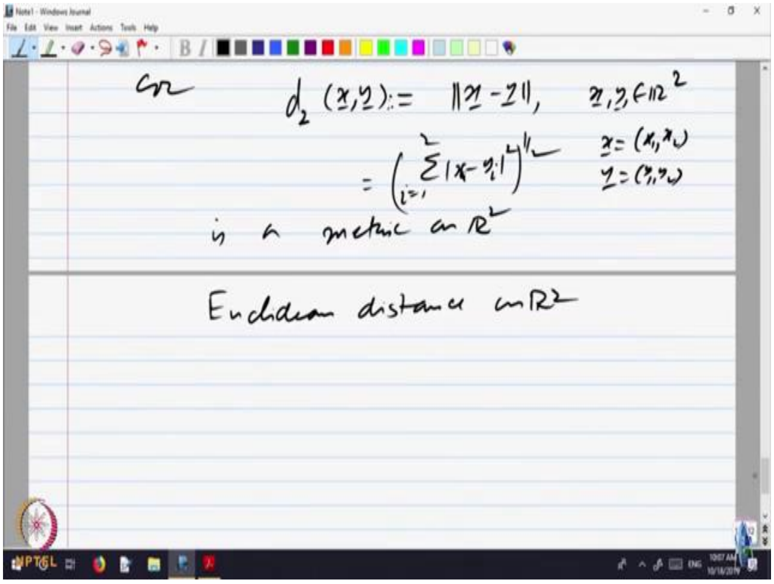
Basic Real Analysis
Professor Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 57
Metric Spaces - Part II

(Refer Slide Time 0:19)

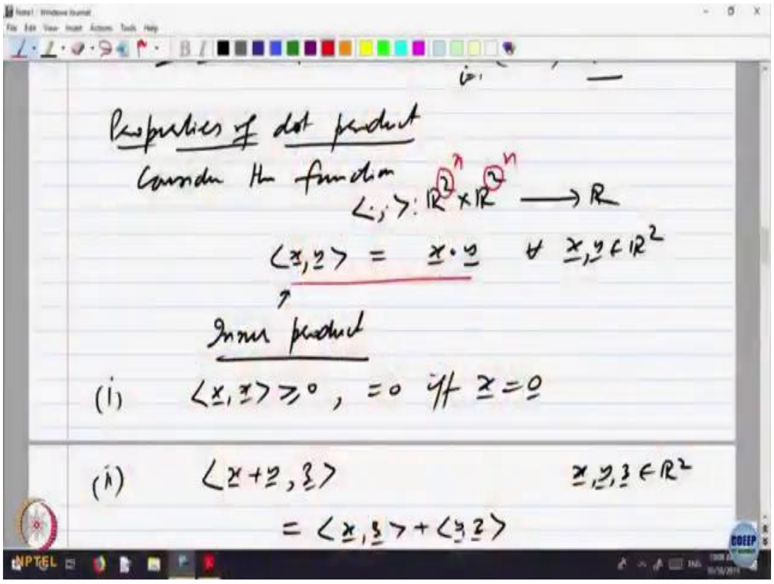


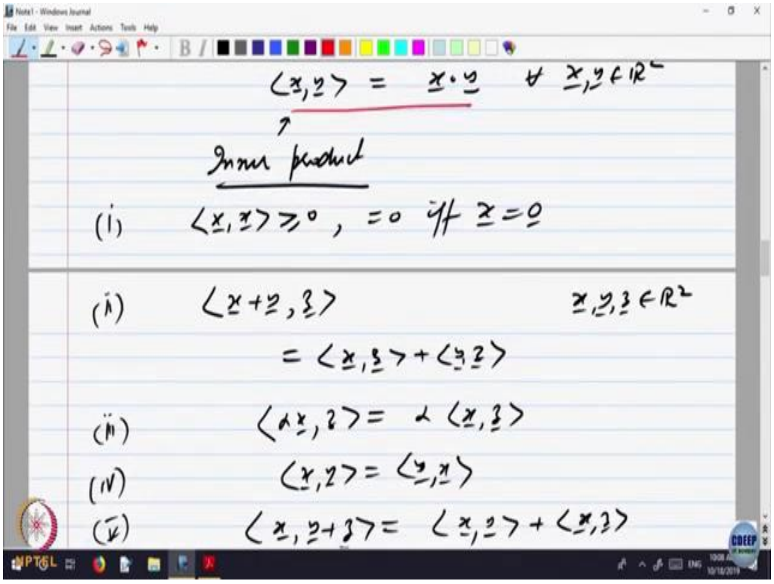
So, this will give me. So, consequence of this corollary that if I write d2 of xy to be equal to norm of x minus y, xy belonging to R2 and what is that? So, if X has got components X1, X2, Y has got components Y1 Y2, then this is nothing but sigma mod xi square minus yi square i equal to 1 to 2 xi minus yi square, right. That was the dot product. So, this raised to power, right. So, that is the dot product is related to the norm. So, it gives you that this is a metric.

