Basic Real Analysis

Professor Inder. K. Rana
Department of mathematics
Indian Institute of Technology Bombay

Lecture 5

Convergence of Sequences – Part II

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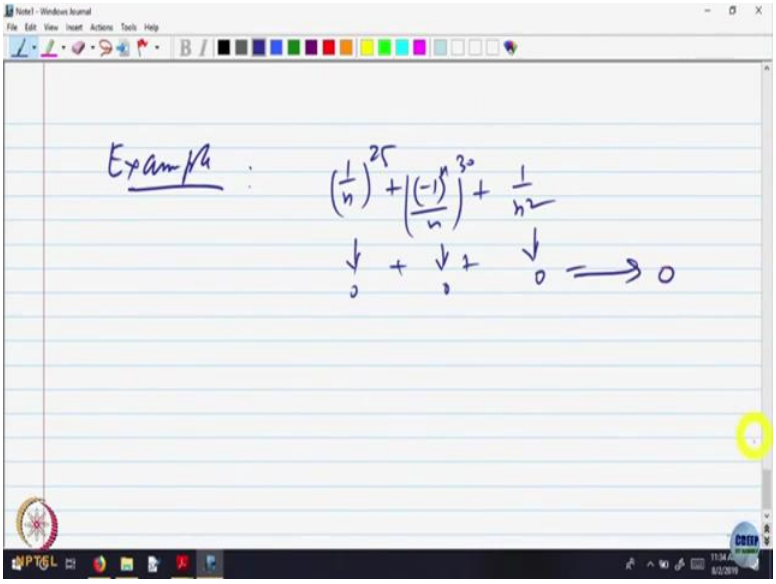

Let us look at say you can write anything. So, let us look at 1 over n raised to power 25 plus something minus 1 by n divide by n, I am just taking some random kind of thing. Let us look at the sequence, I want to know whether it is convergent or not and what is the limit if it is convergent?

I know that 1 over n goes to 0, so it is 1 over n multiplied by 1 over n, so that goes to 0. Minus 1 to the power n divided by n that goes to 0, go multiply it 30 times does not matter, this goes to 0 and this also goes to 0. So, this plus this plus this, so that should go to 0. So, this is a kind of thing you can do, I am just kind of cooking up a example but you will see later on these are useful theorems.

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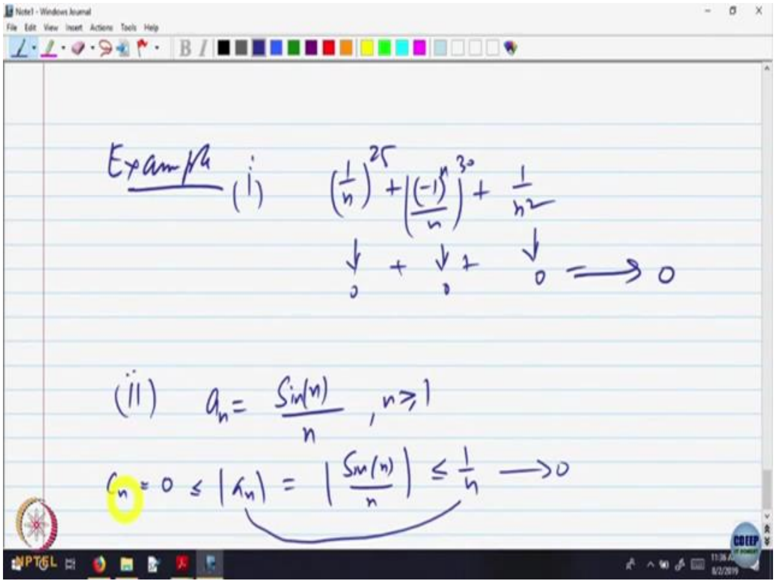

Okay, so one more tool which helps so let me write one, so let me write the example first and then write the, let us write the sequence a n to be equal to sin n divided by, so let us write n, so (())(1:57) sequence a n. What can I say about the denominator? If I do not have sin, only denominator, then 1 over n that goes to 0. Sin n. So, there is something it does not go somewhere, but it does not go anywhere, it remains between something.

So, one writes this as, mod of a n is equal to mod of sin n. I do not know what is sin n, I am not interested also this is less than or equal to 1 over n. Because sin function is bounded by, so, it is something like saying if I have a sequence a n which is bounded and b n which is convergent, then the product will be convergent. So, essentially it says that this goes to 0. So, but what is happening is a n is dominated by 1 over n and this is always bigger than or equal to 0. You can think of left hand side as a constant sequence.

So, this is a sequence c n, where every term is 0. Is a constant sequence convergent? Obviously, because it does not go anywhere with stays put there, so, for every epsilon, all the terms are near, so everything will work out nicely, no problems, constant sequences are always convergent to the term which is a constraint. So, now this is between two convergent things. So, it is kind of sandwiched between two.

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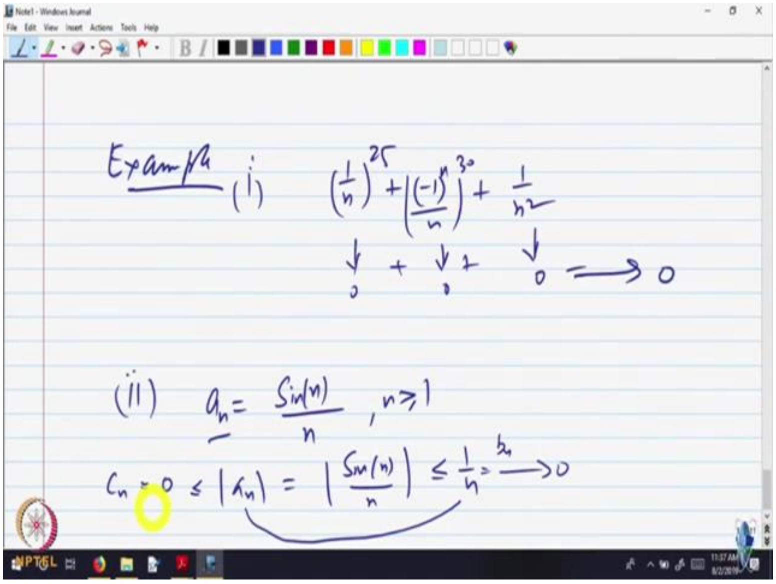

So, the sequence which we have a n is sandwiched between the called it as b n, b n and c n and both b n and c n are converging to 0. So, that indicates something more general. So, this limit of course is equal to 0.

