Basic Real Analysis
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Lecture 39
Riemann Integration Part III

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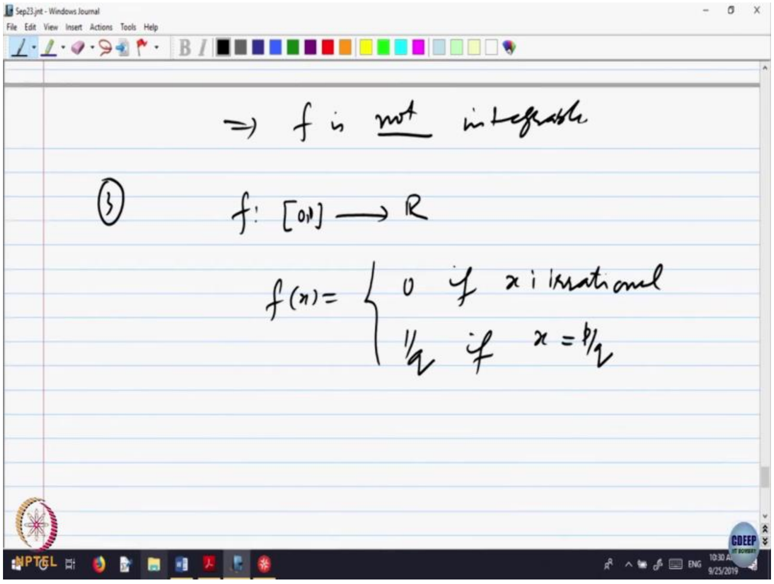

So, let us look at one more example. 3, I do not know whether I discussed a popcorn function. No, let us discuss what is the popcorn function, so f is a function on 01 to R and this function is f of x is equal to 0 if x is irrational like before. In 01, if it is irrational the value is 0. If it is rational it will be a fraction p by q. If it is p by q, where p and q, this is a number between 0 and 1, so q will be bigger than p anyway, between 0 and 1 so put it as 1 over q if x is p by q.

Forget the numerator, take only the denominator of that function, clear what is this function. Can you think of drawing a sort of a graph of this function? What do you think the graph should look like?

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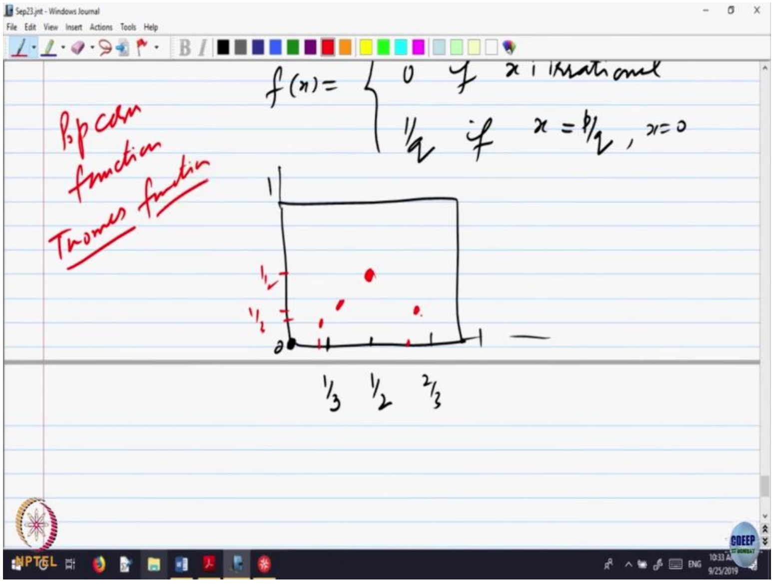

So let us try to plot this to see whether we can do something or not. So this is 01 function will take values between 0 and 1 only. What is the… let us see when it is 0 this is not defined p by q denominator could be anything. So, let us say x is equal to 0, both for x equal to 0 the value is 0 for every irrational the value also is.

So, what is the or you can put value at 1 also, 0 if you like does not matter much actually. So, let us take what is the value at the point x is equal to half, when x is equal to half what is the value of the function? It is half, is a irrational, the value is half, it is 1 by 3 what is the value, value is 1 by 3.

