Basic Real Analysis

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Lecture 4

Convergence of Sequences – Part I

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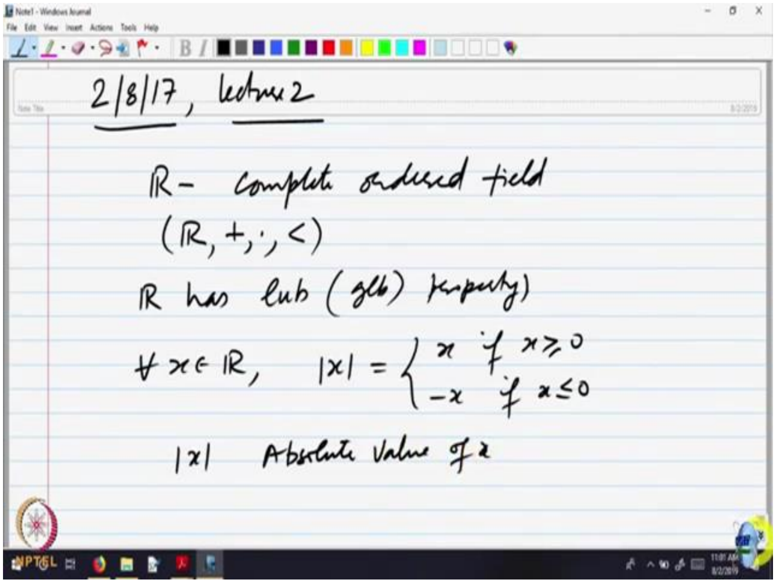

So, let us just recall we were looking at set of real numbers, which is a complete ordered field. So, on R there is addition, there is multiplication and there is an order. And the completeness meant there is R has Lub or glb property. So, every non empty subset of the real line which is bounded above has got a least upper bound and similarly is every non empty subset which is bounded below has got greatest lower bound. So, that is called the lub property and that is what makes it a complete ordered field.

So, we were also looking at the notion of Convergence of Sequences, I just wanted to for the completeness sake wanted to say that for every x, belonging to R, the order gives rise to a notion of a distance. So, we define what is called the absolute value of x to be equal to x if x is bigger than or equal to 0 minus x if x is less than or equal to 0. So, this is called this is called absolute value of x.

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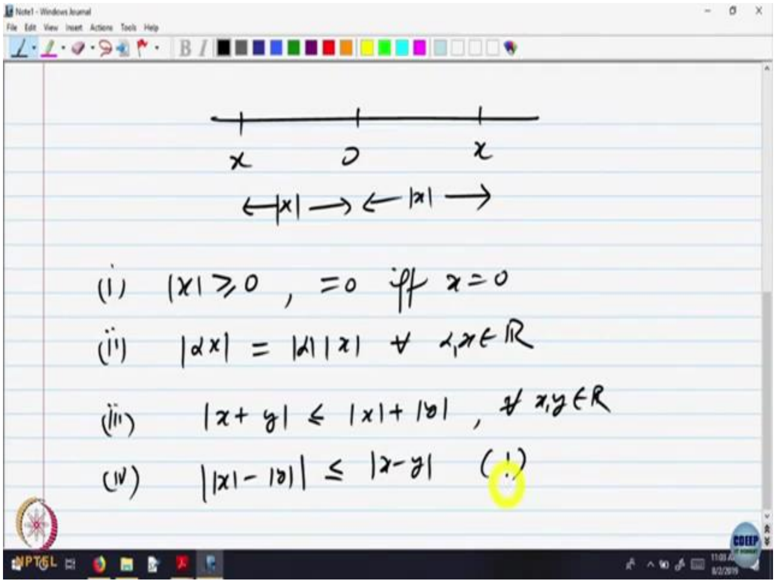

And geometrically it signifies the distance of the point x. So, this is 0 if you look at the geometric representation x could be here or x could be here. So, either way this distance is mod x, okay. There are some obvious properties which you would have come across but anyway in case not you should revise those properties. One, absolute value of x is always bigger than or equal to 0, it is 0 if and only if x is 0. So, that is positive definite.

Then second is alpha times x absolute value is mod alpha times mod x, for every alpha and x belong to real line. So, this is how absolute value behaves with respect to multiplication, okay. And the third how it behaves with respect to addition. So, x plus y is not equal it is at the most best possible is mod x plus mod y it can be equal to sometimes, but not always for every x and y belonging to R.

Fourth, this you might not have come across but try to prove it mod x minus mod y is less than or equal to mod of x minus y. So, here is something if you have not come across check it, why it is so, all these are useful properties.

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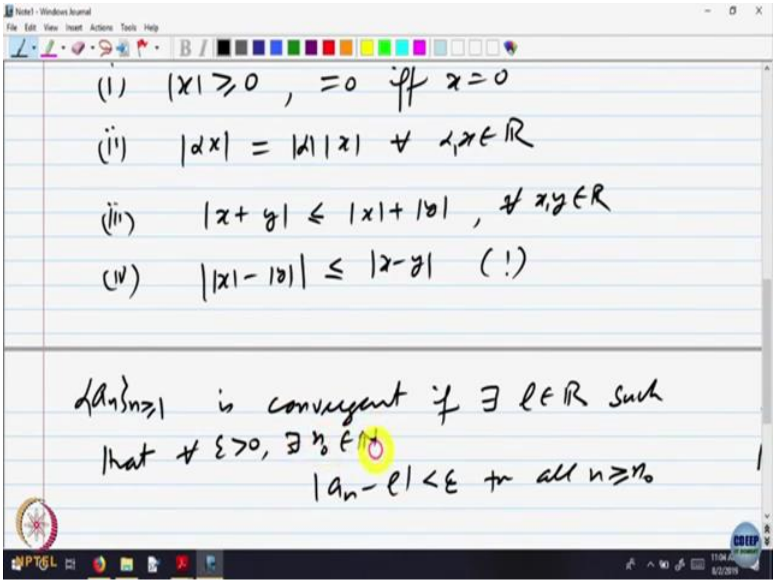

So, let us go over to we were looking at Sequences. So, we looked at a sequence a n, we said is convergent if there exists some value call it L, such that the distance between a n and L or you can say that approximation, an is an approximation to L can be made small, for what? Such that this distance can be made small for what epsilon, such that whatever epsilon I give you I should be able to for all n bigger than some n naught, and what is n naught? There exists a stage n naught such that this happens.

So, think it as a n is an approximation for L and absolute value of a n minus L is the error that you are making and that error you should be able to make it as small as you want, depending on how large and n is. So, that should be for all n bigger than or equal to n naught.