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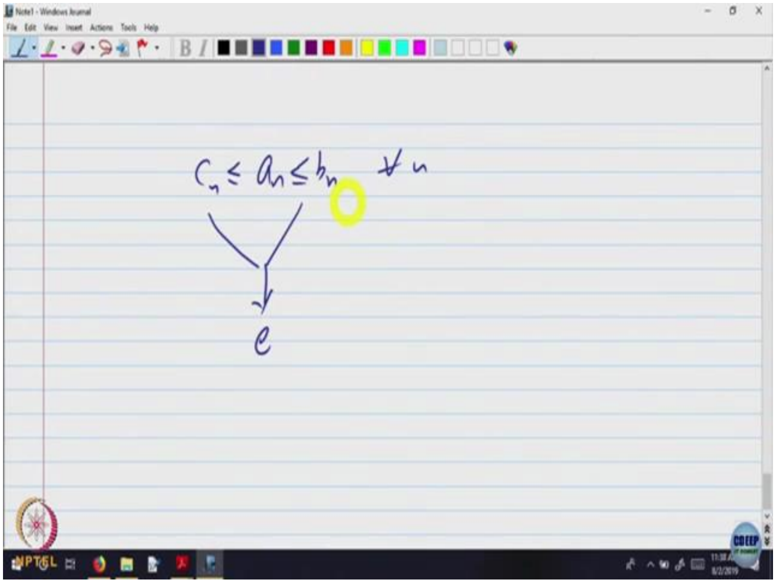

And that probably says, supposing a n is less than or equal to b n is less or equal to c n or or n. Suppose you got 3 sequences a n. b n and c n, I am trying to model on that example. And suppose this and this both of them converge to a limit L. So, convergence means what? b n and c n are going to come closer to, c n is going to come closer to L, b n is going to come closer to L, that means b n and c n are both coming closer to each other and poor a n is sandwiched in between. So, it has to come closer to L. So, the theorem should e.

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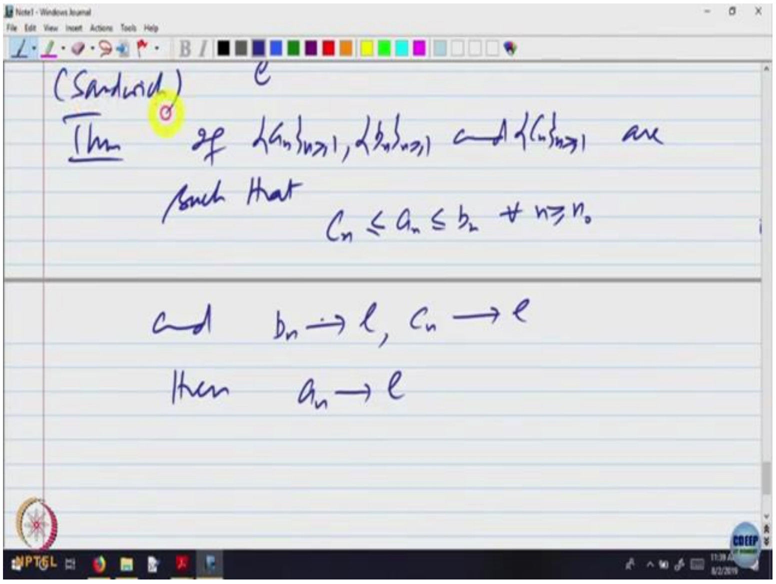

If, so, let me write a theorem. If a n, b n and c n are such that sorry, I wrote c n there, so, let me keep that, such that c n is less than or equal to a n is less than or equal to b n and for every, I do not need for every n because n it has only in the convergence. So, we should write for every n bigger than some n naught, where tail is important, is not what is the thing actually.

Is n naught and b n converges to L, c n converges to L, then a n converges to L. So, this is what is called the sandwich theorem. If something is sandwiched between convergent things, so that has to converge.

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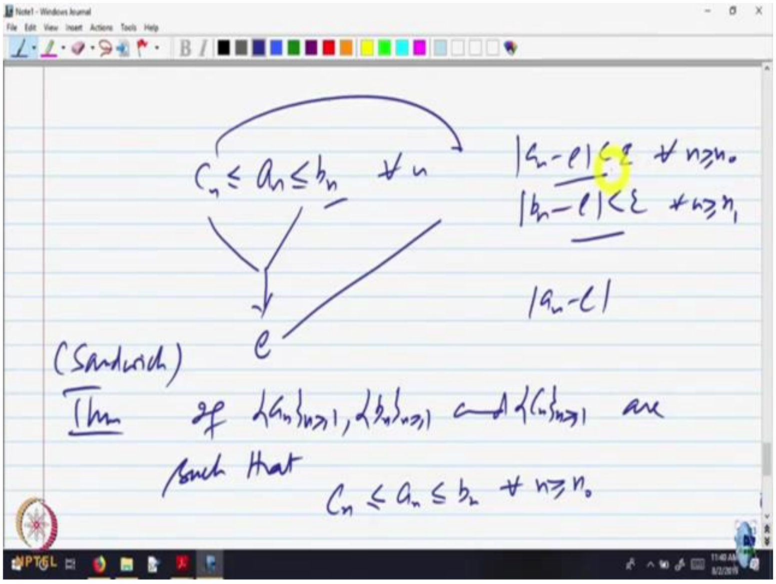

So, shall I write a proof, or you will try to write a proof yourself. I have given you the idea. So, here is the idea. So, please write down the proof yourself and here is the idea, a n converging to L that means given some epsilon there is a stage after which mod c n minus mod c n minus L, absolute value is less than. So, the first this one, this one will give me mod of c n minus L less than epsilon and the second one will give me mod of b n minus L less than epsilon.

For a given epsilon, there is a stage such that this for every n, maybe some other stage n 1, does not matter. But if I look at the stage bigger than these two stages then for that tale this also happens, this also happens, then what happens to, what can I say about, because that is what I want to analyze. Can I say a n minus L will be less than for that stage onwards, whatever is the bigger one, is that okay?

So, this kind of manipulations one has to do, right, in analysis. That means, you have to change your estimates a bit depending on your requirement. So, something happens for one stage, something happens for another stage, if you go beyond that, then what will happen? So, we have got both available to you.