What is the value at 2 by 3? Again forget so again the value is, so this is 1 by 3, this is 1 by 2, so now you can see what is happening at 1 by 4, what is the value at 1 by 4?

Student: 1 by 4.

Professor: 1 by 4, at 3 by 4.

Student: 1 by 4.

Professor: 1 by 4 so here is 1 by 4 so, value at 1 by 4, and 3 by 4 and so on value, so it will be kind of triangular kind of a thing and here the value is smaller, so if you make these dots bigger and smaller these are small just for the sake of visualization.

They look like popcorns when being fried, or baked, they are jumping up? So this is normally called popcorn function. In fact, it was given by a mathematician called Thomas, I think spellings I am not very sure Thomas function or popcorn function anyway, does not matter mathematically. We know what is this kind of function.

Do you think it is a continuous function? No, does not look like otherwise I should have drawn the graph better, but what are the points of discontinuity of this function? At what points it is discontinuous? So, let us look at the way of analyzing this continuity would be because the function is defined differently at a rational differently at a irrational.

So, we should analyze continuity in two cases when it is rational and irrational. Supposing x is irrational, the value of the function is 0 and look at a sequence, our points converge into that irrational, but the sequence I can choose of irrationals then all the values will be 0, so no problem. But I can choose a sequence of rationals converging to that irrational and if I choose this rationals then close to it I can have a point where the value is not 0. So, they will not converge to 0. So, at all irrational points it is discontinuous or at all irrational points it is discontinuous or continuous.

Student: Discontinuous.

Professor: Are you sure.

Student: Yes.

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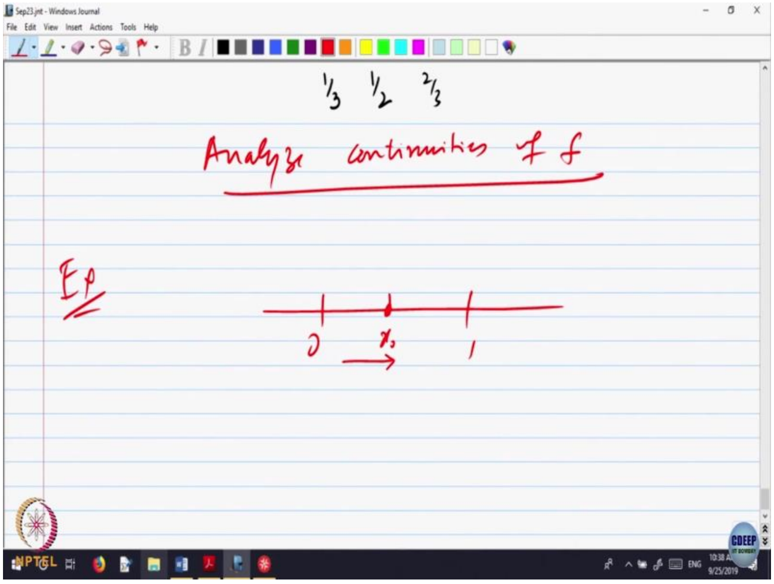

Professor: So, let me analyze discontinuities of f I have you given you some idea. Now supposing let us take x to be rational point, the value will be 1 by q. Now value at all rational point is 0. So, now given a rational point, so here is a rational point here is 0 here it is 1 this is a rational point call it x 0.

I know the value is not 0 for the function at this point. But I can have a sequence of rational converging to it or irrational converging to it, irrationals are dense. So, I can have a sequence of irrational converging to x naught, but the value at every irrational is 0. But the value of the function at that point is not 0.

So, at a rational point it is not continuous. So, that is why I was saying, try to convince yourself that this function has discontinuity at all rational points it is continuous at all irrational points. At a rational it is discontinuous. Is that clear? If you want to say it is discontinuous everywhere, no it is not rationals are the only points of discontinuities. So exercise, I have already given you that at a rational point is it is discontinuous that these are the only points that means what, that means at every irrational point the function should be continuous.

