Basic Real Analysis
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Lecture 50
Integration in several variables-Part 2

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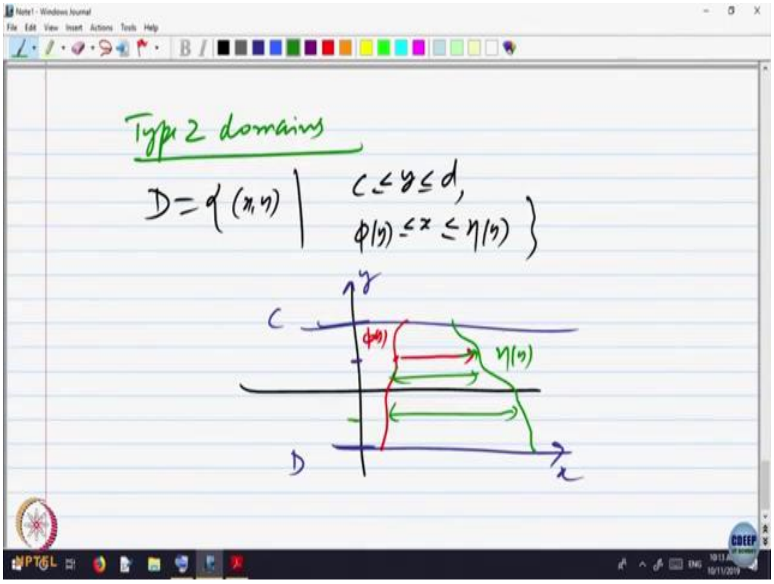

So, let us now also describe what is called other type of domains, so domain of type-2 domains. So, you would have already guess what is type-2 instead of saying x is between some limits a and b, and y varying between something is other way around y is fixed and x is going to vary. So, d is written as x, y such that y is lie between some limit, say c and d and for every such y, x lies between some function. So, let us call it as phi y and less than or equal to some other notations phi eta does not matter we can write eta y itself, does not matter eta y.

So, what does this look like? It look like y lies between some c and d, so let us write here is c and here is d. So, these are the lines in which and for every y, you have to look where does x go from. So, x is going to be horizontally it is going to vary horizontally. So, here is how much for every y effects how much I have to move along x-axis and see how much remain inside the domain.

So, it starts at somewhere phi y, so let us write somewhere as so, this is phi y, so it starts here, and goes up to where? Goes up to some function of eta y. So, it goes up some function let us say this one, so that is eta y. So, this is for any point y so that is limits, where phi y and eta y are functions, let us assume there continuous so that no problem comes continuous functions defined on c to d.

So, in type-1, x so vertical things are fixed, x lies between a and b, and for every x we are going to move vertically how much you move from bottom to the top limit, in type-2 your y is going to be fixed between 2 limits c and d, and for every y you are going to move horizontally, so that you are inside the domain. So, let us look at some previous examples and analyze them.

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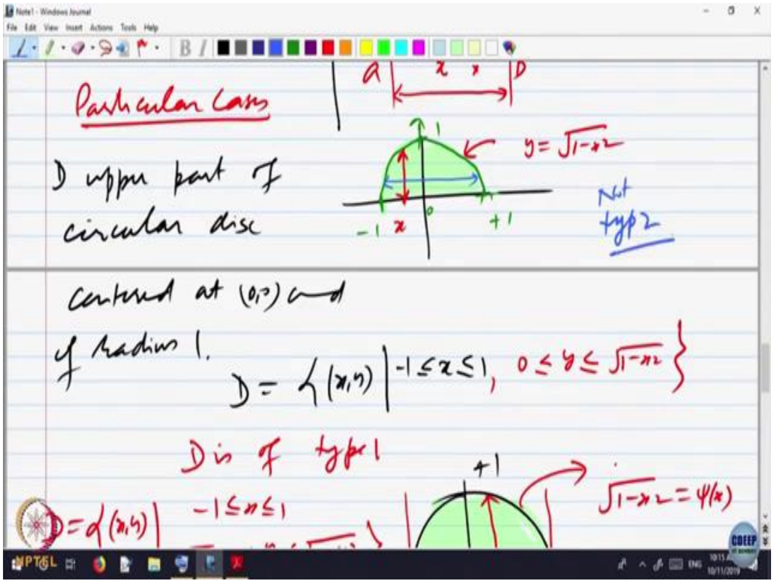

For example, can I say that Semi-circular disc is type-2? Means what, y lies between something and something. So, obviously y lies between 0 and 1. To be inside the domain moving horizontally when you move horizontally you will be going from here to here, but that is a same function, the limits do not change. So, it is not of domain of type-2 semi-circular disc, upper semi-circular disc is of type-1 but it is not of type-2 because, I can say y lies between something but for every y when I move horizontally it does not say it goes from lower limit to upper limit, not able same function it stays.