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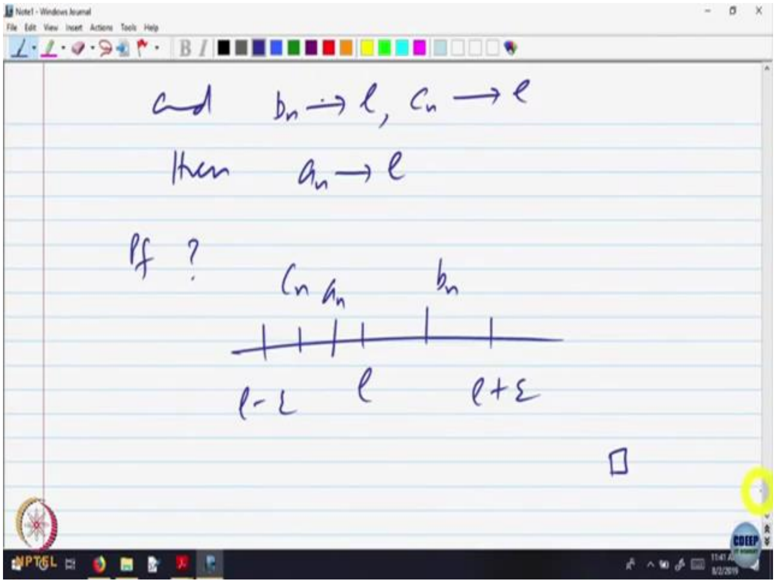

So, write down see if you like the picture here is L, here is L minus epsilon, here is L plus epsilon. Where is after some stage c n is here, after some stage b n is here, and after both the stages both are inside and where a n? a n is here, so, a n is inside L minus epsilon and L plus epsilon, it is over. So, write it in theorems of mathematics, okay. So, this is what is called sandwich theorem. So, in this example that we have done, you can say that this is by sandwich theorem.

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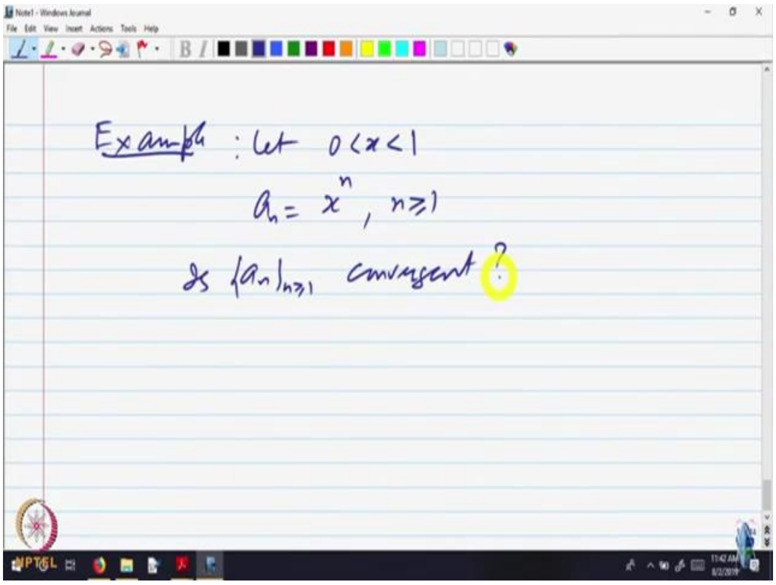

So, let us do one more example to illustrate the usefulness of these things. So, let us look at an example. Okay. So, let us look at, let us look at a number x between 0 and 1. So, let x be between 0 and 1. Okay, let us look at a n which is equal to x to the power n. So, question, is a n convergent? Is a n convergent?

So what we will do, as usual, try to inspect the sequence at least first and this is what George Polya said. There was a mathematician called George Polya, if you have not heard the word, go and Google, George Polya and he has written a very marvelous book called, how to solve it? Which is available on the internet free at least I could get one copy, but it is not very costly Indian edition is available.

So, read it, it says, you want to prove something for every x. He says one thing is try to look at particular cases of that problem. Step one, what is the particular case? For example, I can look at x equal to 1 by 2, 1 by 3, 1 by 5 something like that, less than 1, bigger than 0 and look at what happens to the sequence. It looks like you may take 1 by 2, 1 by 3 or 1 by 4 and go on multiplying them again and again, it seems to be becoming smaller and smaller.

So, there is a pattern in the examples that we are saying and that seems to indicate that this should go to 0. So, that is Polya's suggestion. Look at particular cases that may give you a hint what is true and how to prove it possibly, or for a particular case you may get that is not true. Even then you should be happy because you have solved the problem saying that for a general case it does not have a solution, you are a counter example, so, even then that is good. So, read that book if any one you find time.

So, claim whatever I have said that a n goes to 0. But is x to the power n. So, I should try to now do something, so that I can estimate x to the power n for n large enough. Showing it goes to 0 that means, I should be able to say x to the power n becomes smaller and smaller as n becomes larger and but in a conclusive way not by examples.

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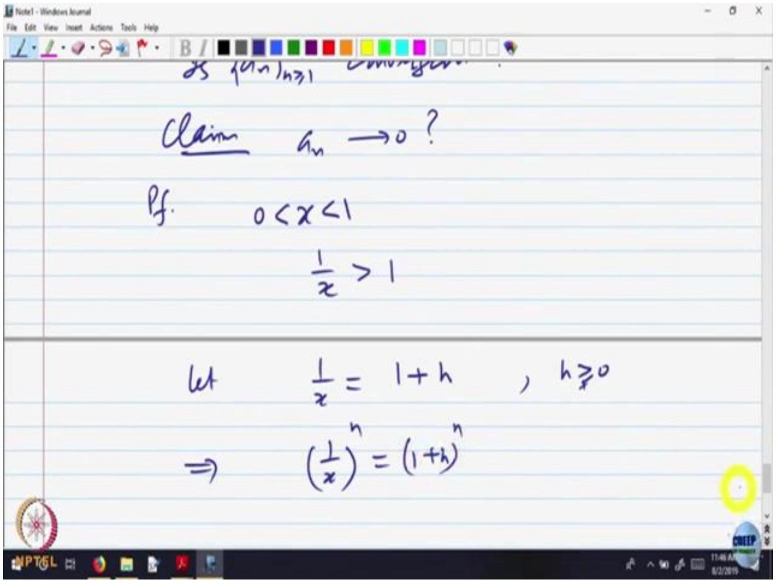

So, let us look at if so, claim so, let us write a proof of that, x is between 0 and 1 then what can you say about 1 over x? That will be bigger than 1. So, let us write, let 1 over x, it is 1 plus something. It is bigger than 1, certainly 1 plus something, some h bigger than or equal to strictly bigger than 0. Okay.