So, check that so mull it over and think about it. I will put it as a problem in the problem session ask them to discuss it with you that this is…

Student: (())(09:15)

Professor: Let us not discussed now, because you will have time to think, let everybody have a time to think about the problem and discuss the problem session that is a better way of doing it. So, anyway, the number of discontinuities of this function are infinite, rationals and every sub interval of 01 will have an infinite number of rationals. So, infinite number of discontinuities, it has infinite number of discontinuities in every sub interval of 01.

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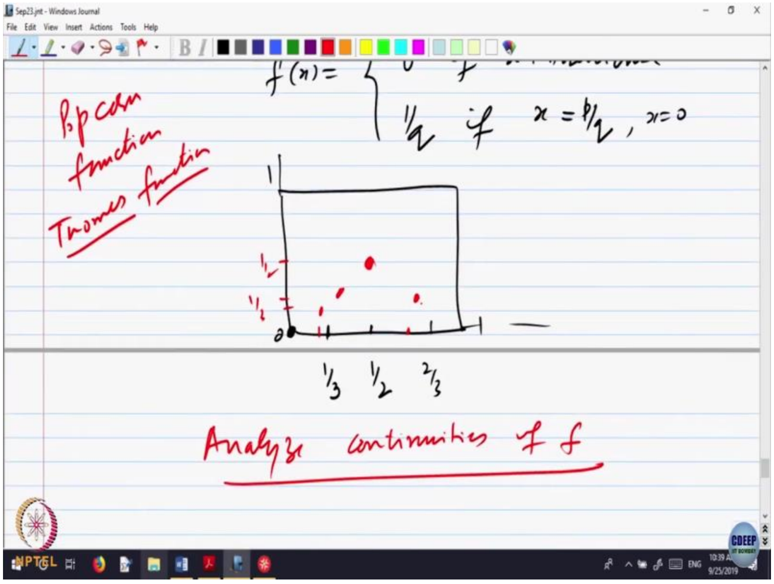

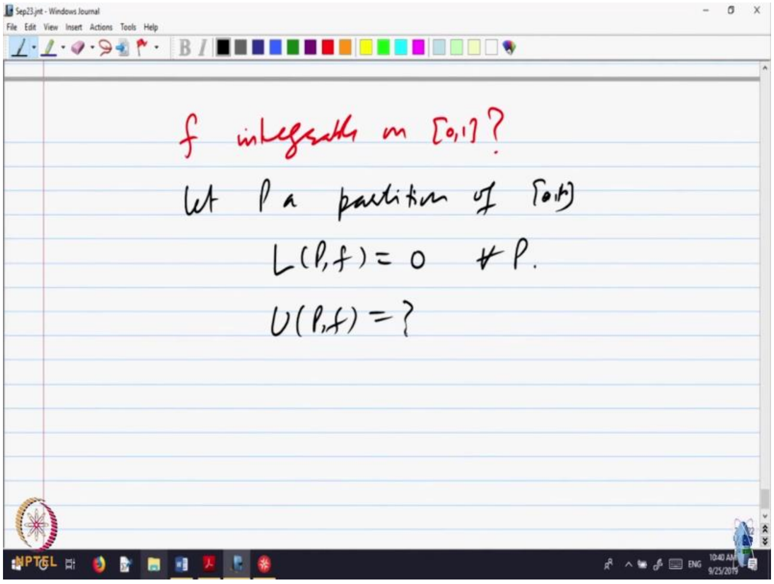

Now, let us try to understand is this function f integrable is this function integrable. It is not differentiable everywhere anyway because it is not continuous everywhere. So, that point is okay. Let us see integrable, can I think of saying what is the graph, area below the graph of the function? So, let us try to see. So, to do that.

So, let us take P a partition. So, if I want to show it is integrable what I should try to do, let P a be a partition of a b 01. What is the lower sum with respect to at a compute lower sum and upper sum, what is the lower sum that will be 0, because every sub interval will have a rational unit. So, it is 0 for every partition and to worry about what is the upper part.