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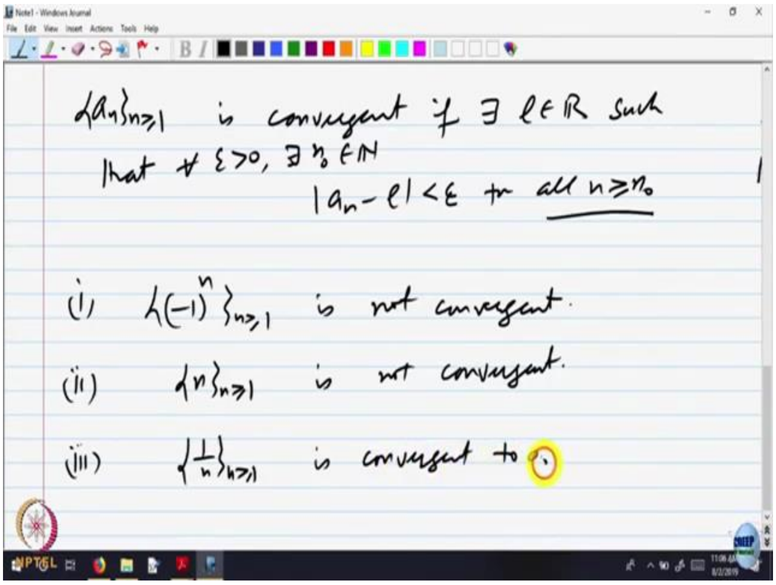

So, we looked at some examples, let us look at one example was that the sequence minus 1 to over n is not convergent because it only occupies the place 1 or minus 1, not only it occupies only these places but occupies very frequently every alternate times. Second, we look at the sequence n is not convergent. So, it does not come closer to any value but keeps on become bigger and bigger.

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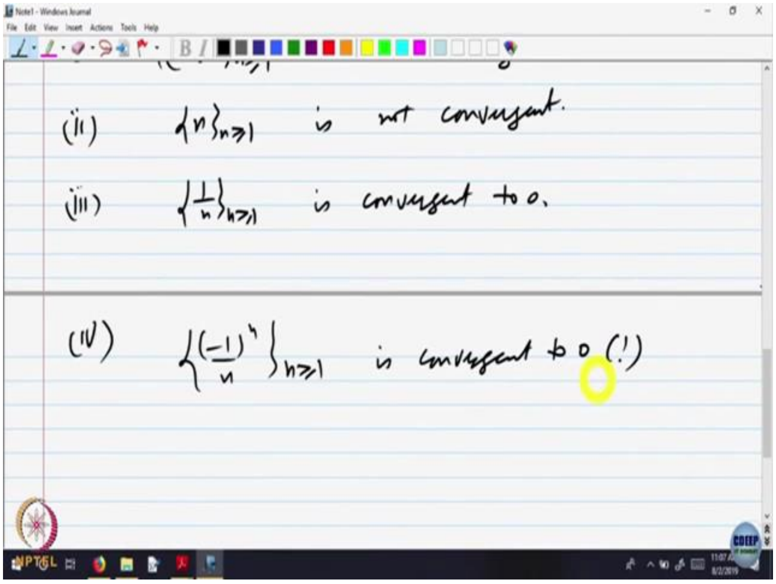

Let us look at another one. So, this is equivalent to saying if I look at the sequence 1 over n, both are consequences and in fact equal to what is called the Archimedean property is convergent and the value is 0. Let us look at one more probably before we go ahead. Let us look at slight variation of the above, minus 1 by n raised to power n, n bigger than or equal to 1.

So, it is the same sequence 1 over n, but now I am allowing it to fluctuate. So, n equal to 1 the value is equal to minus 1, half minus 1 by 3 and so on, but still you can see that it is coming closer and closer to 0. So, one can try to prove it is convergent. So, try to write a proof, try to write a proof of this, okay. So, we will see what, what is important, what is not important.

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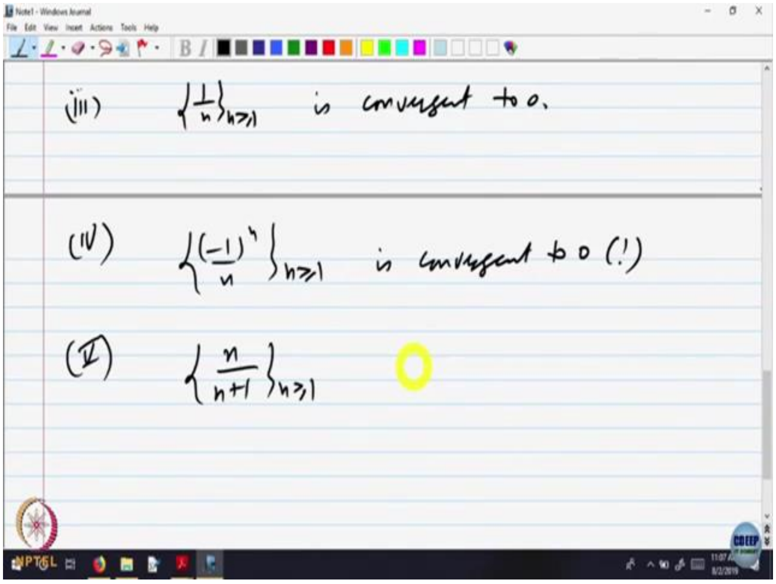

So, let us look at five. Okay, let us look at the sequence n over n plus 1. Let us look at this sequence, seeing all the above sequences let us just observed a few things that this sequence was not convergent, the first one, because it was fluctuating, because it was fluctuating at 1 and minus 1, this was not convergent, because it was in some sense not bounded. You cannot put some bounds that it remains between some kind of barriers.

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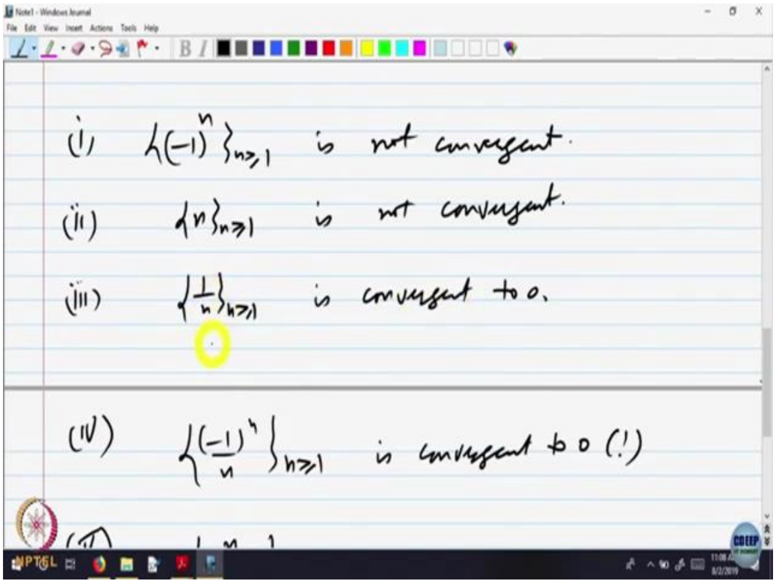

