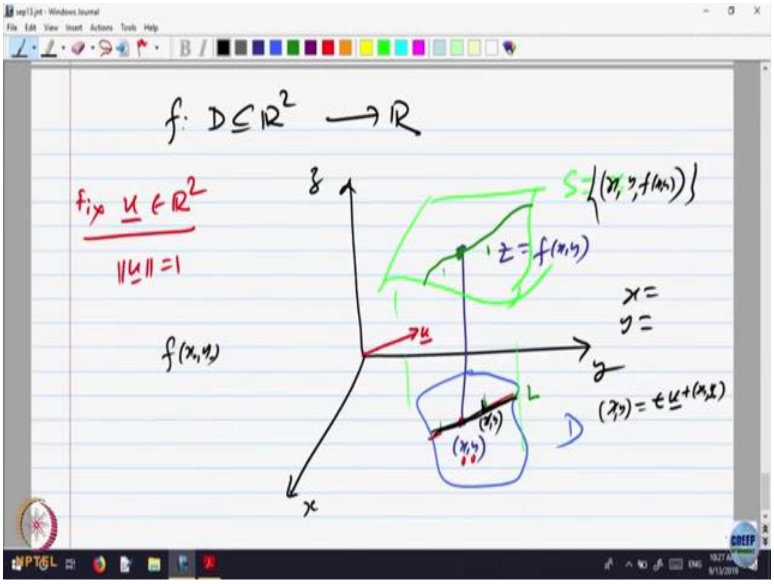

There is a convex function, so, let us look at something more. So, let us take a function F of two variables, domain D in R 2 taking values in R and let us assume F is differentiable, we do not need to assume that, so let us will come to that later. So, let me draw a picture, what we are doing so that you understand.. So, D is a domain so there is a domain and for every point in the domain the function gives you some value. So, Z is equal to F of X, Y. So, this is X, Y in the domain.

So, you get various points so, as I said you will get some kind of a surface kind of, not that far because, so these is kind of a surface will get. So, S which is, not visible yes, visibly visible S is X, Y, F of X, Y. So, that is a surface collection of all size points. Now, we will looked at partial derivatives of this function (())(22:17) .. So, what we did for partial derivative you fixed one of the variables and see whether how much the change rate of change is coming along the other variable.

We want to generalise this slightly, instead of rate of change along X axis or Y axis, we want to go along any vector. So, let us fix a vector U belonging to R 2. So, what is the meaning of fixing a vector? And let us say the norm of this vector is equal to 1. So, let us say this is a vector U that is a direction and the plane, that is a direction in the plane in the X, Y plane.

Now, I want to see what is the rate of change of the function along this direction, along this direction. So, at this point, so let us call this point as X naught and Y naught, let us fix a point also. So, how do I move in that domain, in that direction? So, to move in that direction, so I will draw a line which is parallel to that vector passing through this point X naught Y naught.

So, we will see as the function, as the point moves in this line, we will get a value. So, what you will get? You will get a some kind of a curve, as you move on this line L, L is a line through the point X naught Y naught parallel to that vector U, in the direction of that vector. So, as the point moves on this line, the value of the function will be a point on the surface S. So, as you move along this line, that will give you a curve on the surface, geometrically, quite clear?

So, what I want to know is as you move along this curve at this point is there a some rate of change of the function, how do I find out the rate of change of the function? So, what I have to do the value at this point is F at X naught Y naught. I should see how much is the increment? So, what is a nearby point? So, if I take a point nearby X, Y what is that in terms of X naught Y naught? What is this point X, Y? In terms of X naught Y naught, it is on a line which is parallel to that vector, yes? So, what is this X? Can I say X is equal to something and Y equal to something? In terms of X naught Y naught, what is that? And of course, the vector U because it is parallel to U.

So, what is this? What is X? What is Y? Now, so let me see, so let us if you want to write what is the vector X, Y that is precisely T times the vector U, now plus X naught Y naught, any point on this, so let me demystify it for you. Imagine you are moving on that line, you are moving on this line, at the point 0, at time point 0, you are at this point, you are moving on the line, you are starting your journey at the point T equal to 0 and you are at that point.

Then, what will happen? You are moving in that direction, time T units, how much you would have moved? T times the vector U, depending on whether it is positive, positive or negative, backward or, that is physics.

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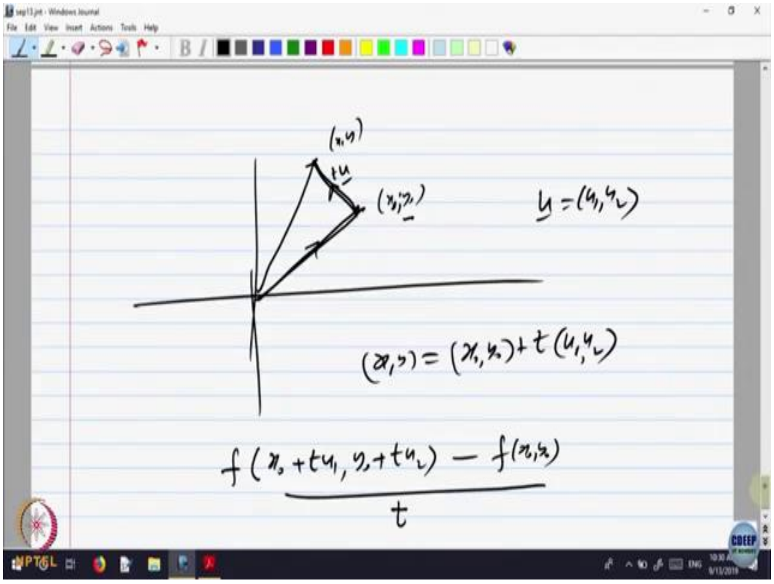

But if you still do not understand it, here is, so this is the point X naught Y naught, this is the point X, Y. So, what are the coordinates of this? This direction is U vector addition now, if this is. So, what will be a point that is T times U, this is T times the vector U, this vector is X naught Y naught. So, what is X, Y? So vector addition if you like, so this X, Y is equal to X naught Y naught plus T times the vector U.