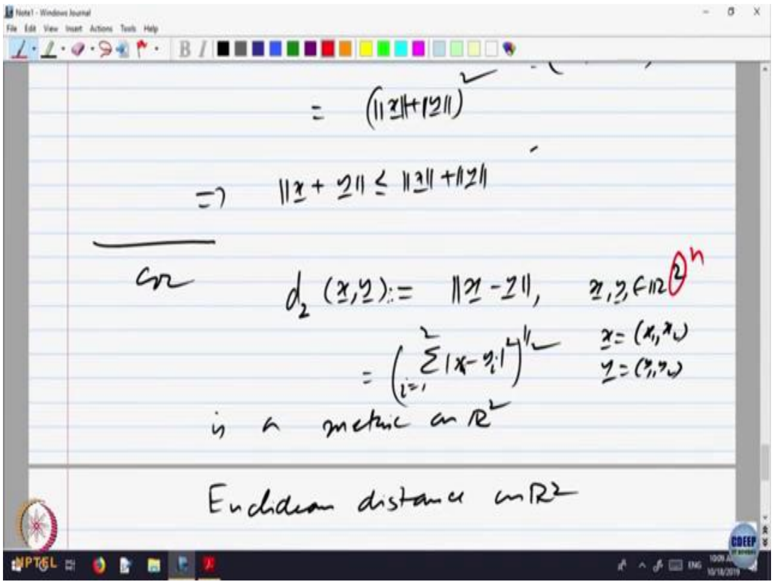
Student: (())(1:31).

Professor: I am defining D2 to be equal to this D2 of xy is defined to be the norm of x minus y, like in real line the distance between two points we define as absolute value of A minus B, distance between, right? Same I am trying to copy now, I am trying to show that how you can, how does one try to extend various concepts from R to R2 and so on. Okay, so what we have done is we have generated a notion of norm and using that norm like absolute value we have gotten this thing, a denotion of distance right. So, this is normally called the Euclidean distance. So, this is called distance on R2.

(Refer Slide Time 2:45)





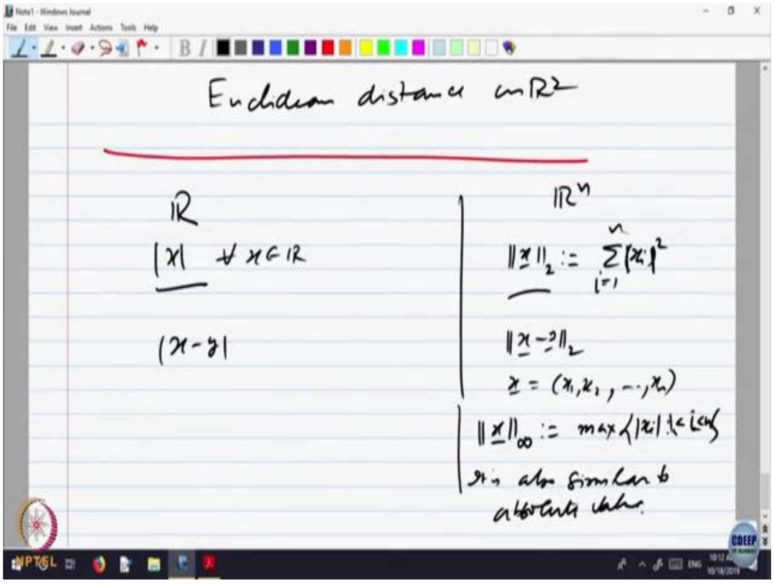


But the interesting thing is, let us go back and look at what we have done is in R2. How does it change if I remove from R2 to Rn? What is Rn? R2 is a vector having two components. So, let us think of Rn as a vector having n components, okay. The dot product. So, what will be the dot product? How do we define the dot product? We can define the dot product instead of 1 to 2 we will have 1 to n, right. So, that will give me the dot product and again dot product will be linear in both because it is sigma ai bi only right.

So, all these properties remain valid, the dot product has all linearity and everything. I can define the norm of a vector in Rn by the same relation, okay. So, this if I take it to Rn right, the same proof Cauchy-Schwarz inequality does not use the fact that we got two components or three components. It only uses the properties of the dot product. The dot product is linear in both the variables and symmetric, that is the only fact to use right.