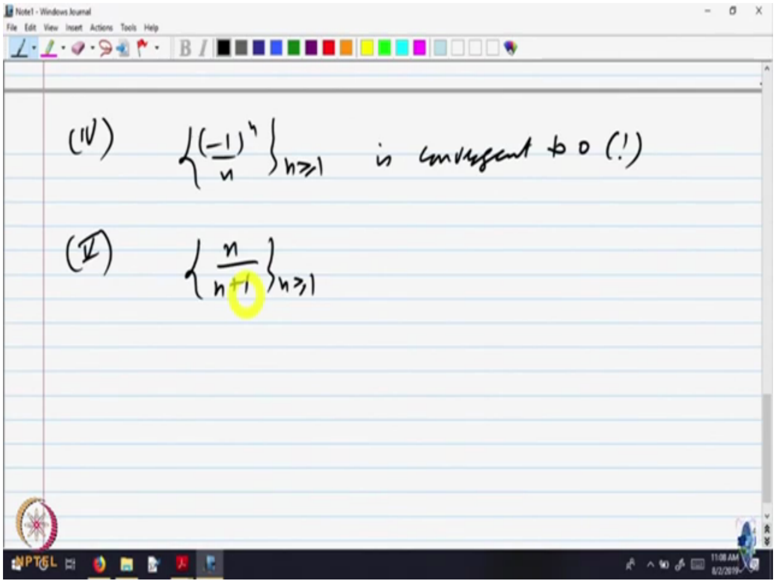

This is convergent to 0 we are going to look at n is becoming larger so 1 over n is becoming smaller. So,mething similar here, but in this thing if you think of n is becoming larger, but n plus 1 also is becoming larger, both are becoming larger and larger. So, it is not very straight forward to guess that whether it is convergent or not convergent. But you can think of that this there in some sense n is increasing and n plus 1 they are increasing at some same rate kind of a thing.

So, what do you expect? You expect that probably it should converge, at the same rate, so, they should cancel out, for enlarge the rate should become same, of the ratio should become same and that should be mean it should converge to 1. So, there is a guess, that is how you guess it is right. Now, try to prove it.

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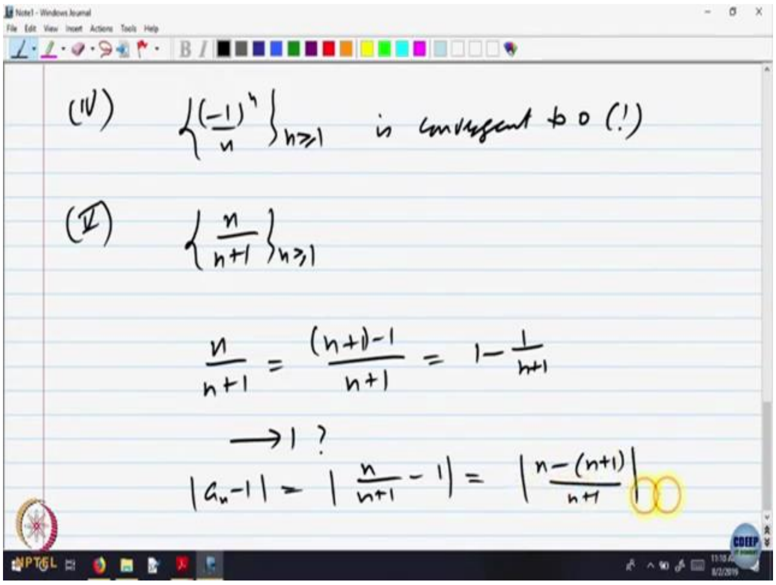

So, let us look at n over n plus 1. So, here is where your thinking will come into picture. If it is 1 over n, I can do something, numerator is n, denominator is n plus 1, let us try to make it same. So, I write this as n plus 1 minus 1 divided by. And why I am doing that? The reason is this. So, that will make it equal to 1 minus 1 over n plus 1. So, that essentially looks like saying that is the constant minus something and that something is becoming smaller and smaller. So, you guess that this converges to 1, so converges to 1.

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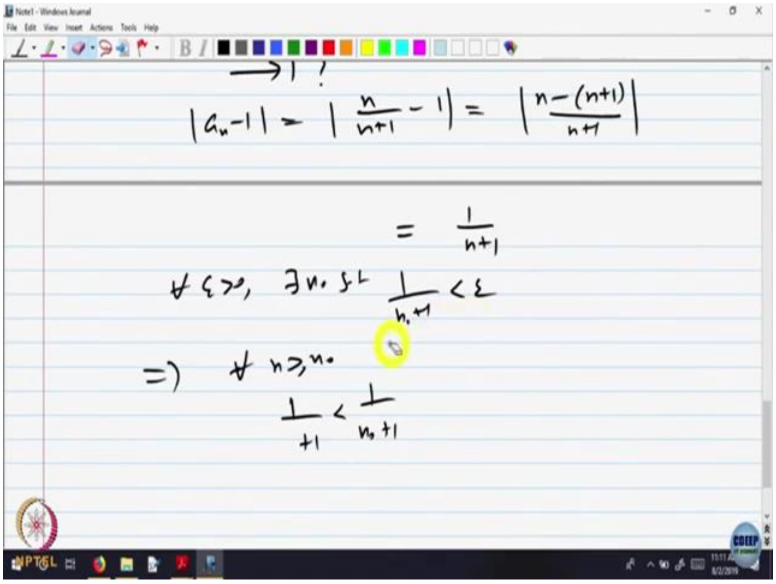

So, let us write a proof of that. So, let us look at a n minus 1, so what is that? That is n over n plus 1 minus 1, a n minus L, I want to make it small. So, what is that equal to, so let us simplify n minus n plus 1 divided by n plus 1 and that is nothing but, so, it is 1 over n plus 1, is it okay? And that I can make it small. So, give for every epsilon bigger than 0 I can find n naught such that 1 over n naught plus 1 is less than epsilon.

So, that we will write so, we can write so, this will imply that for every n bigger than n naught, 1 over n plus 1, if n is bigger than 1 over that is less than 1 over n naught plus 1. So, that is n is bigger so n naught plus 1, n is bigger. So, I am just trying to write this thing for every n bigger than n naught this also holds. Is that okay? So, let us not spend time on that to write it.

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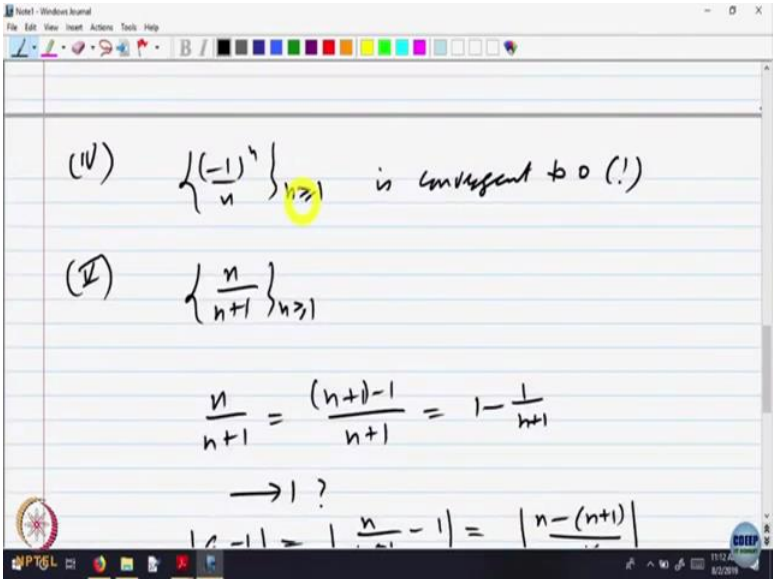

You try to see something that this was convergent right and that meant that the terms of the sequence are 1, 1 by 2, 1 by 3 and 0, so everything lies between 1 and 0. This was also convergent. So, that also lies between minus 1 and 0, okay, this is also convergent 1 so, it lies everything is non negative, so, between 0 and 1. So, it seems to say if something is convergent, that should be bounding. So, what does that mean?