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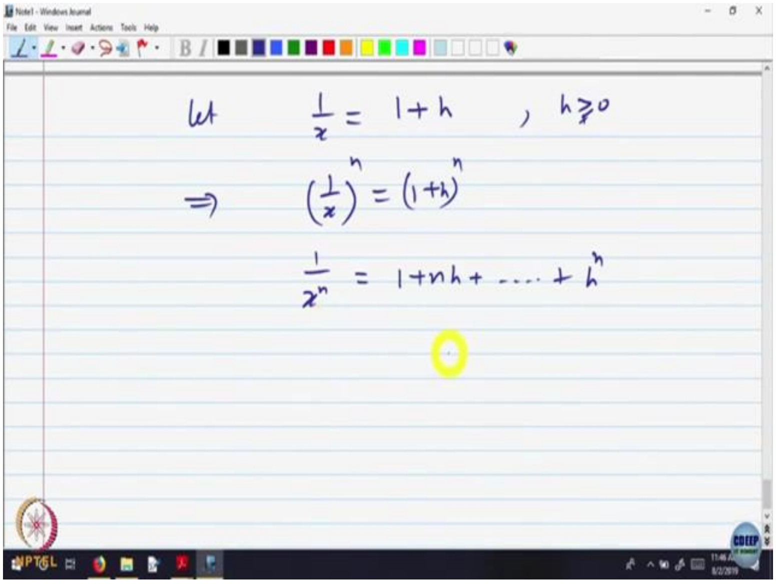

Now, what I am interested in I am not interested in x, I am interested in power of x. So, let us raise the power, so implies 1 over x raised to power n is 1 plus h, raised to power n. So, the left hand side is 1 over x to the power n is equal to, now that I can expand now using my familiar binomial theorem, so 1 plus n h plus h raised to power n. It is 1 over of the required thing. If I make 1 over of x n, which is my required thing less than something, then, so, that will or bigger than something that will give me an estimate.

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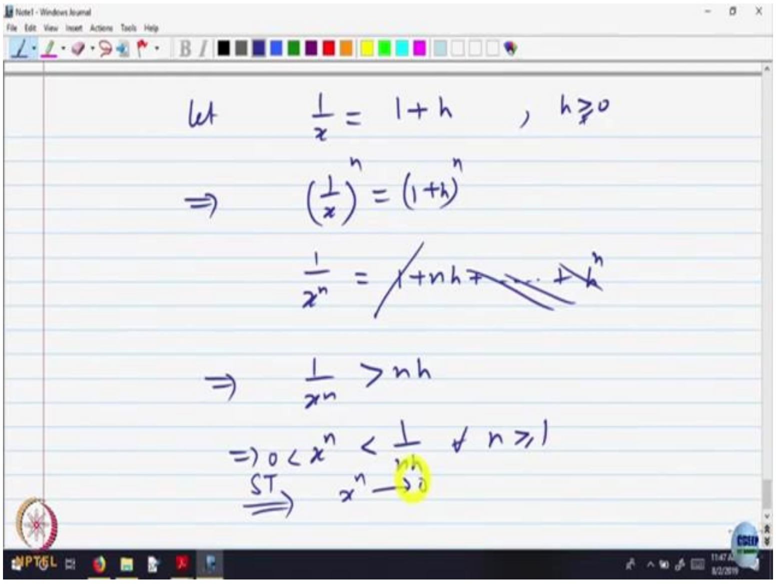

So, now if I forget this one, and I forget all the remaining ones, what will happen? So, this will imply 1 over x n is bigger than n h. Okay, because everything is non-negative, I am forgetting non-negative quantities keeping only one term. And that implies x to the power n is less than 1 over n times h for every n, h is fixed, is a constant.

So, that says, so, this is bigger than 0. So, implies by sandwich theorem, that x n goes to 0, that says this must goes to 0. So, sandwich theorem is useful in this kind of a thing. But you have to make a guess, then try to estimate it by something known possibly. So, this is where our analysis comes into picture. That is why we are doing real analysis, okay.

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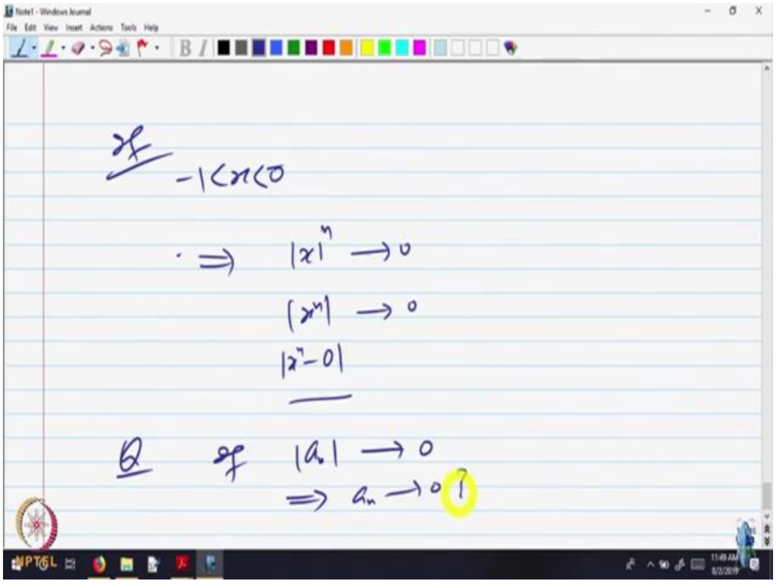

Now, here is once you this is done, can you say something about if x is not positive, but it is x x between minus 1 is less than x less than 0. Can I use my analysis above to say something about this now, is a negative quantity, but still less than, absolute value is less than 1. So, at least what I can say is if this is so, then I can say mod x to the power n goes to or this is same as x to the power n mod, is it okay, both are same? Yes.

So, I have got a sequence whose absolute value is going to 0? Can you see the sequence goes to 0? If you like this itself anyway, it goes to 0, this can be very small, this also proves. But the question is, if mod a n goes to 0, does it imply a n goes to 0? So, keep it as a question, think about it, okay. And the problem sessions try to analyze this question if it is not there in the problem sheet, try to analyze this question.