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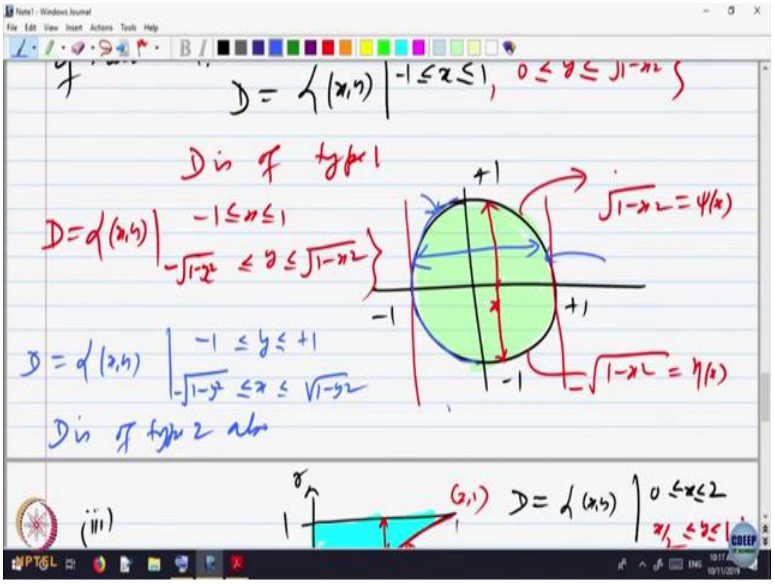

So, it is not type-2, it is not of type-2. Let us, look at the complete circle, this one complete disc, is it of type-2? It was of type-1, we saw it so to be saying it is of type-2, let us write d is equal to all x, y, where does y go from? Again goes from minus 1 to plus 1, how does x vary minus 1 to plus 1 how much the x varying? It goes from the left side to the right side. Can I describe this boundary as a function of y? This as a function of y.

So, x it starts here on the left side, so what is the equation of left side y? For every y you fix what is x, so x is equal to 1 minus y square with a negative sign, is that okay? That is this part so, this part is equation is minus 1 minus y square square root, and what is the right side? What is this side? N That is with a positive sign square root of 1 minus y square, is that okay? I have to write down the left side as a function of y because, what is a domain of type-2 function of y, function of y, for every y you fix what phi y.

So, x goes from so this is of type, so d is of type-2 also. So, this is of both type-1 and type-2 both.

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Professor: You cannot, you cannot write this as because, this equation of this for every y, where does it from it go from? It goes from this equation to a same equation as a function of y, you will have a problem, so that is the reason.

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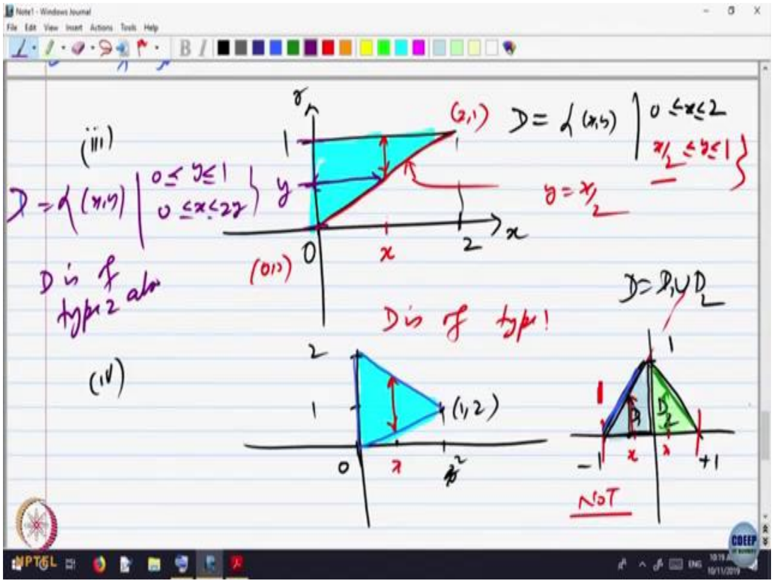

Let me look at this example, this was of type-1, can I say this is of type-2 also or not.

So, what is this domain d x, y, where does y go from 0 to 1 so, y goes from 0 to 1. Now, x for any point y, I have to see x varies so, x starts at this line, vertical line and goes up to so it starts at x is equal to 0 and goes up to 2y, I have to write x as a function of y this is eta y, it should be function of y. So, this is also of type-1 as well as type-2. So, d is of type-2 also.