So, using that Cauchy-Schwarz inequality remains true and Cauchy-Schwarz inequality remains true, that means you also have the triangle inequality. So, the all the proofs, same proof everything is same instead of 1 to 2, summation 1 to 2, summation 1 to n if you write same proves work, Cauchy-Schwarz inequality works and that gives you the notion of distance okay in Rn.

(Refer Slide Time 4:31)

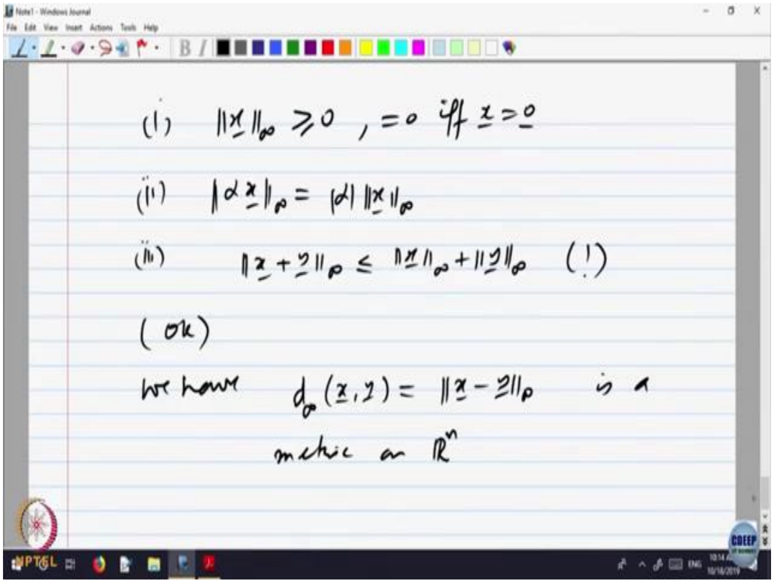


So, now here is something interesting that is happening, let us observe that. I want you to keep track of here is real line and here is Rn and the real line, we had the notion of absolute value right for every x belonging to R and here we have defined, okay, so let me put 2 here, just to later on we will see why that is. So, that is sigma of xi square i equal to 1 to n if X has got components x1, x2, xn and that gives me both have the same properties.

This also behaves like absolute value and here this gives me the notion of a metric and this also gives me the notion of a metric okay. So, that is ordinary metric in earlier line and here is the, okay. Now, let us look at something more. You can think of this xi square as mod of xi square that does not matter, right? Square of a number is same as square of the absolute value of that number.

Okay, right. Okay. So, here is something interesting. Let us define, let me put infinity below it. Okay, what is that? So, that is look at the maximum value of mod xi, i between 1 and n. x is a vector with components. So, x is a vector with components x1, x2, xn, right. Let us look at the components' absolute values which is the largest of them. There are a finite number. So, maximum of that, right. Claim this also is, it is also similar to absolute value.

(Refer Slide Time 6:55)