You can analyze both ways if a n goes to 0 does mod a n go to 0 and converse, if mod a n goes to 0, does a n go to 0? And you can even improve your question, if a n goes to some L, does mod a n goes to mod L and is the converse true? So, you can generate your own questions and think and analyze that is how you progress in solving.

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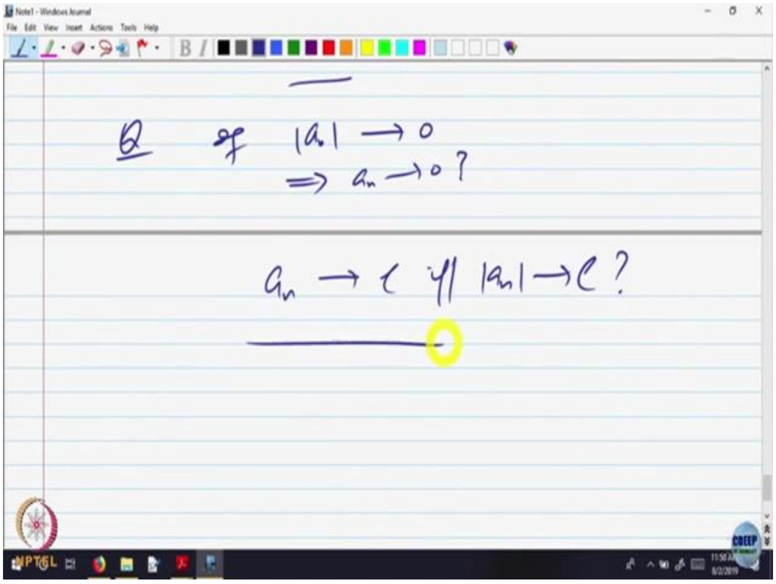

So, you can try to analyze another one more question, that a n converges to L if and only if mod a n converges to L, I do not know. So, you have to analyze this for L equal to 0, I am saying it is true, for L equal to not equal to 0 maybe it is still true, but you have to analyze that. So, you want to prove give a argument. If you want to disprove then you have to a counter example for anyone. So, it is the two way statements, if an only if, okay.

So, what we are done is, till now we have looked at convergence of a sequence, we have tried to analyze possible ways it can converge or diverge. It can diverge, if it is not coming closer to a value that is divergence. It is unbounded or it fluctuates possibly. If convergent, it must be bounded. So, not bounded will imply not convergent and for analyzing convergence of sequences we have got some tools, algebra off limits, sandwich theorem. Till now any doubt? No.

So, let us go ahead a bit more look at something, okay. There is a some (pat) till now we have not analyze that example, when we said they will look at try to find out the area of the unit circle, that approximations, by inscribed and gones and making the sides bigger and larger and larger. They said eventually it should give me area of the circle. Still now we do not have any mathematical tool which says yes, you can do something. So, let us try to formalize that.

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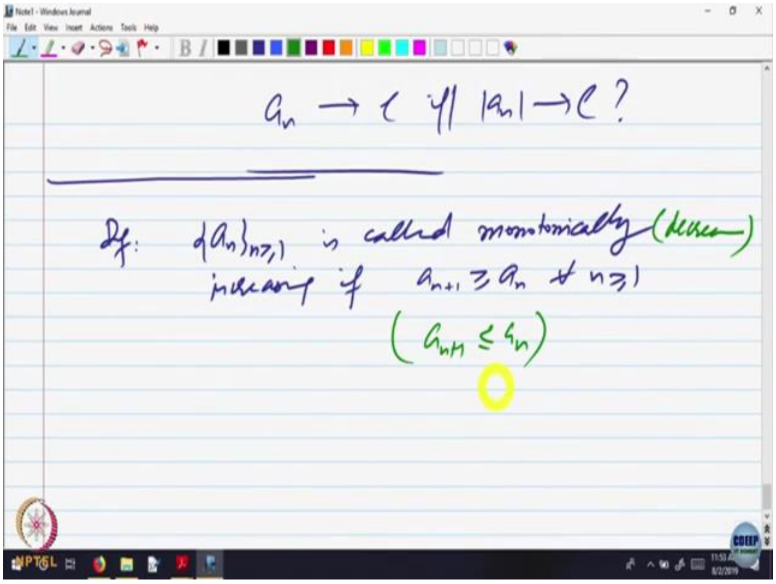

So, let us define, what is called a increasing and a decreasing sequence? Which is obvious, in fact you maybe knowing. So, a sequence a n is called monotonically increasing if a n increasing. So, next one is bigger than a n plus 1 is bigger than or equal to a n for every n bigger than or equal to 1. And similarly, you can write decreasing, so, what will be the condition a n plus 1 is less than or equal to a n. Two parallel definitions increasing every next term is bigger than or equal to the previous one decreasing and so.

So, here is here, pardon, may be, I am just saying less than or equal to, bigger than or equal to if you want to say no if you do not I am not happy with this, then you will add the word strictly monotonically increasing or strictly monotonically decreasing, if I do not say that some terms could be equal, tenth and eleventh may be equal, but then the other ones may be less than, possible.

So, monotonically increasing means a n plus 1 is bigger than or equal to. Strictly will mean this inequality is strict, that is all, is it clear? There is only one way you can interpret a statement here, is a mathematical statement. Okay? You can say that by this definition, yes, okay, no problem, what is wrong with that? A constant sequence is both monotonically increasing and decreasing, what is wrong with that? It causes a problem to you in your mind and it will clear cobwebs in your mind probably, okay.