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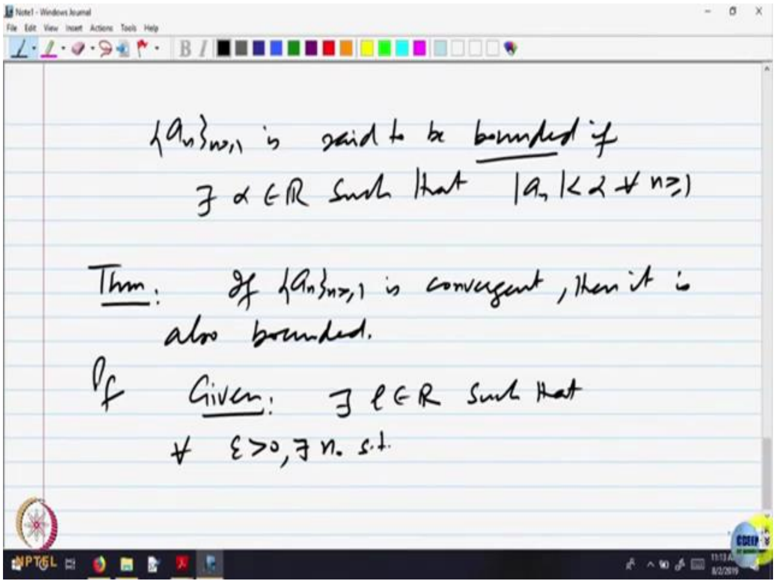

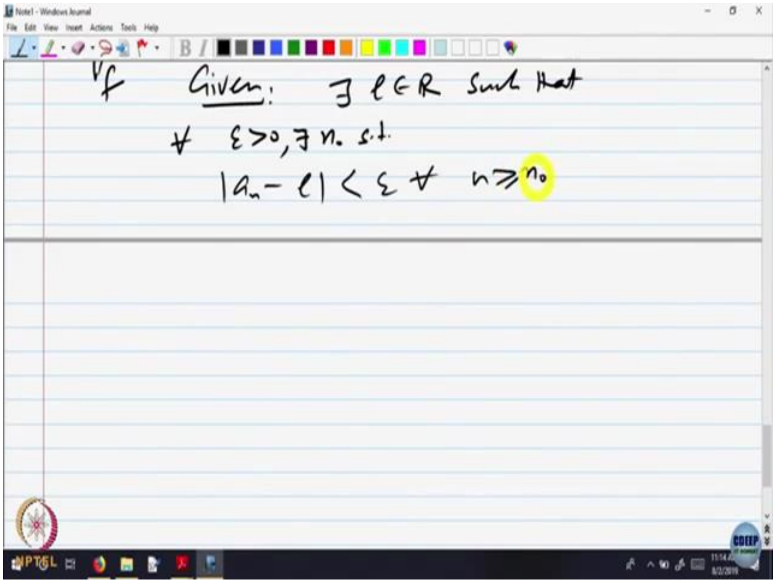

So, let us make it precise that is sequence a n, so, we will say a sequence a n is said to be bounded if there exist some alpha belonging to R such that mod of a n is less than alpha for every n, bigger than or equal to 1. All the terms lie between some barriers, okay, so that is bounded, so this is called bounded.

So, we should have a theorem based on our experience of earlier results, that if a n is convergent then it is also bounded, so let us prove. So, what we want to do? We are given it is convergent, so, what is given? There exist number L belonging to R such that for every epsilon bigger than 0, there exists some n naught such that mod of a n minus L is less than epsilon for every n bigger than n naught, that is convergence.

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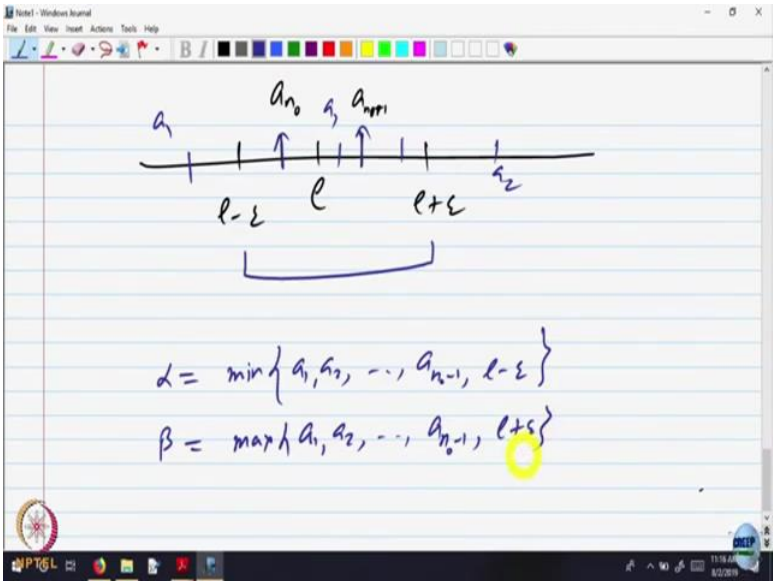

So, let us try to draw a picture of this, this is my L, this is L minus epsilon and this is L plus epsilon. And what is this condition saying, that a n lies here, a n naught lies here, a n naught plus 1 that lies somewhere here, so let us just probably, this is this, this point, this point and so on.

So, for every n bigger than n naught everything else is, so, what is outside this interval? What terms of the sequence are outside, possibly possibly we do not know some of them may be inside, a1 maybe inside, it may be outside a2 and so on up to a n naught minus 1 they can be out they can be in.

So, maybe a1 is here, a2 is here, a3 is here and so on. So, in any case, there are only finitely many of them which are outside. So, let us look at the smallest of a1, a2 up to a n naught minus 1 and L minus epsilon. There are only finitely many. So, let us write alpha equal to minimum of a1, a2, a n naught minus 1, these possibly can be out. May be on the left, may be on the right and but remaining all are inside L minus, they are bigger than L minus epsilon.

So, let us call this alpha and similarly look at on the right side some of them can go beyond L plus epsilon, only finitely many of them. So, let us pick up the largest of them, so let us call it beta equal to maximum of a1, a 2, a n naught minus 1 and L plus epsilon.