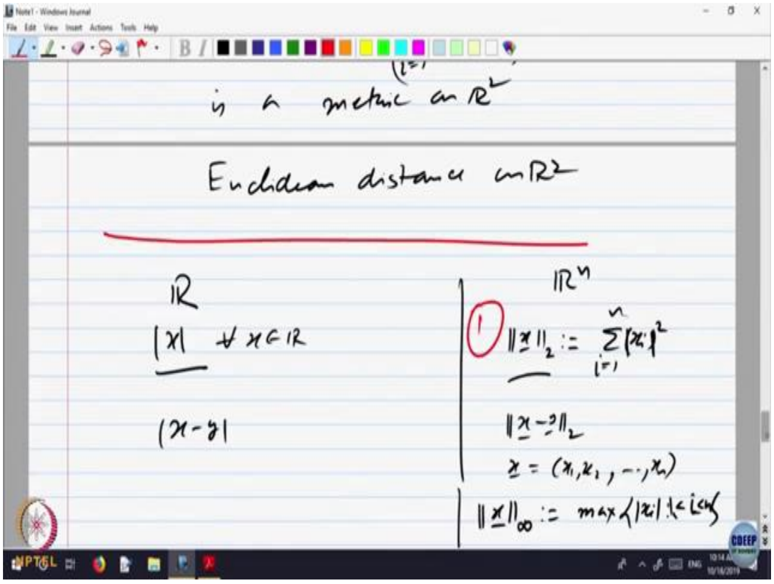


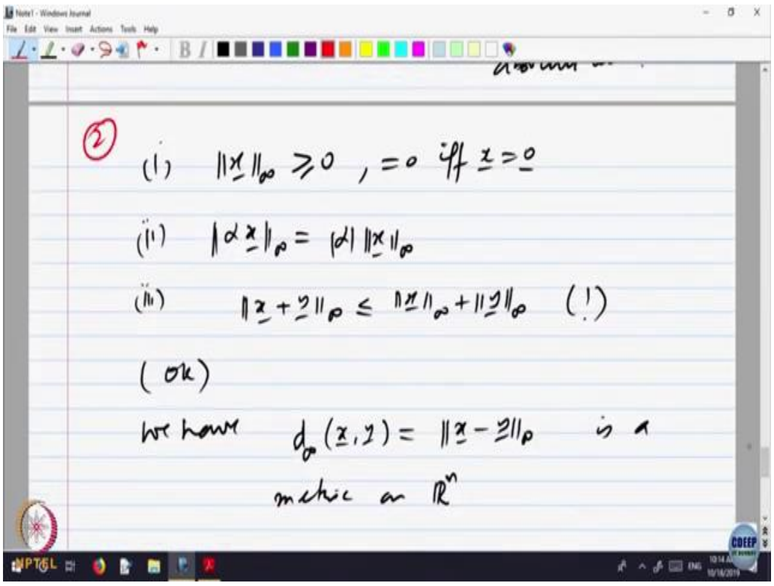
Meaning what? So, let us write what that similarity means. One, is bigger than or equal to 0, equal to 0 if and only if x is the 0 vector. Is that ok? Because the maximum is 0 every other component is 0, right. So, all the components are 0. So, this property holds. Second, alpha times x is equal to mod alpha times, right? That is also okay because if you take components of alpha x will be alpha x1, alpha x2 and alpha x n, right.

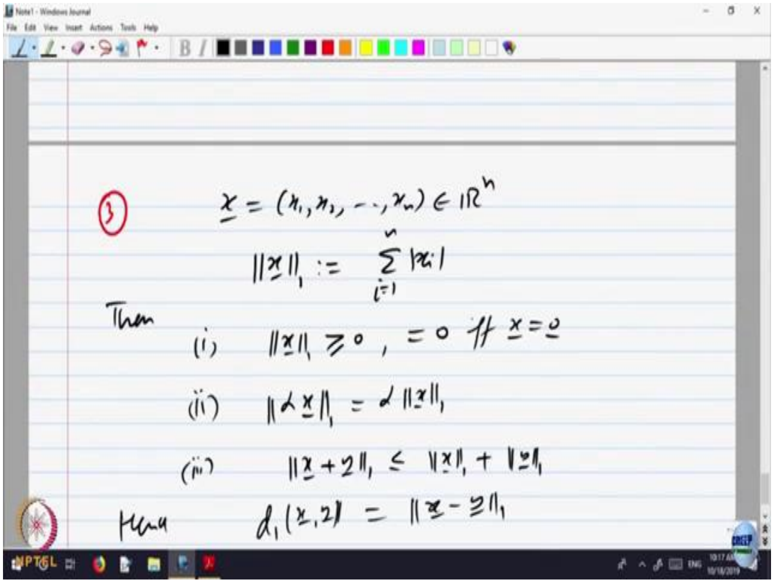
So, the largest of mod alpha times xi is same as largest of alpha times mod of xi, right? So, no problem. Claim that this also has triangle inequality because mod, if you are looking at maximum of a and, a plus b, that is less than or equal to maximum of a plus maximum of b, only that property is required to prove this because the left hand side will give you maximum of xi plus yi, that is less than or equal to maximum of xi plus maximum of yi.

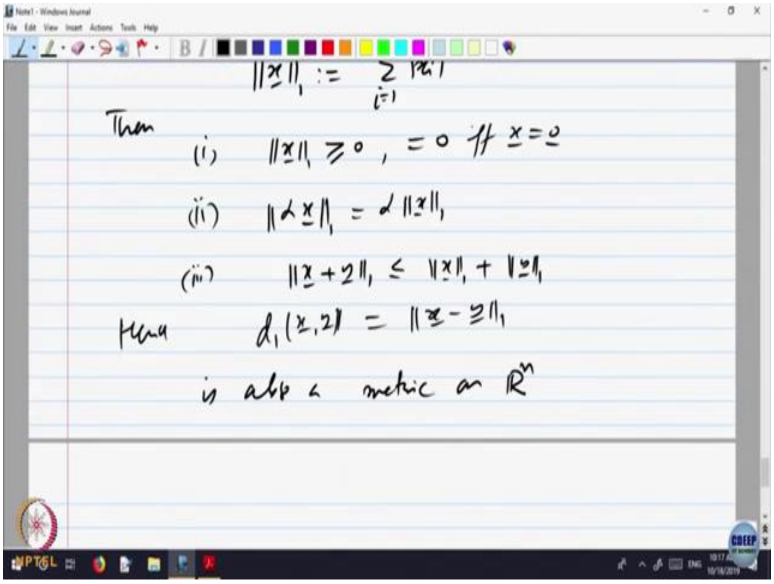
So that is again obvious, right. So, these properties are okay. So, we have, let us write D infinity xy equal to is a metric, is a metric on Rn. So, on the same set Rn, we have got two different metrics, L2, right D2 and D infinity.