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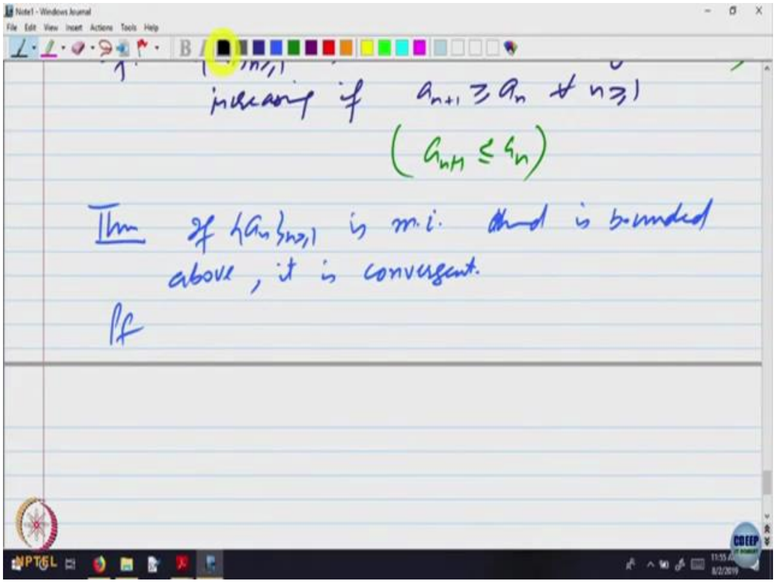

Okay, so here is a theorem I want to prove, which says that, if a n is I will just write m i okay? Mathematicians have a very good bad habit of shortening everything, okay, they do not want to write English at all. So, m i will mean monotonically increasing if the sequence then and is bounded above it is convergent, then it is convergent. So, let us look at a proof of this. Keep in mind real numbers is a complete ordered field that means it has the lub property.

Now, if so, let us see, how will the proof you think will go? If a sequence is monotonically increasing, it could be a constant sequence does not go anywhere, then it is convergent anyway. We are trying to prove a monotonically increasing sequence is convergent.

So, let us assume that is not the case it is at some stages it may be constant, but then it will be increasing, a n plus 1 will be strictly bigger than a n at some places, but we know it is bounded, it cannot go beyond something it is bounded above. So, if I try to plot the terms of the sequence on the line, they will be somewhere on the line and there is a barrier.

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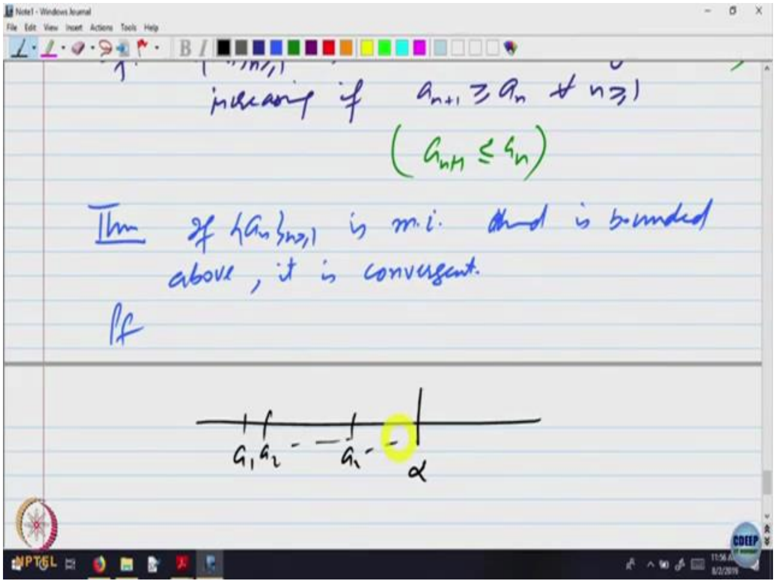

So, essentially what we are saying is here is a line here is a barrier say alpha and all the terms a 1, a 2 and a n, and they are all on the left side of it and going nearer and nearer. As it is as if you are walking to the wall and you are taking a step positively, you are not standing there. If you keep standing, you will never reach the wall, you must take some step, whichever smaller step you want. Then eventually you will reach somewhere, that is the claim that is what the theorem is saying.

Now, what do you think possibly could be the limit of monotonically increasing sequence if it is bounded? So, it should be the some kind of a largest value of the sequence, but a sequence may not have a largest value. So, there must be something which is the largest, all the terms are smaller, but they do not go beyond it, but they come closer to it. So, in some sense what we are saying is limit possibly is the least upper bound for that sequence, that is our guess.

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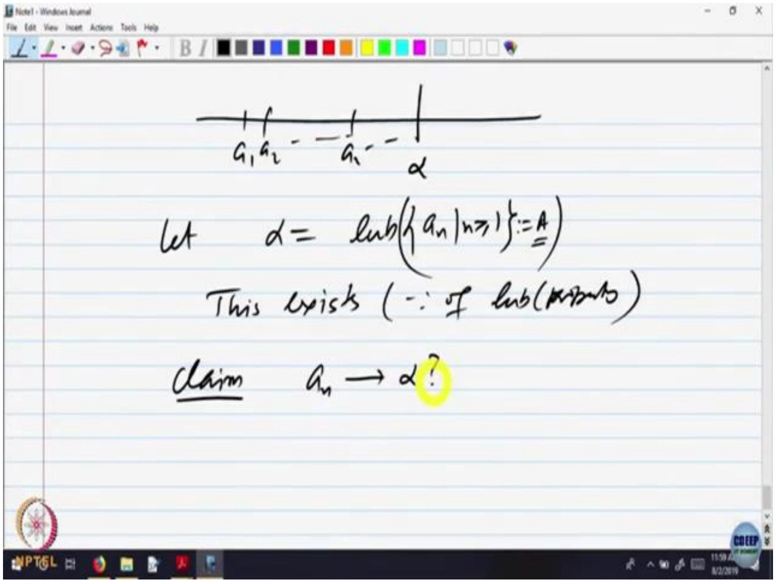

So, claim, so let us write claim, so, let alpha be equal to lub of the sequence, not as a sequence but sequence as a set, when you are plotting it on the line, you are taking it as a set. So, it is a n, n bigger than or equal to 1, I am looking at that set this is a set A, if you like. Define the set A to be the image of the sequence you can call it if you like at n, a n is a number, that is a real number.

So, collect all these real numbers, put them in a box, call that as a set A. That set A so has so this exists because of lub property. Because it is a bounded set, all the a n's are on the left of something. So, this is a set which is bounded below by anyway bounded below by 0 or below by a 1 or not that is not important. It is a set which is bounded above. It is a non-empty set, because a 1 is there, a 2 there all the terms are there.

So, this set A the image set of the sequence is a non-empty subset of real numbers which is bounded above. So, what should happen? It should have least upper bound. So, let us call that least upper bound as alpha. So, alpha is the least upper bound of that set, okay.