So, let us try to see the upper one, the upper sum, the contributions will come from the rational points because at rational the value will be non 01 over q into the length of the interval. So, I had to bother about at how many points the function, what value it takes.

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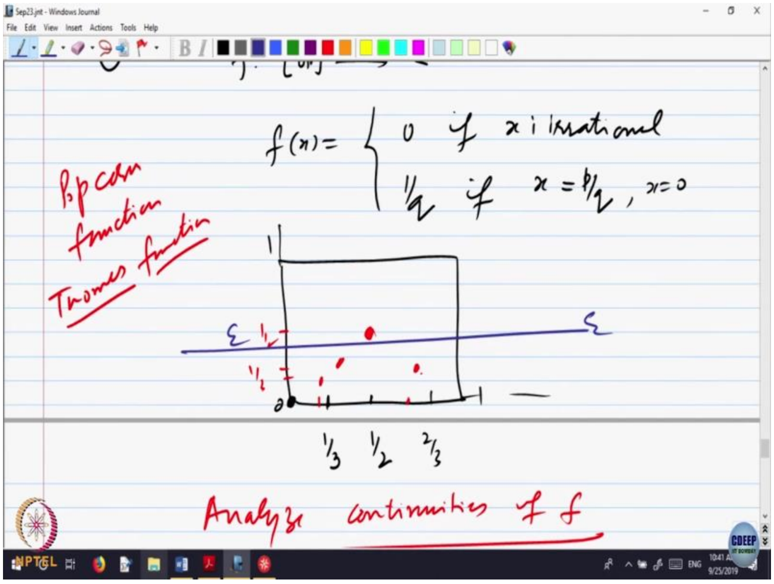