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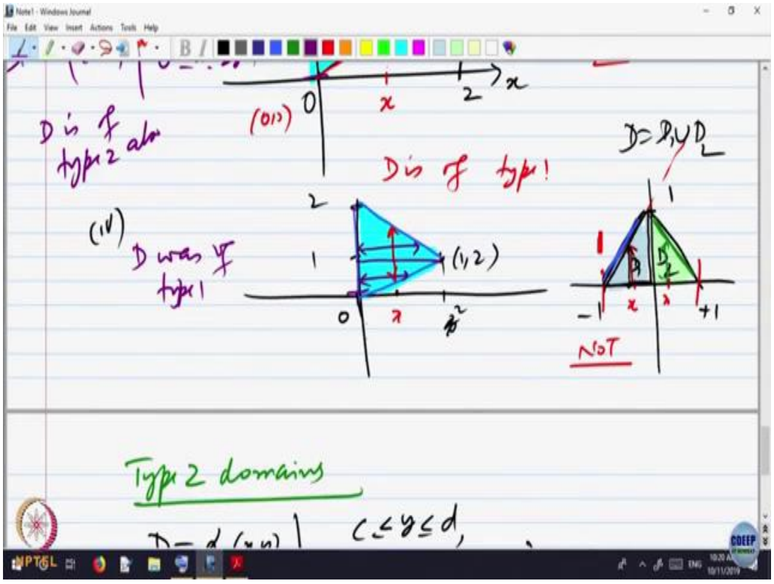

Let us look at this one, this was a domain d of type-1, this was type-1 because we wrote it as it goes from 0 to x goes from 0 to 2 and vertically we know it goes from one limit to another limit on equation 2.

Can I say it is of type-2? No, why not so, y goes from 0 to 2 and if any point if I look at here I have to go from here to here but, that depends on whether I am there or so, the function upper limit changes depending upon the point y again, if I want I can cut it into two parts as union of two non-overlapping domains of type-2 I can do that if I want to, otherwise domain type-1 is also fine.

So, is it clear what is domain type-1 and type-2? Type-1 x is fix and y varies between limits, and type-2 y is fixed and x varies between limits. So, depending on your convenience you have to interpret the domain as type-1 or type-2 or cut it into parts so that it becomes union of domains of type-1 or type-2. Now, the question comes why I am discussing all these domains of type-1 and type-2? Because there is a theorem which helps us to compute double integral.

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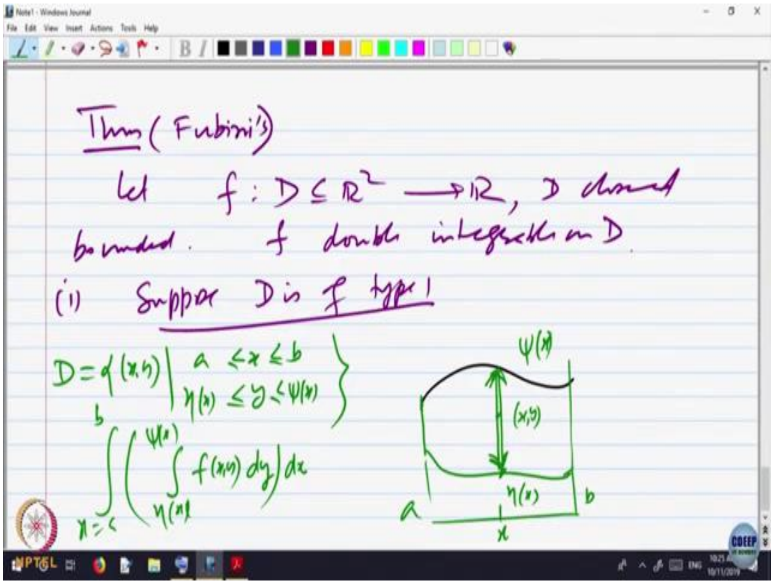

So, this is the theorem it is called Fubini’s theorem. So, this one helps us to compute the double integral. So it says let f, d contain in R2 to R, f d close bounded, f double integrable, double integrable on D. The basic idea of this theorem is very simple. So, let me just explain that in the picture first. So, let us suppose d is of type-1, so that is a first assumption in case of type-1, what does the picture look like? The picture will look like so, that is a, this is b and for every point in between x goes from here to here.

So, let us give what was the name we got it psi x, does not matter what actually eta x so, d is equal to all x, y such that x lies between a and b, and y lies between eta x and phi x. That was domain type-1, it says if it is of type-1, basically what we are looking at, see what we are looking at is what is the volume above this domain of something. Now if, I fix a point x here, if I fix the point, now this is a line which is in the domain, this is a line which is in the domain, where x is fixed, what is varying here y is varying so, all these points are x, y where y is inside y goes from eta x to phi x.