(Refer Slide Time 9:13)









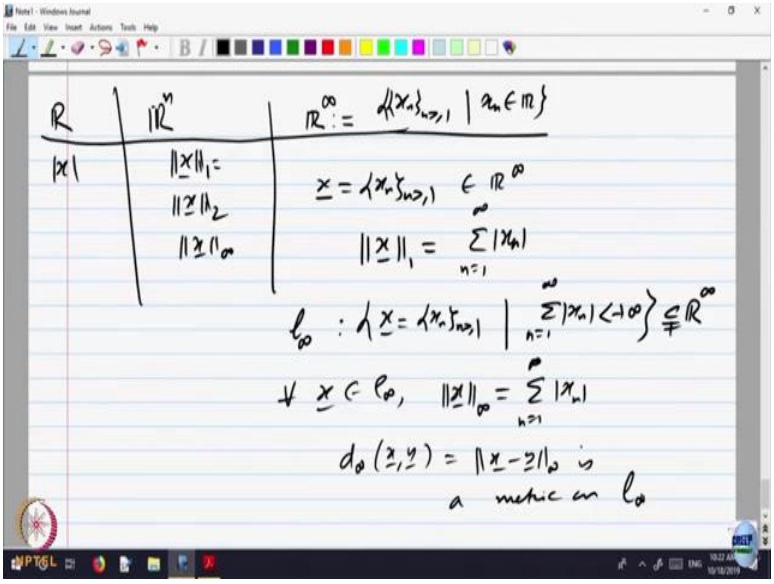
Let us look at one more. Okay. So, this was one metric and this was the second one and let us look at a another one, the third. So, how is that defined? See, how absolute value of the components are used to define something? In D2, we are taking squares and adding and taking the square root. Here we are taking the maximum of the components. Another one for x, x1, x2, x n in Rn.

Let us define, let me call it as 1 to be sigma mod xi. Instead of taking squares and then square root, let me just add the absolute values, nothing more than that. Okay. So, then let us check once again, is bigger than or equal to 0 because summing non-negative quantities, equal to 0, non-negative numbers sum is equal to 0 if each component, each term is 0, that is each mod xi is 0, and that means if and only if x is equal to 0. Obvious property by definition itself.

Second, alpha times x. So, we will be multiplying summation mod alpha xi, alpha will come out. So, that is alpha times, right? Again obvious property because in the alpha times sum, alpha comes out, so it is sigma of alpha. So, third property. What about. What will be this quantity? This will be sigma mod of xi plus yi plus but mod of xi plus yi is less than or equal to mod xi plus mod yi by triangular inequality in real line.

So, it is less than or equal to norm of 1 plus 1. So, that again says, so hence I get a new, so D1xy equal to okay, is also a metric. So, we have got three different metrics on Rn, okay.

(Refer Slide Time 12:24)



So, let me just write here. R, Rn, so here is absolute value, here is 1 and then we had 2 and then we had infinity, right. See, how one generalizes things. We are just copying. What do you think should be the next R? Rn, what should be the next thing if I want to generalize? R to the power natural thing infinity. So, infinity. What does R to the power infinity mean? As I set, I have to tell what is it as a set.

So, what is it as a set? One can say it is a set of all vectors with infinite components and that is same as there is a finite component, there is a first component, there is a second component, there is a third component. That means, it is a space of all sequences. Instead of writing as a component dot dot dot, it is a set of all sequences each xn belonging to, all the squares of all real sequences. x is a set of all real sequences.

Now, let us try to extend this 1, 2 and infinity to them. Right? So, what we will try to do? So, for x, which is a sequence, we would like to define what is, so let me copy, what was this thing here? We took the component. And we took the sum, we took the sum in, when it was 1, we took the sum 1 to n of each component.

Sum up all the components. So, sum up n equal to 1 to infinity. That should be the natural generalization. But as soon as 1 does that, you will end up into problem. So, what is the problem? This 1 to infinity sum, what does it mean? It may not exist, right. So, if you have done series of numbers, you should understand that this is a series which may not be convergent.