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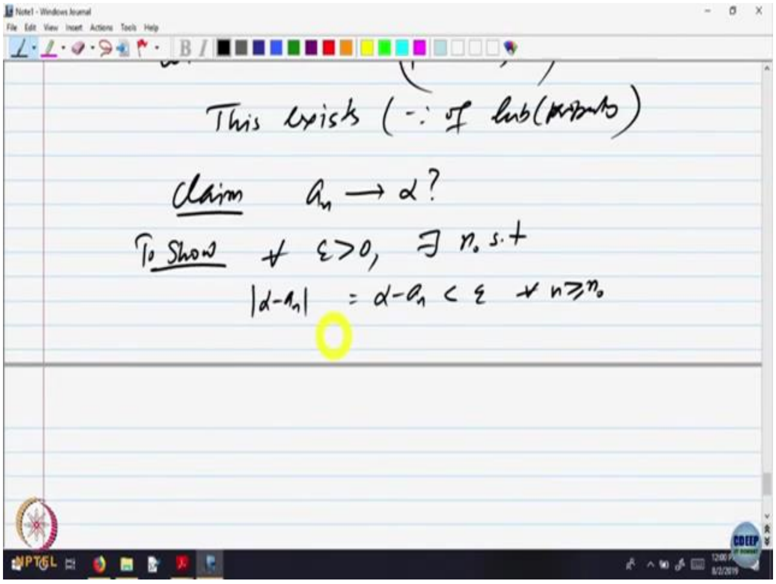

So, now, we want to claim that a n converges to alpha. So, what should I have to what I have to show? To show for every epsilon bigger than 0 there exists some n naught such that where is a n, alpha minus a n that is a non-negative number because alpha is least upper bound that is same as the mod is less than epsilon for every n bigger than that is what I have to show. Is that okay? If you like this is a same as mode of alpha minus a n, is it clear?

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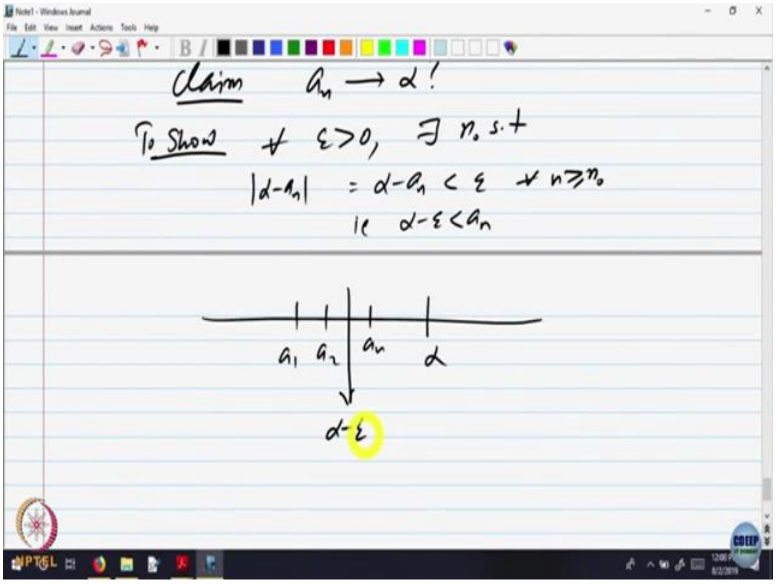

So, I am fixing my target now. So, here is the line, here is a 1, a 2, a n, and here is my alpha and I have given a epsilon. So, that is a same as saying, alpha is alpha minus epsilon is less than a n. That alpha minus a n less than epsilon is same as saying, I take a n on the other side and alpha on this side, is that okay? Yes, okay. Now here is alpha, so here is somewhere alpha minus epsilon, that will on the left side of it.

Now, can I say that alpha minus epsilon cannot be an upper bound for the sequence. Alpha is given to be an upper bound, least upper bound, if I subtract something from it that cannot be the least upper bound, nothing smaller can be, we already taken the least upper bound as alpha. So, if this is not least upper bound, that means what? There must be term of the sequence on the right side of this at least one.

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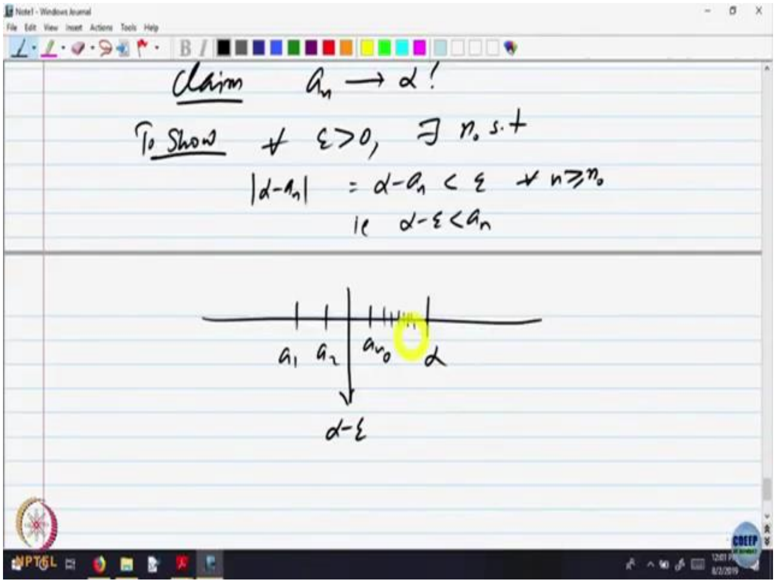

So, there must be some a n not which is on the right side of it, is it okay? But if one a n naught crosses over L minus epsilon all will cross over, because it is monotonically increasing, all the remaining ones will be here. That means what? Given epsilon, I have found a n naught such that all the a n's after n naught are inside L minus epsilon and L plus epsilon, what does that prove? Given epsilon I have found a stage n naught, such that all the a n's after n naught are inside L minus epsilon and L, that precisely says limit has to be equal to L, proof is over.