Now, when I want to rise it above, I can think of at a every point what is the value of the function and integrate. So, I want to integrate over this line, so what will that give me if I integrate over this line, so integral of bottom is eta x top is psi x f is a function, and what is fix here? x, so, I am integrating with respect to y, is that okay? On this line which is in the domain x is fixed and y is varying from eta x2 on this line my function is defined because, the function is define on the whole domain.

So, if I look at this integral what will this represent? That will represent look at the graph of the function above this line, so there will be some graph, it will give me the area below the graph of the function, it will give me the, so this integral will give me the area below the graph of the function so, you can think it of with this line as the base there is a sheet, paper sheet top is f of x, y x is fixed, y is varying.

Now, if I add up all these sheets I should get the volume, if I add up, this is a sheet is area I know, what should be the volume if I take a small thickness that will be dx, I want to add up so, I should integrate x is equal to a to b. That should give me the volume, is that clear to everybody? Yes or no picture, this is my domain, let me probably if I can draw 3 dimensional picture so let me go back to the my original picture and see whether we can add something there.

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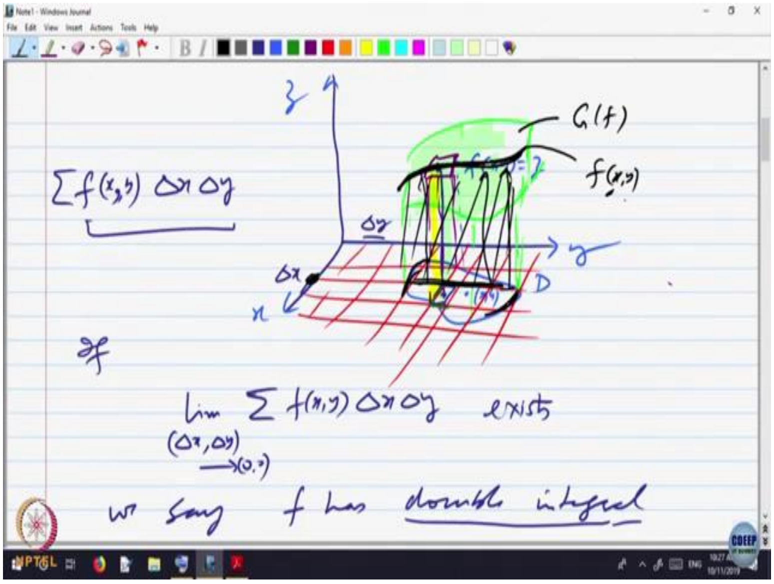

In this so, this is my surface at the top I think I can (())(15:30) so, this is my surface if you are fixing, what are we fixing in this? x, we are fixing x, and letting y vary, so let us fix an x, in this let us fix an x, some color which is good, so let us fix an x so, this x is fix so I am looking at this line, is it okay? That is a line x is fixed y is varying, is it okay? This side is my lower limit, this side is my upper limit for the domain, when I integrate as x varies in this way it is the function value is going to vary, so they are going to vary like this on the surface.

So, here the values f x is fixed, so y is varying, x is fixed for every point there will be height, for every point there will be height, for every point there will be height, so that will be curve, so that will be a function. For, that function x is fix y is varying, so what happens when I integrate this what should I get? When I integrate I should get the area of that sheet, I should get the area of that sheet, and as x varies I will get that different sheets add up all the sheets of small thickness you will get the volume.

If you like you can think of a book, what is the volume of the book? Volume of the each page added together, that is a simplest way of looking at it.

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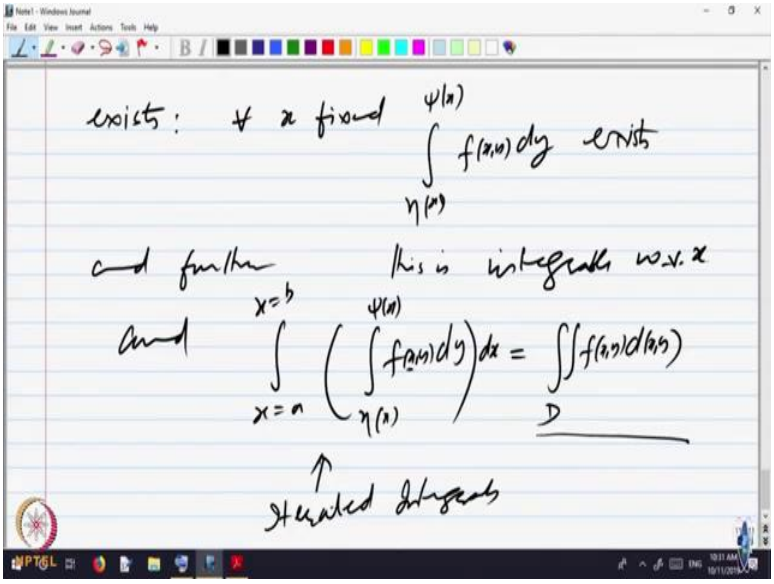

So, Fubini's theorem says the following that so, suppose d is of type-1, so this is then this quantity exists meaning what? That means it consist of looking at the function f x, y so, exists meaning what, for every x fix integral eta x to psi x of f x, y dy exists. This integral exists and further.