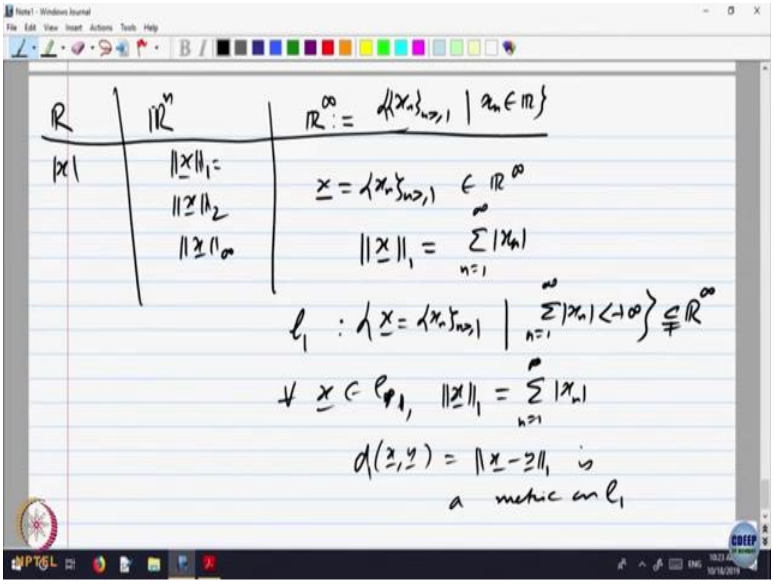
So, one cannot define for every x belonging to r infinity, one has to look at a subset. So, look at all sequences such that sigma mod xn 1 to infinity is finite. That let us call it L lower infinity, that is a subset of R infinity. So, we cannot just extend taking the sum of all the components, right. That does not make sense. We have to respect to those sequences xn such that mod of xn summation is finite.

For every x belonging to L infinity, right, one can define to be equal to sigma mod xn and equal to 1 to infinity. So, it will have the same properties right. It is bigger than or equal to 0 and this will be equal to 0, it is sum of a non-negative series, so each term must be 0. Triangle inequality obviously follows because of the absolute value. So, D infinity xy equal to is, this is a metric. Not R infinity on the subset L infinity.

Student: (())(17:15)

Professor: Yes, mode of xn. So, oh, this is oh oh, I have got confused, I just took it that way.

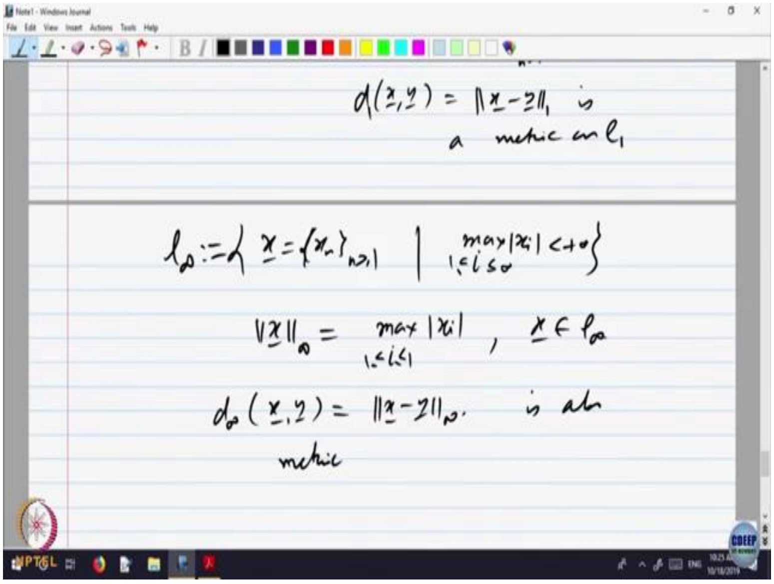
(Refer Slide Time 17:24)



So, that is yes it is 1, we are taking a sum, sorry sorry. That infinity is R infinity. 1 and this is also 1, yes you are right, we are taking the sum of all the components, so D1. So, is a metric on L infinity. So, this is also 1. So, actually then we should not be calling it as L infinity. We should call it as, that also we should call it as L1, okay.