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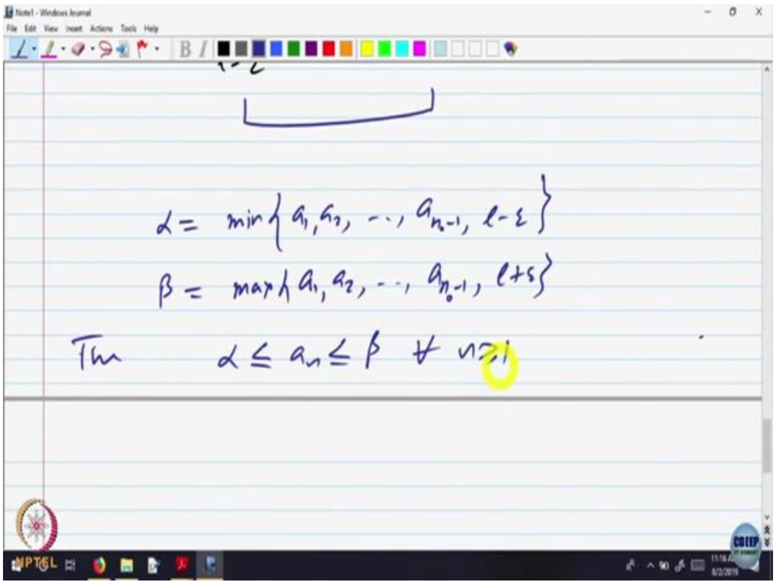

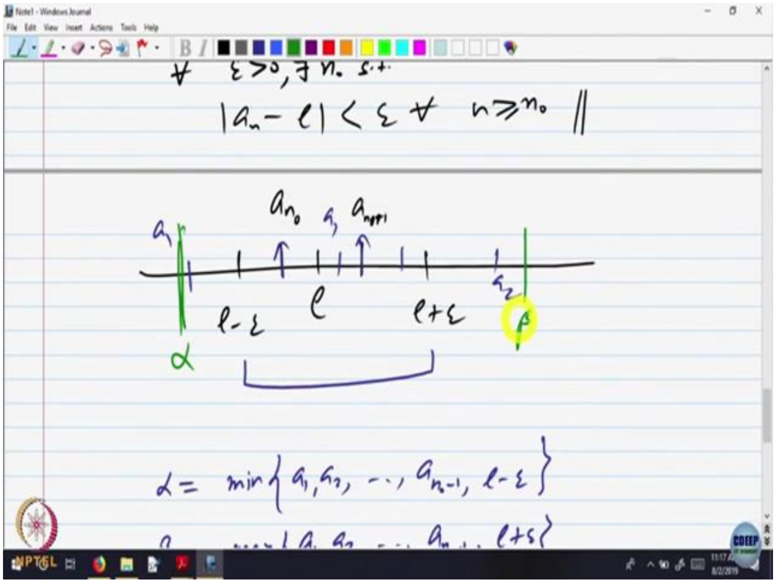

So, then what should happen? So, then alpha is less than or equal to a n is less than or beta for every n bigger than n1. Is that okay? So, everything else bigger than n naught is inside the finitely many which can go out or in we have taken the smallest and the largest. So, in the picture if you look at it, so, this will look like here is probably my alpha and here is something like beta, alright, and what is alpha? Alpha is the minimum and beta is the maximum of this finite number of things, whichever is the largest pick up that.

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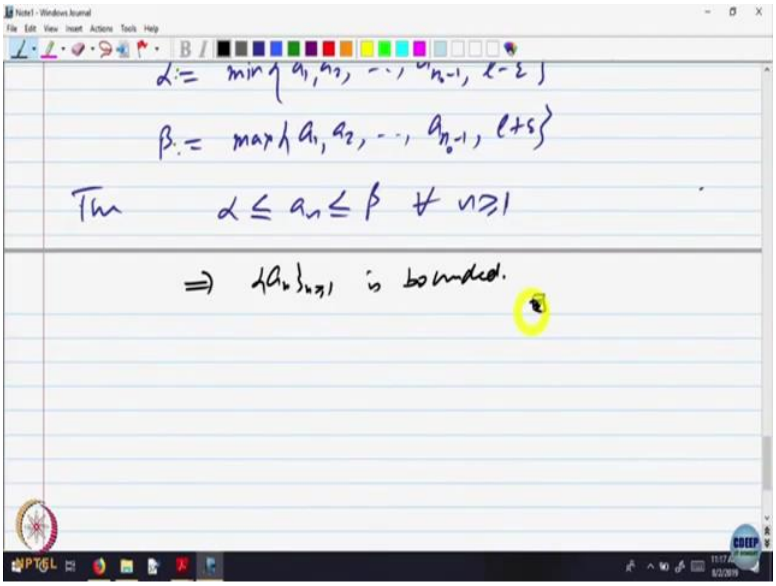

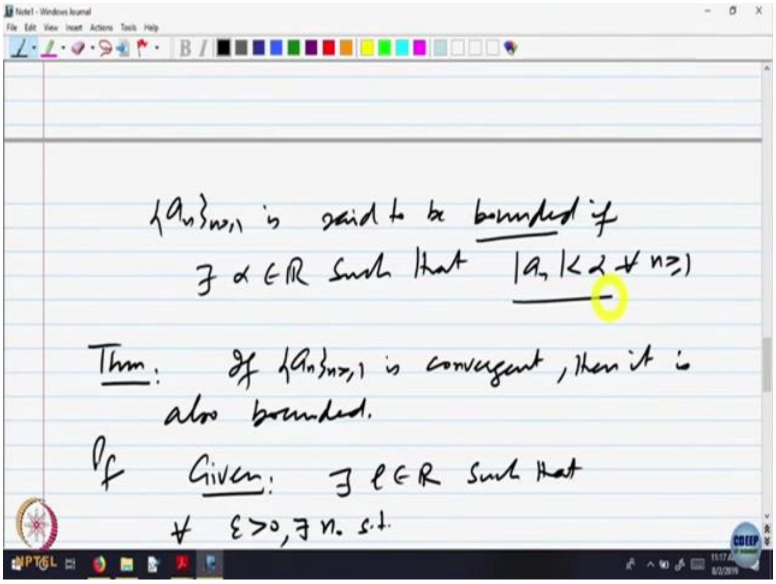

So, that implies. So, this implies a n is, is it clear that why is this implies a n is bounded? See, our definition was something slightly different. We said bounded when mod of a n is less than, but here I am saying a n is between alpha and beta, but that does not matter. I can always have a number if you like such that alpha and beta lie between minus of that number and plus of that number.

So, mod of the a n will be between, is it okay? Clear? Or you can pick up whichever alpha is, mod alpha is bigger or mod beta is bigger then all a n will be less, mod a n will be less than or equal to that mod of that number. So, both are equivalent ways of saying.