So, f x, y, x was fixed, so what does this, what is the value of this integral? That will be a number which will depend upon x. Now I sum it up, sum it over x, further this, further this is integrable with respect to x, it is also integrable with respect to x and, so x going from a to b we are saying this is integrable, so eta x, psi x, f x, y dy that is the area of the sheet into small thickness dx that gives the volume of the thin sheet added up together that is same as the volume that we are starting with.

So, the double integral of a function over a closed bounded domain, if the function is integrable if the domain is of type-1, then the double integral can be computed by looking at function f x, y one variable at a time, here we are fixing our x, x is fixed here as a function of y integrate one variable, and then integrate this as, so that gives you another function of one variable that is also integrable and that integral is same as the double integral.

So, this is what is Fubini's theorem for domains of type-1. If your domain is of type-1, then you can calculate the double integral as one variable at a time. So, the left hand side these two integrals are called iterated integrals, these are called iterated integrals, you are iterating the double integral by one variable they are iterations. So, that is why each one the inner integral is called iterated integral with respect to y and that is integrated you got the second iterated integral.

So, Fubini's theorem says, if your domain is nice type-1, then double integral can be computed as iterated integral, so that is the type-1. Similarly, there should be type-2, so, let us write type-2 also so that we understand again.

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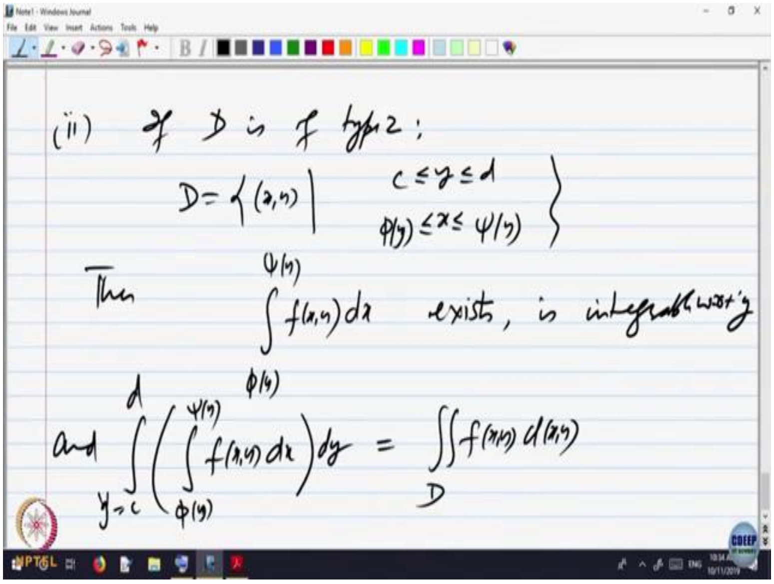

So, second if d is of type-2, so what is type-2? So, d is equal to x, y say that y is between some c and d and for every y x lies between some function say phi y and psi y, where phi and psi are nice continuous functions. One can relax this conditions but, let us assume they are nice is of type-2 then.

So, let me write then integral of f x, y dx, I am fixing y now we are fixing y, fix y so it is between phi y and psi y this exists is integrable with respect to y and when I integrate this with respect to y so, phi y psi y f x, y dx integrate this y was fixed, so this is a dependent on y so, dy integral y goes from c to d exists and is equal to the double integral f x, y, then again it is equal to the double integral.

So, computation of double integral by Fubini's theorem becomes easier in the sense that you can push the problem to one variable at a time, depending upon whether your domain is of type-1 or of type-2 or you can cut it into pieces of types-1 and type-2.

So, basic fundamental things are building blocks are integrals over domain of type-1, domains of type-2. Then everything is nice you can compute the integrals. So, this is a computational aspect of Fubini’s theorem.