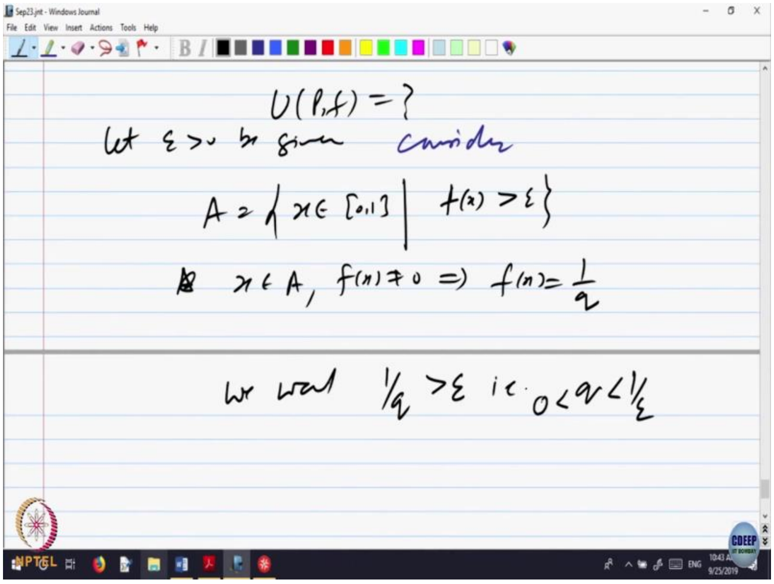

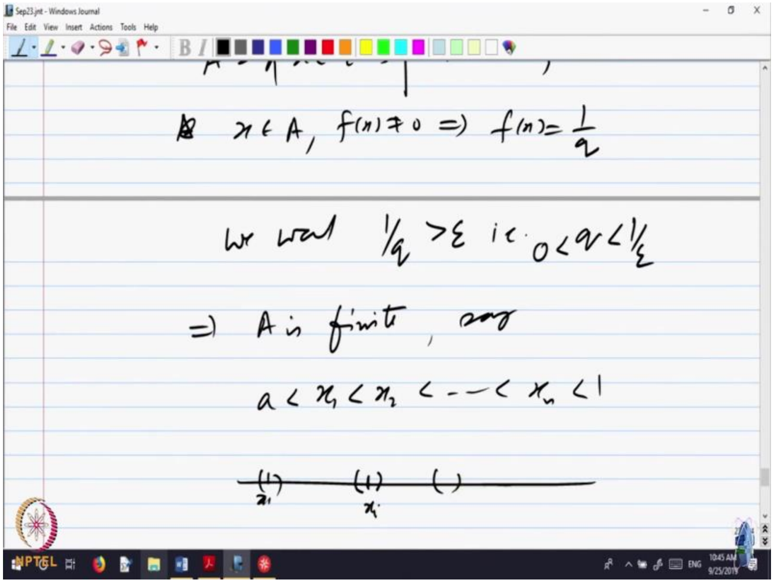

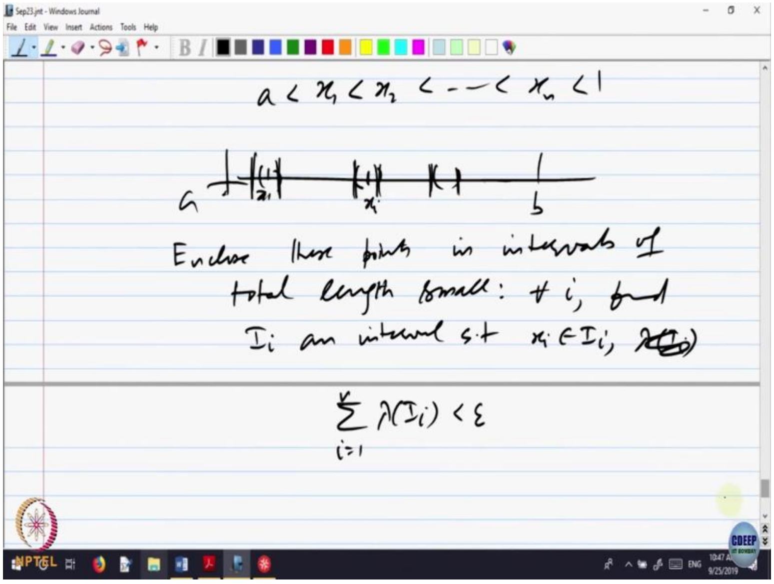

So, let us look at the set A. So, let us if I want to show it is integrable I had to make upper minus the lower small for some partition. So, let us start with let epsilon be given, so I am looking at the values of the function, I want to look at here is my epsilon. At how many points the value can go above epsilon. At how many points in the domain, the value of the function can go above epsilon.

So, those are the rational points and at the rational points the value is 1 over q. So the only points possible are when 1 over q is bigger than epsilon. So, I am looking at the set, consider a to be all x belonging to 01 such that the value of f of x is bigger than or equal does not matter epsilon. What can you say about A? It is bigger than f of x is bigger than epsilon when f of x is 1 over q. So how many qs are possible, so that 1 over q is bigger than…

Student: Countability many.

Professor: Countability many? I do not know I am asking. So, x belonging to A f of x not equal to 0 implies f of x is equal to 1 over q. So, we want 1 over q bigger than epsilon that is q is bigger than 1 over epsilon, other way around q is?

Student: Less than.

Professor: Less than, good, q is less than 1 over, how many qs are possible between 0 and q has to be bigger than 0 anyway? Between 0 and bigger than 0 and less than epsilon is fix, how many are possible?

Student: Finitely many

Professor: Finitely many, there are only finitely many, not countable only finitely many, q is positive. So, how many are possible only how many natural numbers because also A is, so that means implies A is finite. So, the set of points where it goes above, so points where it is less goes above it is only finite.

So, let us call this finite number something say x 1 less than x 2 less than finite sum number say x n. Bigger than a less than, now, so here are the here is a point xi at this point the value goes above epsilon. So, I want to make them in some sense I want to enclose them in a box, so that these points are inside intervals so of length small. So, around every point I can have an interval.

So, it is x 1 x 2 x i and so on, enclose each one of them in a interval, so that total length of this intervals is small, there are only finitely many, there are only finitely many, so let us support. So, enclose these points in intervals of total length small, so meaning what, this is all English, what is the mathematics of that, that means for every i find interval Ii an interval such that xi belongs to Ii length off Ii is small, sigma length of Ii, i equal to 1 to any small, so how small you wanted let us make it less than epsilon again.