So, let us call it U 1, U 2 the components if U is equal to U 1, U 2 vector addition only, nothing more than that. So, I want rate of change so, F at nearby point X, Y, so that is T, U 1 sorry, what is nearby point? So F at X naught plus T, U 1, Y naught plus T, U 2 there is a point nearby minus the value F at X naught Y naught divided by how much is a change?

How much distance you have moved? That is T, because we had taken the vector to be the unit vector, T times U 1 square plus U 2 square, square root that is a distance. So, that we have taken it as 1, what is it? We are going other way round is that okay? We are taking the norm to be equal to 1.

So, along this, this point X, Y is T of U plus the vector X naught Y naught. So, what is a value at this point, nearby point? Value at this point, so there is this, there is F at the nearby point F of X, Y. These F of X naught Y naught. So, the increment divided by the distance, rate of change I want to find out.

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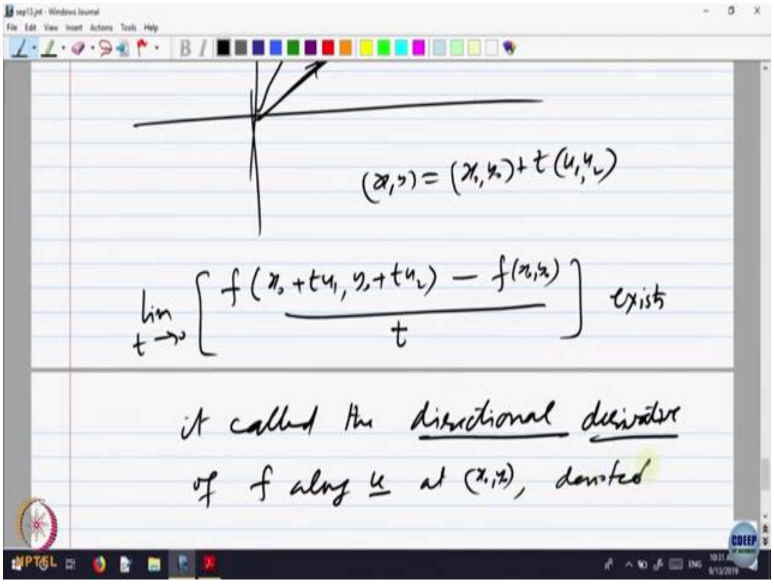

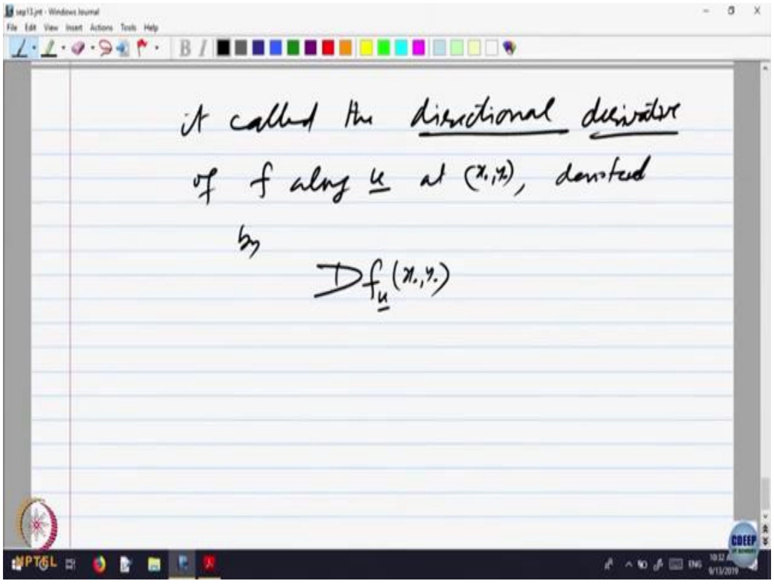

So, there is what precisely I am looking, I am writing yeah, F at nearby point minus the value at that point divided by how much a increment? I should take the limit of this as T goes to 0, if this exists, it may not exist. We are trying to find out what is the rate of change, if it exists, it is called the directional derivative of F along the vector U at X naught Y naught.

So, it is of the function F along that direction U at that point denoted by, so let us given denoted by directional derivative capital D of the function F in the direction U at the point X naught Y naught, so that is a notation we will use. So, can you say what is a partial derivative with respect to X, what is U there? Direction only.

So, partial derivative what is a vector U? If you want partial derivative with respect to X, it is nothing but the vector U is 1 comma 0, unit vector in the direction 1 comma 0, Y direction 0 comma 1 so, those are the partial derivatives. So, they may not exist even partial derivative we have seen may or not exist, so if they exist so this is the notation for that.