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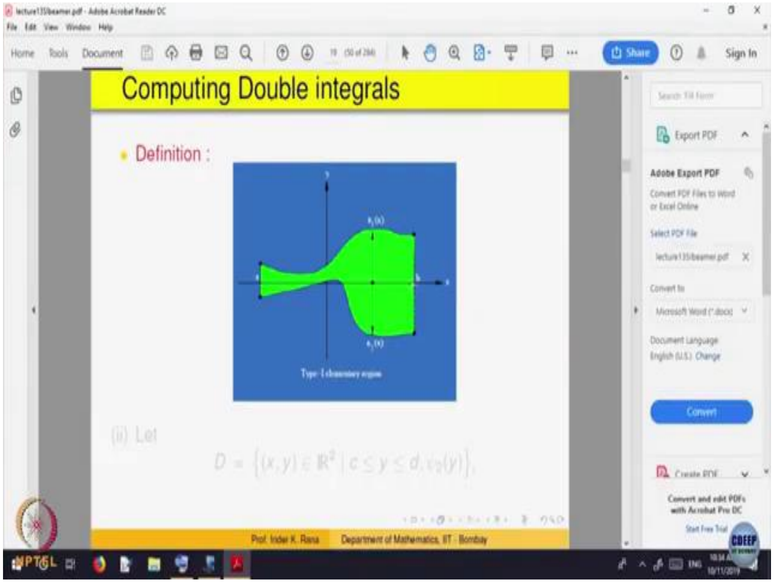

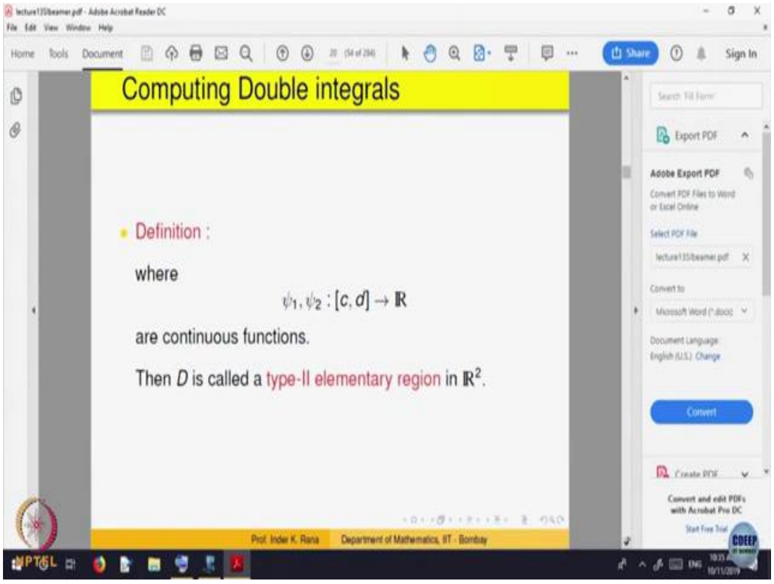

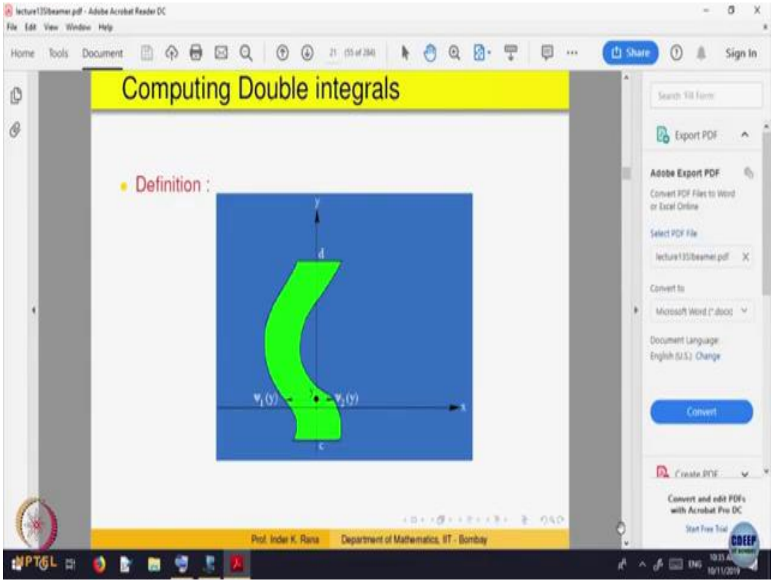

So probably let us look at some examples, so whatever I have said this is type-1, x goes from a to b vertical goes from one function to another functions, so let us just revise anyway and similarly, type-2 will look like this, horizontally there are limits c to d and for every moving along the horizontal line it goes from one function to another.