Around every point find interval of length epsilon by n. So, when you add up all these finite number total length will be epsilon by n plus epsilon by n, n times that will be epsilon, here is my a and here is my b. Now, these intervals have got some end points like these are the points. So, these points give me a partition of the interval a b. So, what are the partition points a 1 point here 1 point here next point here, next point here, next point here and so on.

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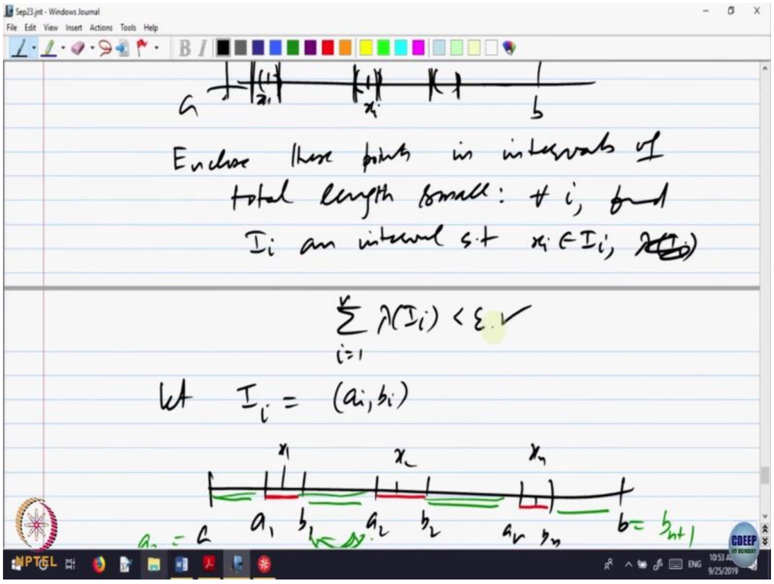