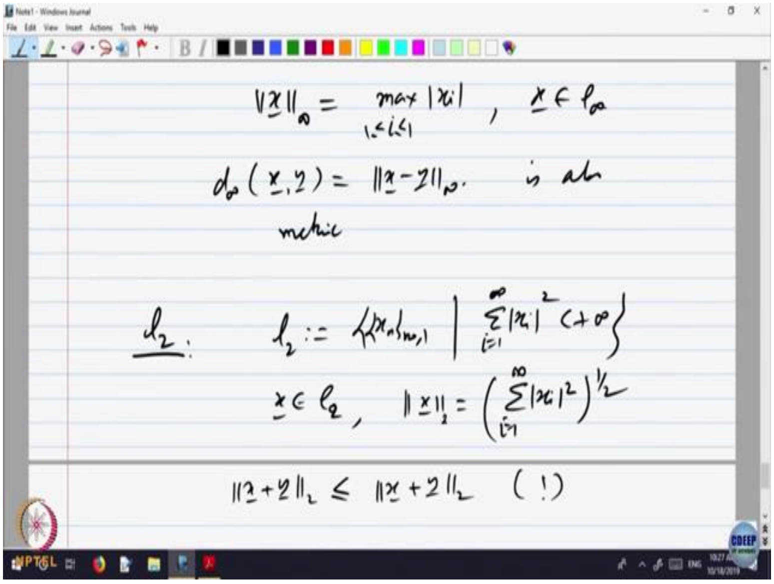
So, let us, good thanks for pointing out that we are dealing with this L1. So, L1 is the set of all sequences whose absolute values of the term summation is finite. Okay. Now, you can now guess what should be L infinity.

(Refer Slide Time 18:37)



So, L1, you can define L infinity to be equal to all sequences such that maximum of I1 to infinity is finite. Supremum exists, infinite collection now. So, that exists. So, on this, you can define to be equal to maximum of mod xi for x belonging to L infinity. So, you see how smoothly things go on, but you have to put appropriate conditions okay. So, D infinity xy, you get a metric now on L infinity, so that is x minus y infinity is also a metric. No proof, no change, other than instead of saying 1 to n, you have to go to 1 to infinity, same things essentially work, right.

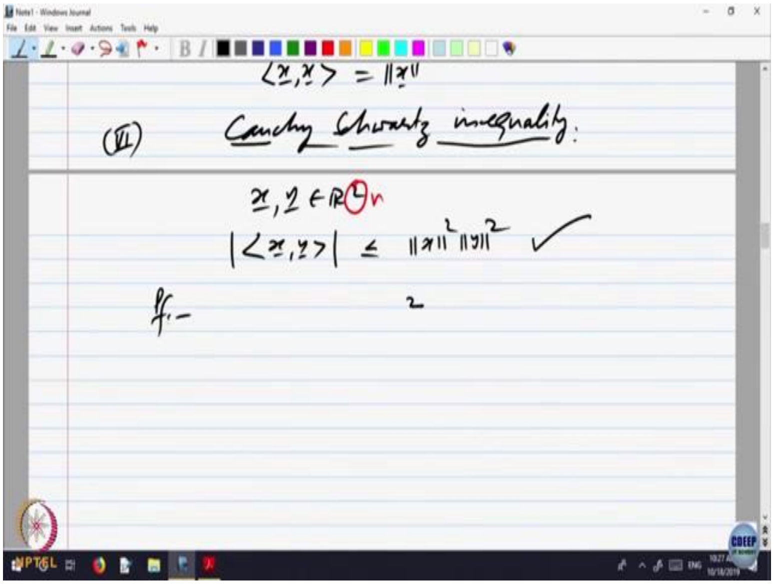
(Refer Slide Time 19:53)