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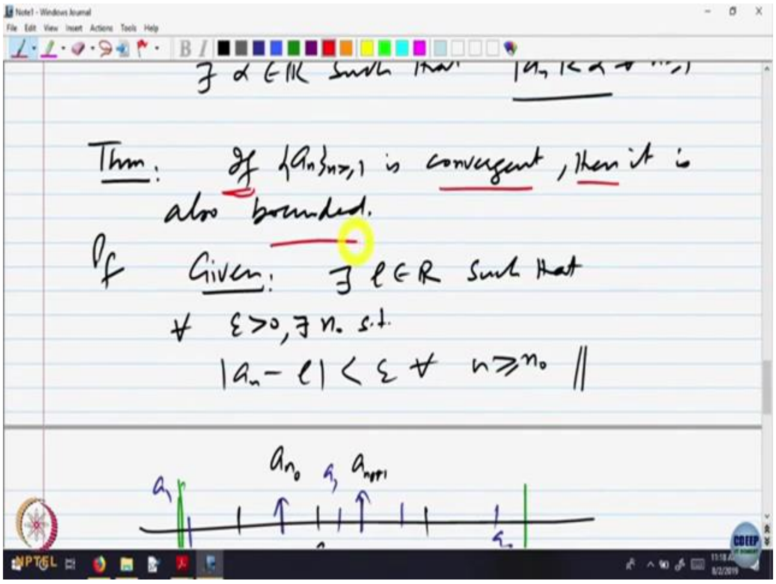

Saying that a n bounded is this or all the a n lies between some alpha and beta, the two statements are equivalent ways of saying a n is bounded, okay. So, what we have said is, if see, now this is what is important is if is convergent, then it is bounded. That means boundedness is a necessary condition for a sequence to be convergent, boundedness is a necessary condition for a sequence to be convergent.

So, if I given a sequence, the first thing I should look at is whether it is bounded or not. If it is not bounded, then it is not going to be convergent, so problem is over. If it is bounded, then I will go ahead. So, all necessary conditions are useful in verifying something is not true. If sequence is not bounded, then it cannot be convergent, so, that is the reason. In your calculus you might have come across a statement probably will prove it again later on, that you had done calculus local maxima, local minima, of functions.

If a function is differentiable at a point and that point is a local maxima, then derivative must be equal to 0 that is also a necessary condition. That for a differentiable function if it is a point of local maxima or minima the derivative is 0, that means, at that point the tangent should be horizontal.

So, that is why when you want to analyze local maxima or minima of functions, what you do? You first look at derivative, if possible derivative equal to 0. That is why that theorem comes and those are possible points where local maxima, minima can occur.

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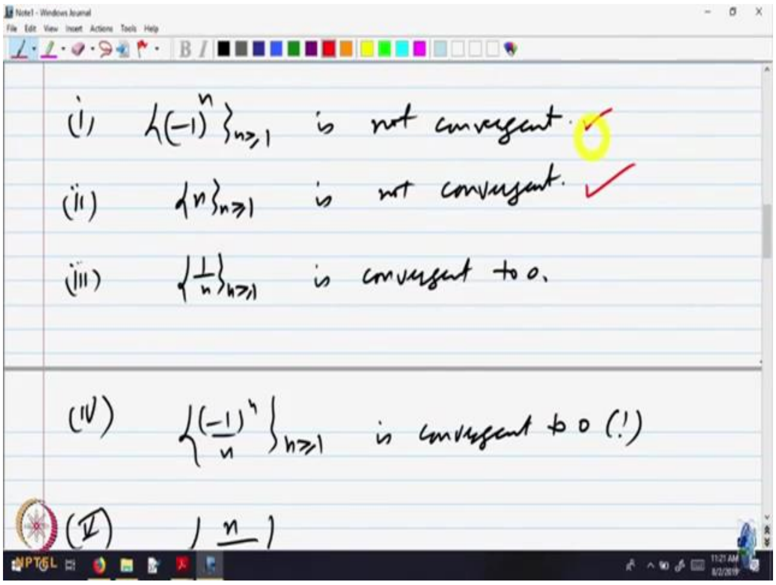

So, necessary conditions are useful things even though they give something negative it not true. So, same way, a n is convergent, implies a n is bounded. So, not bounded will mean it is not convergent. So, that could be the reason for example, you can look at that sequence, this is not convergent, because it is not bounded. You can say that way or. So, but this one is bounded but not convergent, so, converse of the statement is not true. A sequence may be bounded, but may not be convergent and this is a obvious example of that, okay.

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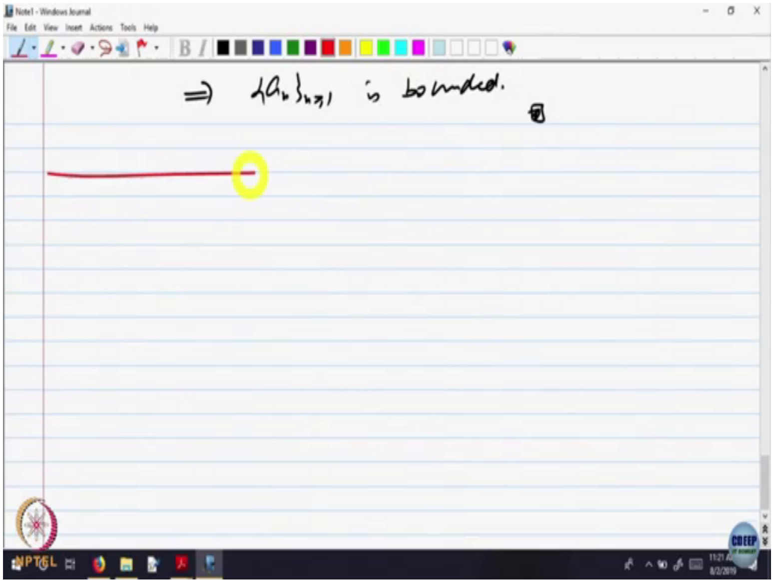

So, let us keep that in mind and go ahead. Now, how does one try to analyze whether a sequence is convergent or not? Are there any tools, one is obviously that if is not bounded not convergent, so, that is not one of them. Other possible tools are what are called algebra of limits of sequences, we will not prove all of them, we will just indicate. So, that helps in analyzing limits.