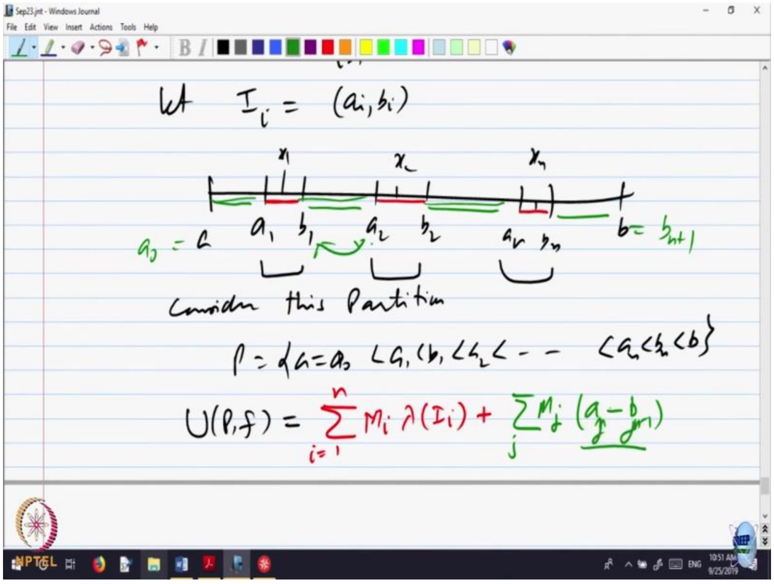

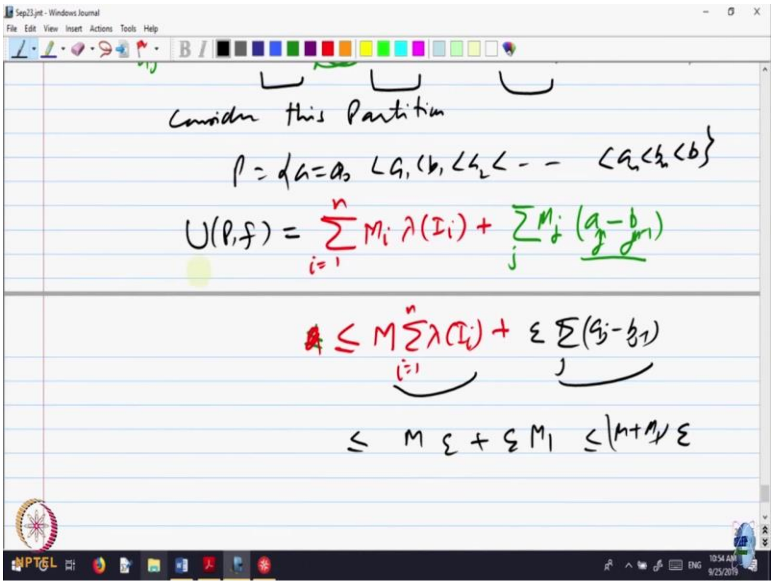

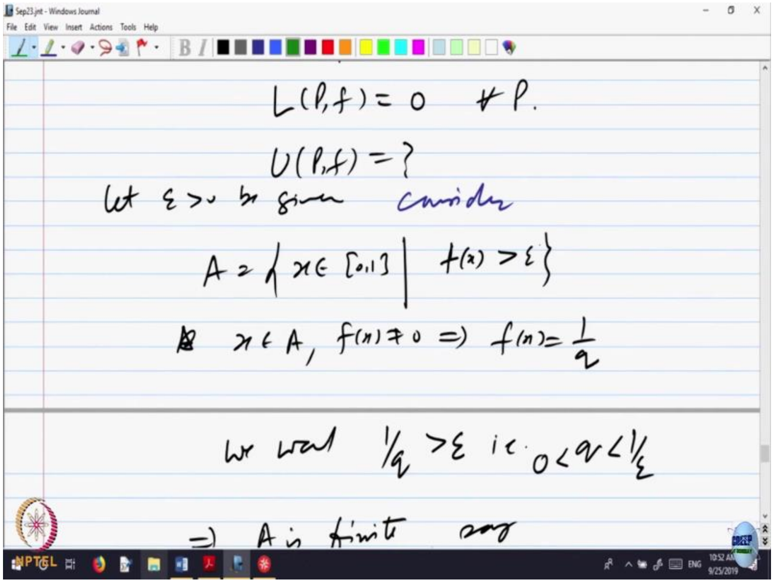

So, if you want let us give them a name so let us Ii be you want to give a name, so let us ai, bi. So, what does it look like? Here is a1, here is b1 here is a2 here is b2, x1 is somewhere inside x2 is somewhere inside and so on xn an bn and here is a and here is b. That is a picture it looks like, length of a 1 b 1 plus length of this plus length of this put together is less than epsilon.

So, this gives me a partition. So, consider this partition. So, what is this partition? So, P is a equal to a 0 less than a 1 less than b 1 less than a 2 less than so on less than a n b n less than b. With respect to this partition lower sum is 0 we know that already, lower sum for every partition is 0, I want to calculate what is the upper sum with respect to this partition.

So, the upper sum I will divide it into 2 parts over the intervals which are a 1 b 1 a 2 b 2 a n b n, what is upper sums maximum value in a sub interval times the length of the sub-interval. So, let us divide it into 2 parts. So, 1 part is with respect to this ai bis. So, let me call it as maximum value in ai bi. So, length is, length of Ii, I equal to 1 to M plus the other parts, so these intervals so, what is this sigma? Let us call it Mj if you like. So, what would be the length now?

It will be a 2 minus a 1 a 3 minus, so it is a n minus b n minus 1 what should I write n minus 1 b 1, it is just only the way of writing, so this we call it as a 0 and this we call it as b n plus 1. Then what I am looking at is so, a 1 minus a, is that okay over j. You understand what I am writing here? I am writing over the sums over these green portions. So, what will be that, the maximum value Mj whatever that may be into the length of the interval. So, length of interval will be like here will be a n minus a 2 minus what is the length here a 2 minus b 1, the length next one will be a 3 minus b 2, so that is what I have written here.