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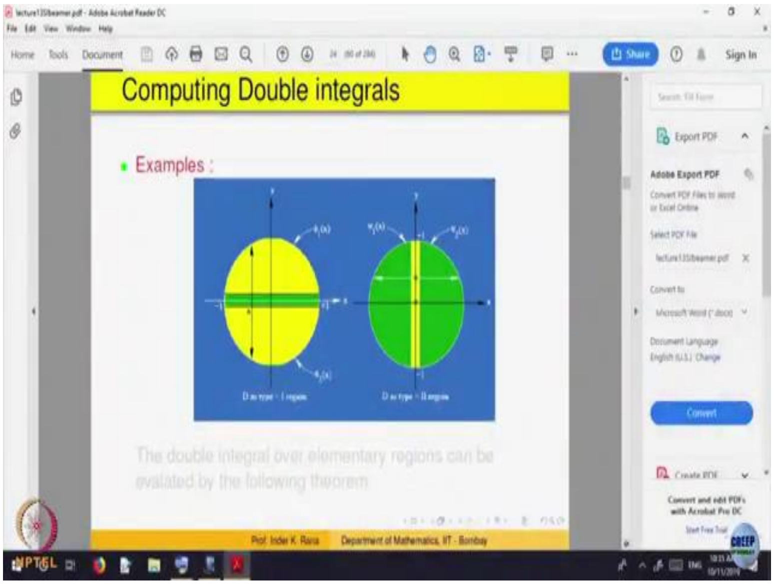

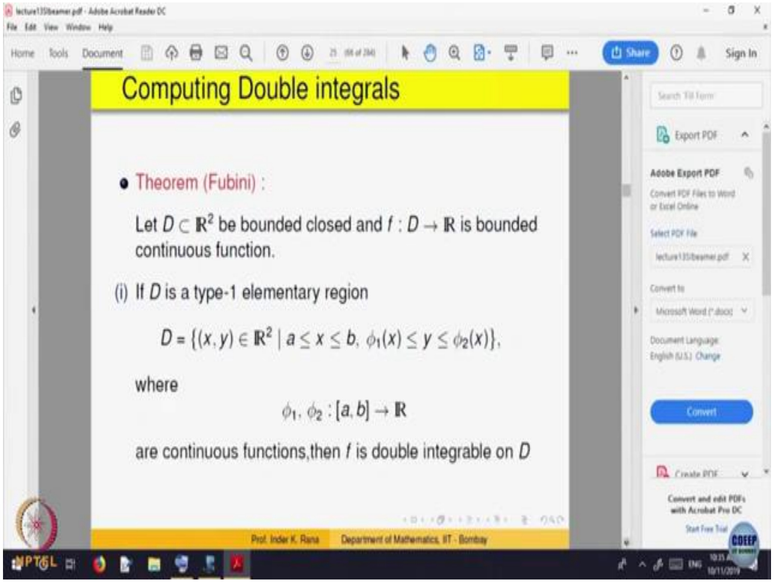

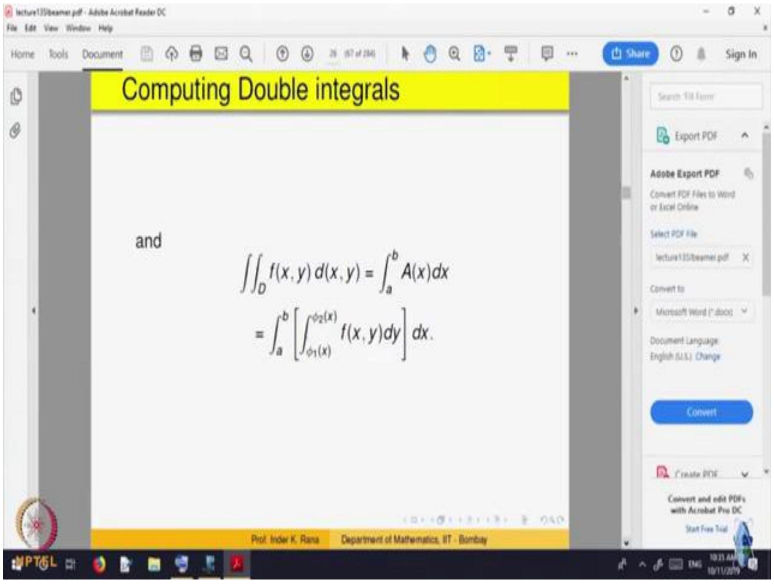

So, you go from this function to this function that is type-2. So, example of the disc and so on we have looked at so, circular disc is both of type-1, type-2 and Fubini’s theorem say that if it is elementary region of type-1, then the double integral looks like, integral x goes type-1, x goes from a to b the iterated integral is f x, y phi1 x to phi2 of x dy. So, integrate with respect to y with x fix, whichever is the finite limit c to d or a to b that goes out, that is a outside one, inner one is a variable thing that you so, that is type-1.