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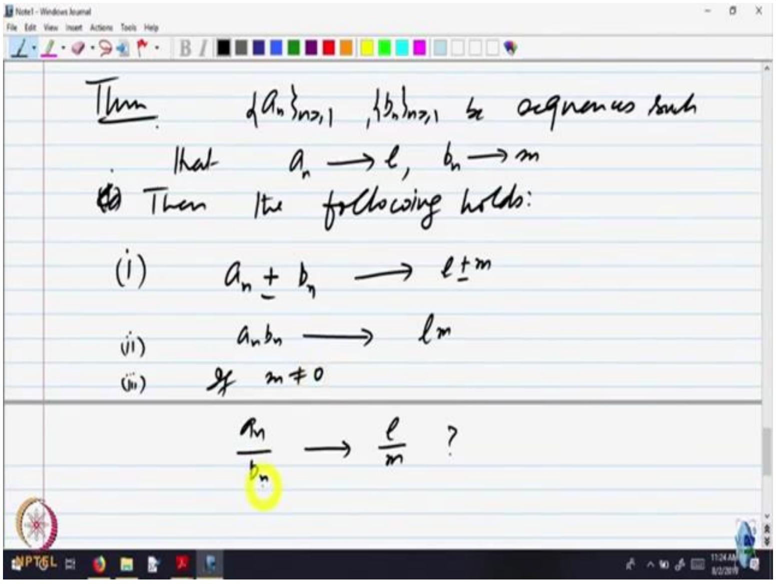

So, so let us write theorem, let a n, b n be sequences. So, okay, such that let us write the conditions here also such that most sometimes I will just write is equal this way b n converges to m. So, this means the sequence a n is convergent and the limit is L, sequence b n is convergent and the limit is something m. Then the following holds, one, see given a sequence a n, that is a number, b n is a number. So, you can add and nth terms of the two sequences.

So, you can form a new sequence, which is a n plus b n, so that is a new sequence. So, the theorem says if a n converges to L, b n converges to m, then this must converge to L plus m. So, this is algebra, we are adding sequences. If we are adding sequences algebra says you should be able to subtract also. So, minus that also holds, okay. Can I multiply? Yes, why not, a n, b n, nth terms we multiply.

So, let us follow our notation that this one converges to L and we have written it as L. So, a product of limits is, limit of the product is equal to product of the limits, you can interchange the two. Limit of a product of sequences is nothing but the product of the limits provided both converge. Difference subtraction all are okay, you can okay. Let us look at, what about division? I want to say that a n divided by b n converges to L by m. I want to say that, not possible?

Always, because obvious thing can go wrong m could be 0. So, L by m does not make sense. So, let us write condition, if m is not equal to 0, at least then L by m makes sense, okay. Now, what if m is not 0, what can you say about b n? Because I am going to define a n by b n, for that also all should also be not 0. Or at least from some stage onwards, they should not be 0, because I am looking at the convergence, so I am looking at only at the tail of the sequence. So, I should be able to say that b n are not 0, if m is not 0 from some stage onwards, can I say that? Yes, Paul, we can do that.

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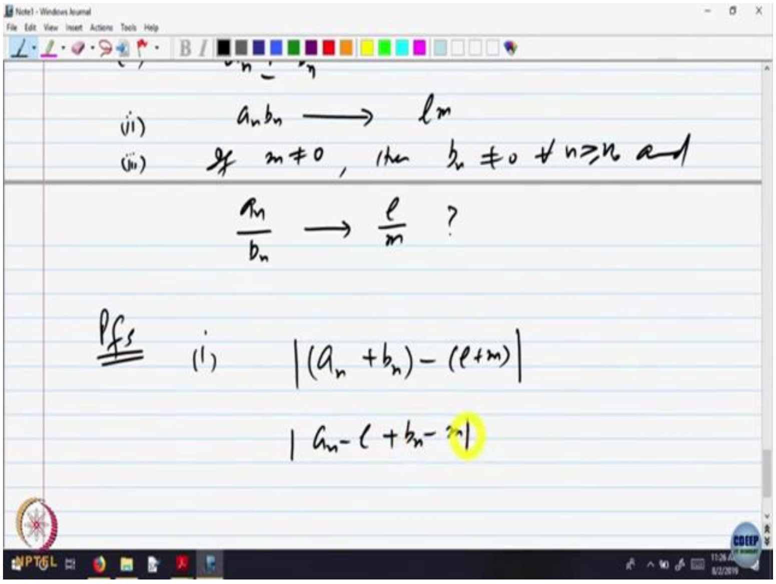

Then let us write b n is not 0 for every n bigger than n naught. And so I will indicate why it is so because that is an important thing. I will just indicate the proofs. And that will help you to understand why and how we are doing things. So, first one, I want to look at a n plus b n. I know a n converges to L, b n converges to m. So, this, sum minus L plus m, I should be able to make it small. But what do I know? I know something about a n minus L, I know something about b n minus m. So, I should separate them out, if possible, somehow or the other.

So, this is what you want to reach somewhere, keep the target in mind and somehow you have to go in that direction, that is how problem solving and writing a proof means. Not by hook or crook by logical reasoning, okay. So, this one, I can try to write at least a n minus L plus b n minus m, and then these two I know that I can say something about them. And now here comes the triangle inequality. It says this is less than or equal to plus b n minus m. So, that makes everything work, bn with the negative sign same thing will work.

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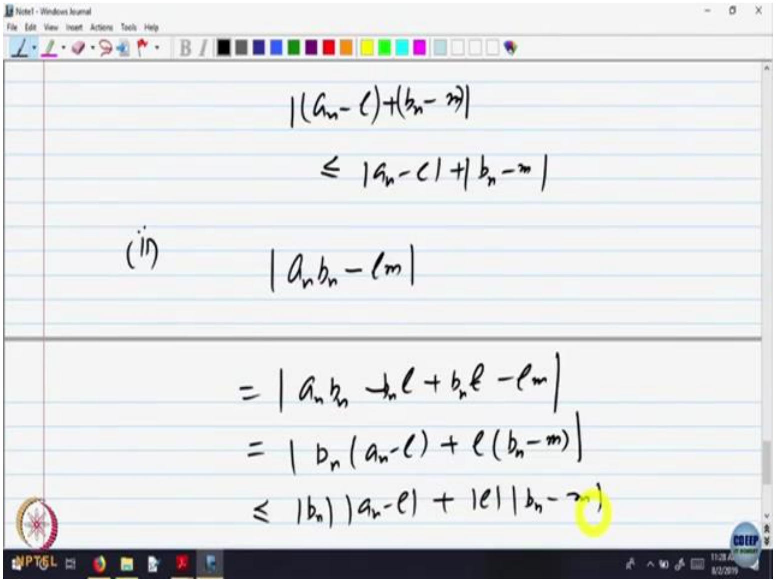

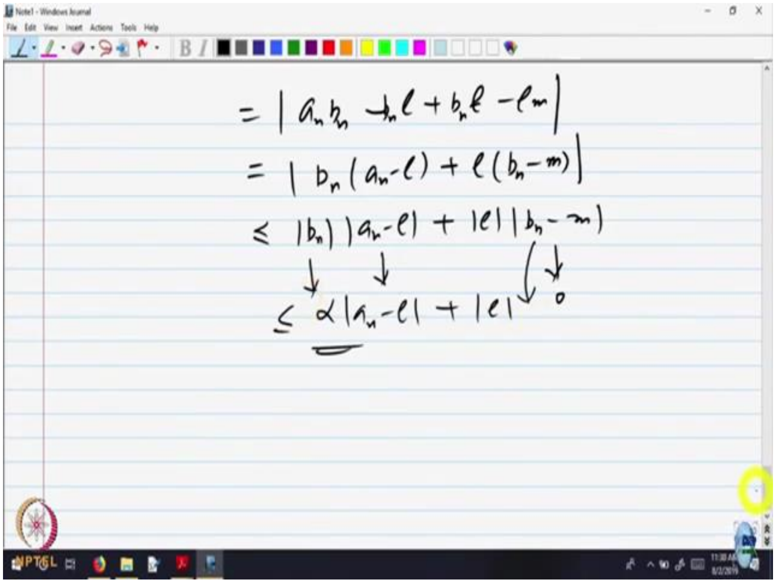

Let us look at product, what is happening about the product? I am just trying to indicate how the proof should go. I have to look at a n, b n minus L m. Now, once again I know only that mod of a n minus L can be small, mod of the b n minus m can be small. So, how do I separate out those things now, here a n is multiplied by b n. So, I have to do some kind of manipulation to separate out.