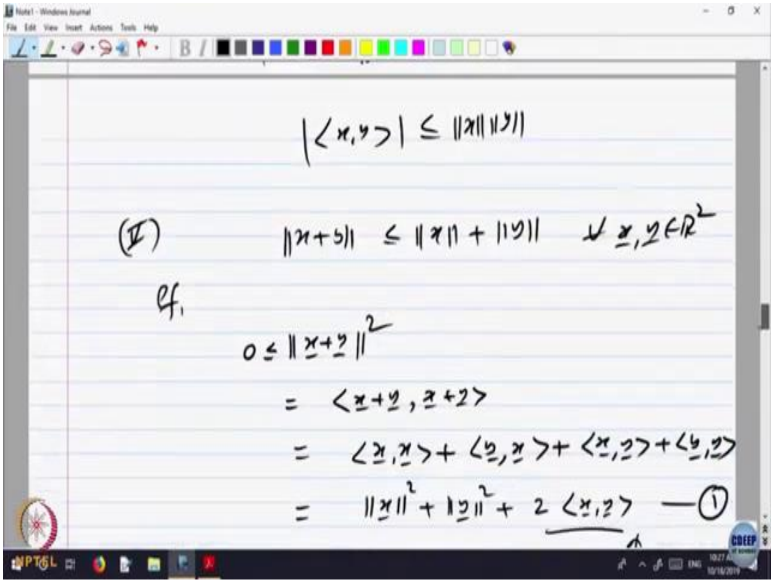


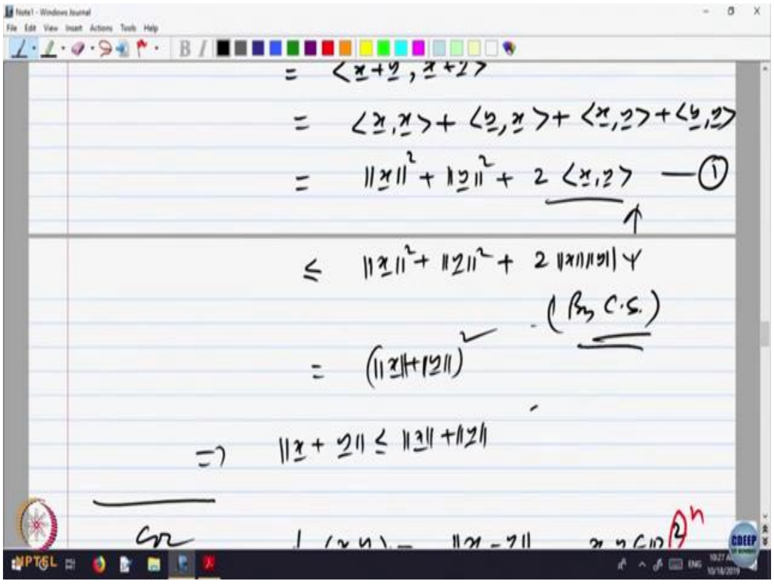
Now, comes our L, we should also have L2 corresponding to the Euclidean distance. So, what should be, so L2 is the set of all sequences xn such that sigma mod x i, i equal to 1 to infinity square is finite and whenever that is the case for x belonging to L2 define norm of x to be equal to sigma xi square 1 to infinity now, Rn to infinity raised to power 1 by 2, okay.

Of course this will be bigger or equal to 0 or it is equal to 0 if and only if each xi is equal to 0, right, alpha times, alpha square, square root, alpha comes out. The problem arises when you want to prove Cauchy-Schwarz inequality, it is 1 to infinity. You want to prove Cauchy-Schwarz inequality and then use that to prove your Triangle inequality. Right. So, this is 2 here. So, is less than or equal to okay. So, what one has to do?

(Refer Slide Time 22:07)

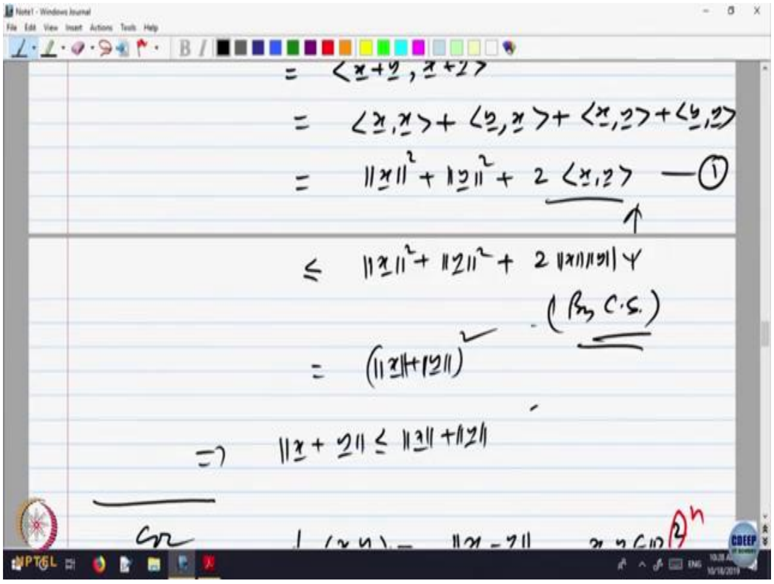






See, for R2 to prove for this property for R2, what was our root? Our root was proof Cauchy-Schwarz inequality and using that Cauchy-Schwarz inequality we proved, so this is all proof of Cauchy-Schwarz inequality and using Cauchy-Schwarz inequality here, we proved triangle inequality okay.

(Refer Slide Time 22:39)



Same route one follows to prove it for L infinity also or sequences whose squares are summable L2. Actually much more generalisations are possible. Okay. So, I think it is a good idea to prove those things because they will be useful for you later on also. See, okay, in L1, what were we doing? We are looking at the absolute value of each component mod xi, and summing it up.

In L2 we are looking at squares of them, and it was realized that we need not do only for 1 and 2, you can do it for any real number between 1 and infinity. Right. So, let me define the problem and define the set and then proof for at one go for everything.