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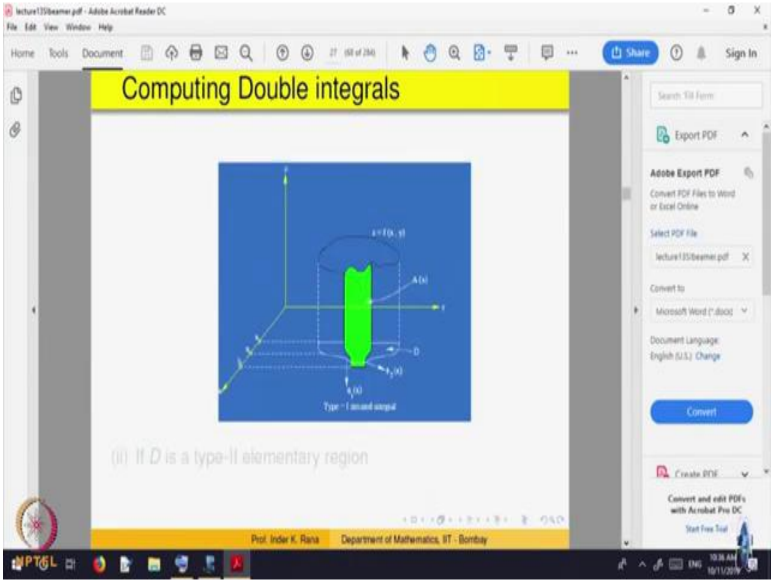

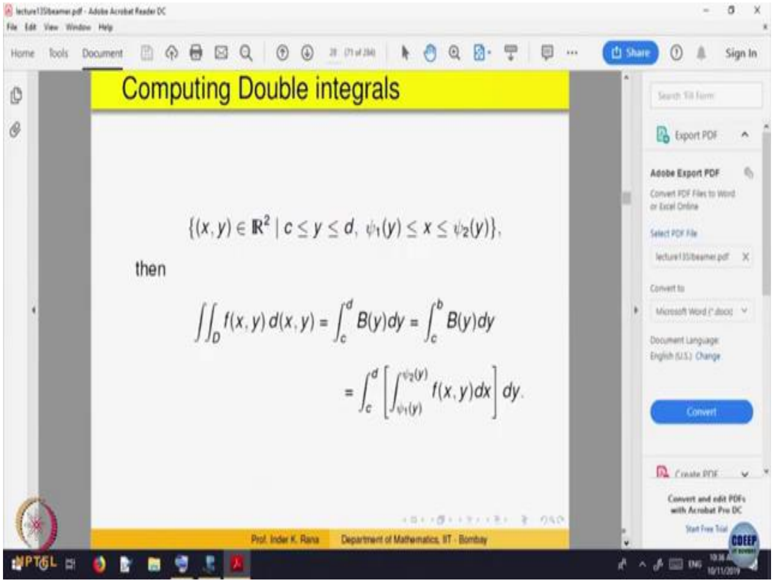

So, that is what if that picture makes sense you are looking at the sheet and the sheet is moving. Similarly, of type-2 y is between c and d, y between c and d, x goes from a function of y to another function of y. So, that is the double integral Fubini’s theorem says it same as integrate the variable x with the variable limit psi1 y to psi2 y, that depends upon y because, y is fix, so sum it up with respect to y, so that is double integral.

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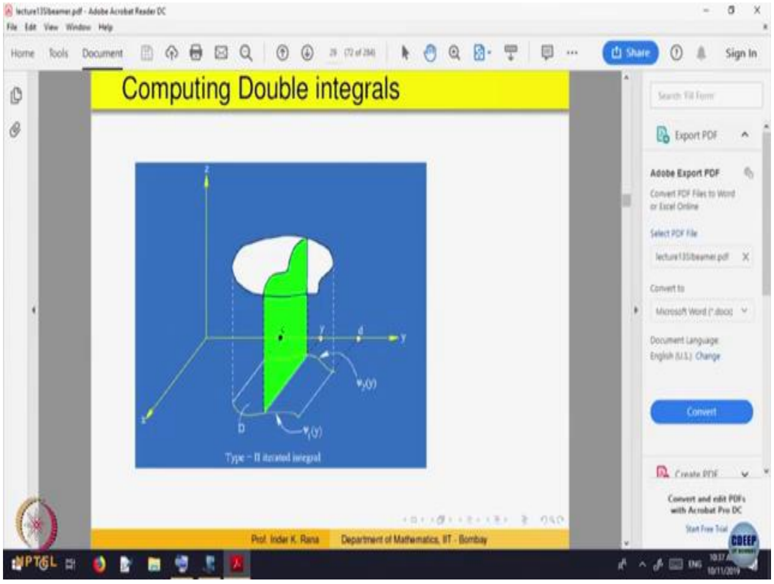

So, this probably looks more clear picture you can see that sheet, and that green thing is going to sort of move and cover up everything.