So, let us write this is equal to a n, b n, I want a n minus L. So, let us add b n minus L and add that also, add and subtract n minus L m, so I have not change anything. Only I have added and subtracted this because then I can take out b n out and I will have something a n minus L. So, my target is in mind, what is it? So, this is b n, a n minus L plus L into b n minus m.

Now, at least I can separate out and worry about it. So, mod of b n, mod of a n minus L plus mod of L into mod of b n minus m. So, at least I have separated out.

It is something like if you have a 200 rupee note and you want to pay somebody 10 rupees you have to get a change, you should have that denomination with you otherwise you cannot pay. Same way I want to use mod a n minus L, mod b n minus L. Somehow I have to bring in those things in my analysis.

So, now, I know that this will become smaller. In intuitively this goes to 0 this goes to 0, multiply something with constant and that is going to 0 so that will also will go to 0. Is it okay for everybody? Now, here is a problem this b n may not go to 0 or b n is not a constant. As n changes, I know b n multiplied by something is 0, but this b n is not 0. Yes, so here at times, that if I can say it remains between a constant, it does not blow up, it does not become infinite kind of a thing.

So, boundedness, so it is less than or equal to some bound, so let us call it is alpha times a n minus L, where here I am using the fact, the remaining thing. So, here one is using that fact that if a sequence is convergent, then it is bounded and everything is coming out naturally that requirement is coming keeping what I have in mind and what were I want to reach. So, this will go. So, this will be the second part proof essentially.

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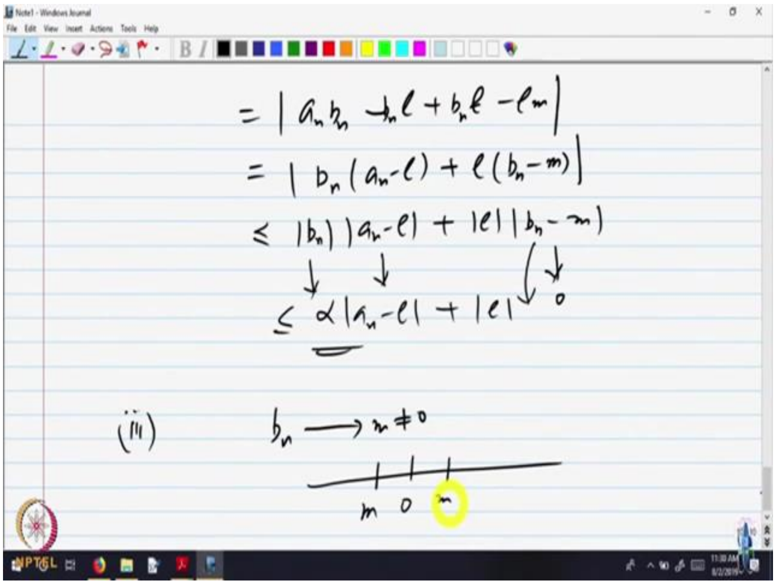

Let us look at the third part. So, we are saying that b n converges to m which is not equal to 0. So, let us draw a picture, here is 0 and limit is either here or here, either m is here or m is here on the left or on the, we do not know where, positive or negative, does not matter, but we know that b m is going to converge. So, it is going to come closer to the limit eventually, after some stage onwards, the tail should be near the limit. So, and I want to avoid.

So, let us take this neighborhood if it is on right hand side. So, take a neighborhood that means what? m minus epsilon and m plus. So, choose epsilon such that m minus epsilon is strictly bigger than 0, I can do that because convergence says for every epsilon something happens. So, a tail will come inside now, a tail will come inside now here. That means, all the b n after some stage will be bigger than 0, if m is bigger than 0, is it clear?

So, that is a idea of limit, limit is not evaluating at a point it says what is a behavior of the tail of something. So, that is precisely being used here, that if the limit is positive, then the tail of the sequence will be positive, for some tail and if negative then the some tail similarly you can go on inside also. So, a n by b n will be defined and then again you have to manipulate a n by b n minus L by m, LCM and do some (())(31:12) things, arrange those things, okay. So, do that.

By the way, as far as the examination is concerned, this will not be asked, the proof, okay? But try to understand what is happening, try to write out a proof yourself that will help you to understand mathematics behind this, okay, do not worry about exams too much, try to understand things.

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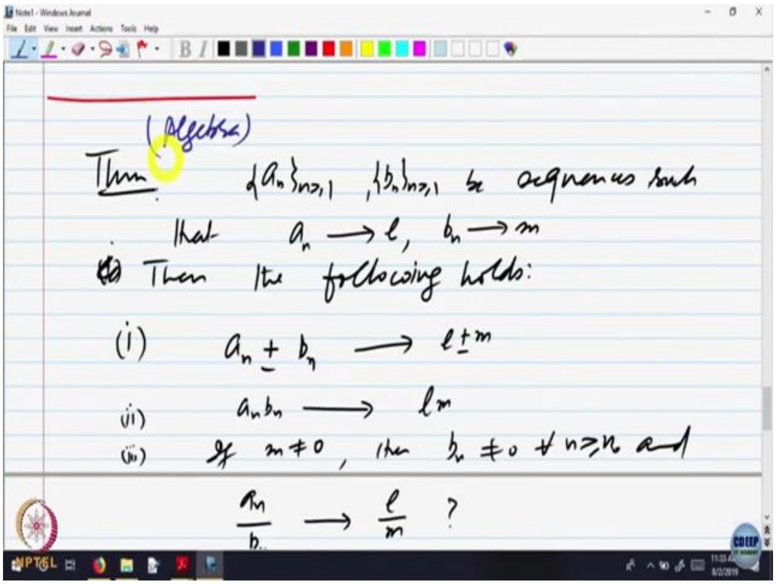

So, what we are saying is, here are the, what this is called the Algebra of Limits. Why Algebra? Because I am adding, I am subtracting, I am multiplying, I am dividing that is what Algebra is. So, it says essentially that, so this can, this helps now. How does it help? It helps in analyzing limits of sequences, because given is a n, I can break it into probably smaller parts, which are either addition of something or multiplication of something or like that. For example, let us write some example of this why it is useful.