Student: (())(22:32)

Professor: Oh sorry aj, yes. aj, Now, in the red ones, I do not know what is the value of the function but I can sight the red ones. I can write some constant M the function is bounded by 1 into sigma length of Ii all the intervals maximum value each Mi the maximum in the sub interval is less than equal to maximum in the whole interval plus what can you say about the other part? Mj? What is Mj here, that is the maximum in the portion where the green ones, what is the value of the function in the green portion? In the red ones the value was bigger than epsilon. So, green ones, those points where it is bigger, whether or finitely many so it will be less than epsilon.

So, let us write this value is less than epsilon times sigma aj minus bj minus 1 whatever that j may be and this length this one we have seen it is the way we found this intervals were less than epsilon. Let me put that as epsilon, so less than equal to M times epsilon plus epsilon time some call it as M 1 you see this is a constant.

So, total length, so is less than or equal to M plus M 1 times epsilon, so the basic idea of the proof. So, I have found a partition say that the upper sum is less than epsilon time a constant. So, the constant does not matter in our analysis. So, I have found a partition, so that the upper sum, lower sum is 0, so that the difference is less than epsilon that means the function is integrable.

So, let me revise a bit the idea of the proof. See, this is the crux of the proof says upper sum will arise from the values where the values are non-zero. So where the values are non-zero I look at those points were the values exceed epsilon and less than or equal to epsilon.

Where they exceed epsilon the value of the function they are only finitely many such points, those points where the value can tend to become larger I enclose them into small intervals of total length small. So, I that a person with respect to those intervals I make it small by enclosing each point in a small interval of total length small.

In the other intervals, the value or the function itself is small. So does not matter because the total length of sub intervals is B minus A. So, that is how I contain the upper sum minus the lower sum I make it small by making the upper sum small in the intervals where the value exceeds something they are only finitely many, other part does not matter with our value is less.

So, this is try to read the understand this proof yourself later on when I will send you the slides, the basic idea is that this function is a function which has got discontinuities infinite in every sub interval, but still it is a integrable function. Now earlier function rational irrational 01 that was discontinuous everywhere.

So, that was not integrable and if the number of discontinuities are finite that is not really a problem. Once again we can define the integral and separate parts and add, so the function is integrable. So, integrability of the function has to do something with continuity of the function.

We have got different examples constant functions continuous everywhere integrable. Monotone function discontinuities are countably infinite at the most, we do not know how many but still integrable. This function is a function with infinite discontinuities in every sub interval still integrable and the other extreme is that 01 function discontinuous everywhere and not integrable.

So, somewhere continuity were to say how many points you were allowing discontinuity for the function to be integrable. So, next time we will prove first one theorem that every continuous function is integrable like every monotone function is integrable, every continuous function is also integrable and there is a deep theorem which says, you can actually give a measure of how many discontinuities are allowed and that is the beginning of a story called lebesgue measure. That is the beginning of probability theory lebesgue measure, measure theory and so on.

That basically says the set of discontinuities of a function if they do not contribute much in the length of the interval, then that function is going to be integrable. For example, for a finite set in this we were able to unclose each finite set in a interval total length small. But if I take a countable set, even that I can enclose each point in a interval of total length small level x 1 x 2 x n is accountable infinite set.

x 1 enclosed in a interval of length epsilon by 2, x 2 enclosed it in a interval of length absolute by 2 square. Next one epsilon by 2 q, so what is the total length of these intervals sigma epsilon by 2 to the power n and that is a geometric series, sum is equal to epsilon, so you can enclose countable infinite set of points in small intervals of total length small.

So, in that sense, you say that the countable set do not contribute anything towards the size of the interval you say it is countable number of points from a set of length 0 because you can put them in any small collection of sub intervals. So, there is a deep theorem says that a function is integrable if and only if setup discontinuities are small size set of length 0.

We will not prove that theorem I am just saying this may come across some time somewhere, but you have seen a lot of examples of that. On one hand continuous everywhere, constant function slowly we are coming to monotone infinite but integrable. Then this function popcorn function infinite in every sub interval still integrable and then discontinuous everywhere not integrable. So, let me stop here.