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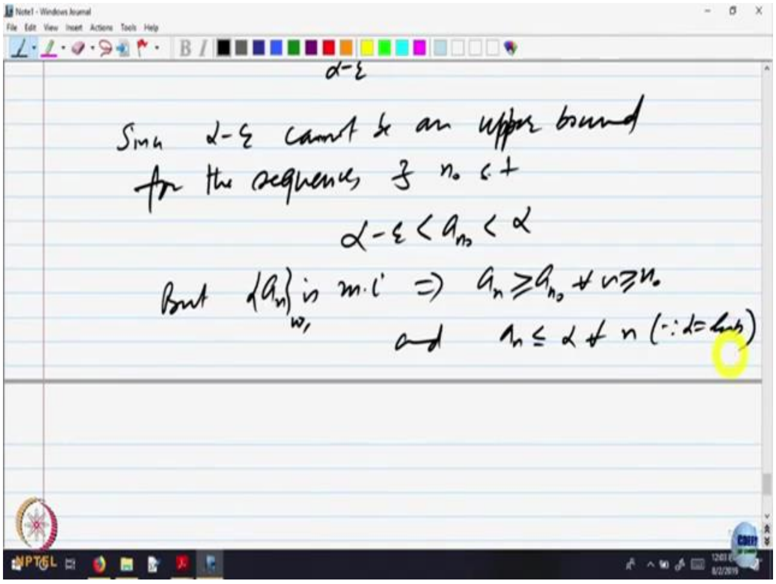

So, let us write that this whatever we have thought about. So, since alpha minus epsilon cannot be upper bound, I should write better cannot be an upper bound for the sequence the sequence there exist n naught such that L minus epsilon is less than a n naught is less than alpha. What was where where is L, there is no L, there is only alpha, alpha is the one we are looking at.

So, alpha minus epsilon right. Is it okay? But in a n is monotonically increasing implies a n bigger than or equal to a n naught for every n bigger than n naught. Because it is known and a n is less than or equal to alpha for every n, because alpha is equal to lub. So, nothing can go beyond lub anyway.

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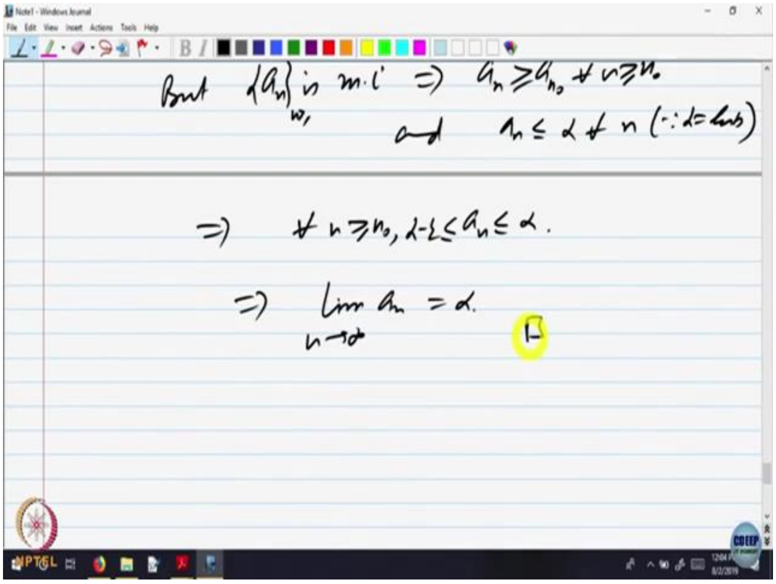

So, what does these two implies? So, imply for every n bigger than n not, a n is between alpha and alpha minus epsilon. Whatever we saw that picture, I am translating that picture now into the mathematical language, that is all nothing more. So, that means, implies limit a n, n going to infinity is equal to alpha. So, that proves it.

So, what we are saying is every monotonically increasing sequence which is bounded above, must converge. So, another tool in our kitty of analyzing sequences, algebra of limits, sandwich theorem and all that and plus one more is there. And correspondingly you can write the other part, if a sequence is monotonically decreasing.

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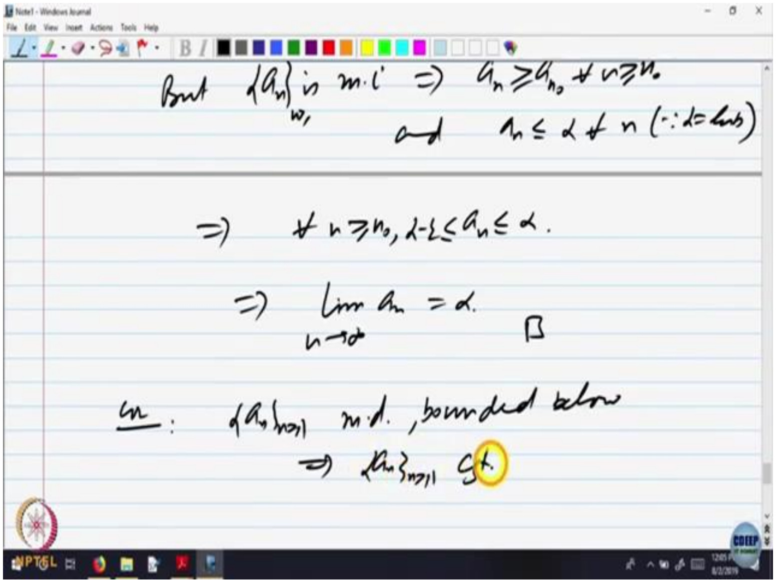

So, let me write as a corollary, a n monotonically decreasing, bounded below implies a n convergent, okay. So, increasing to decreasing you can just go by, if a n is increasing minus of that sequence will be decreasing, because minus reverts sign inequalities. Bonded above negatives will be bounded below, that will converge and now, you should use that limit theorems negative of a n is converging, so a n must converge. So, that combined with the earlier one will give you this also.

So, what we have shown is the least upper bound property implies every monotonically decreasing bonded sequence is convergent. In fact, the converse is also true, I will not prove it, that if you take a ordered field in which every monotonically increasing sequence is convergent, then you can improve it should have the least upper bound property, that is an equivalent way of describing least upper bound property, will not do that, because this is a elementary course for analysis for you.

So, what we are saying is? We are taking definition of completeness as lub property. One, we have shown that lub property implies convergence of every monotonic sequence which is either bounded above or increasing or bounded below if it is decreasing, this is okay. So, now for example, as a consequence of this as a consequence of this, I can define what is pi.

Now, I can define what is pi, what is pi? Look at the sequence of increasing sequence of approximate areas, square, octagon, those I can compute, numbers those I can compute numbers, because radius is 1, diagonal of the square is 1, radius is 1, so, all those octagons, sixteengon all are made up of triangles, all are made of triangles, whose areas I know the formulas, half base into height, I can compute.

So, those numbers a n's I can compute. I know they are monotonically increasing. I know they are bounded above because I can have a square circumscribing the circle. So, that must converge and that is my mathematical definition of Pi. So, I will call Pi as the limit of those areas, so that is one advantage of going to real line, completeness property. Okay.