Basic Real Analysis
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Lecture 59
Lp Spaces – Part I

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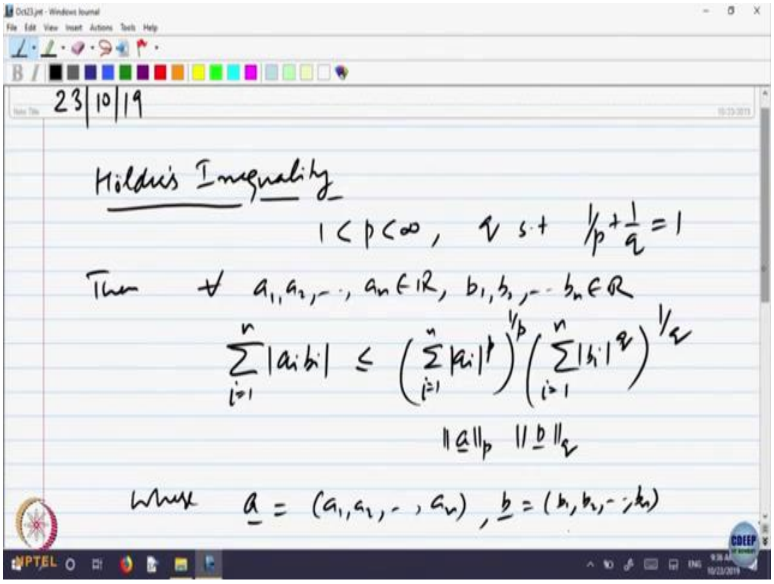

So, let us recall what was Holder's inequality. So we had a number of P between 1 and infinity and q such that 1 over P plus 1 over q is equal to 1, then for every a1, a2, some a n, belonging to R, and b1, b2, bn belonging to R. Sigma of mod a i b i, i equal to 1 to n was less than or equal to sigma mod a i to the power P raise to power 1 over P and i equal to 1 to n sigma mod b i raise to power q, raise power 1 over q. So, this quantity we had called it as norm of the vector a pth norm and this was called the norm b the qth norm where you consider the vector a to b, a1, a2, a n, and b is the vector b1, b2, bn. So, this was the Holder's inequality which is generalization of (()) (2:21) inequality, which is when p is equal to q equal to 2.

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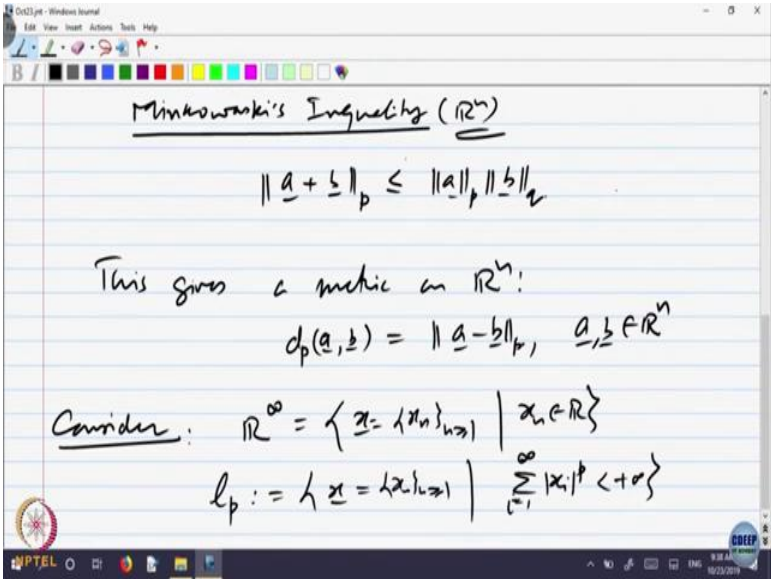

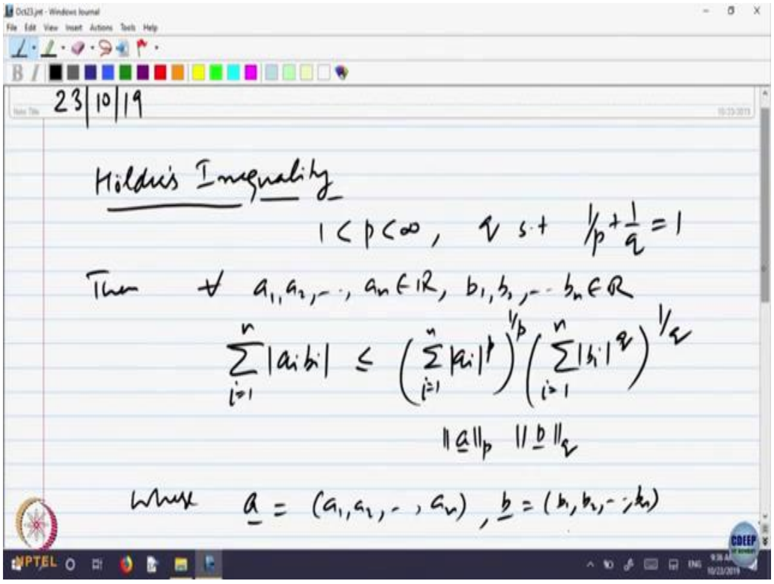

Using this we prove Minkowski's inequality. Of course on Rn, namely, norm of a plus norm of b, b is less than norm a norm b. So that gave us as a consequence of this, this gives a metric on Rn namely, distance P, the vectors a and b is equal to norm of a minus b for a, b belong to Rn. So that gave us the notion of distance, generalizing the notion of the Euclidean norm when p is equal to 2, so for every P between 1 and infinity, one gets a notion of a distance.

What we want to do is, we want to extend it further than Rn. So, we had started doing that, so let us consider R infinity so that is all sequences. So, you can consider this as a space of all sequences real sequences. Now of course, if you want to copy this notion of the norm, which may not make sense because a number of terms in that summation become infinite. So, one has to restrict as we saw, so we lo at what is called l p so that is all x in xn, all sequences such that norm of mod x i to the power p on to infinity is finite then this sum is finite.

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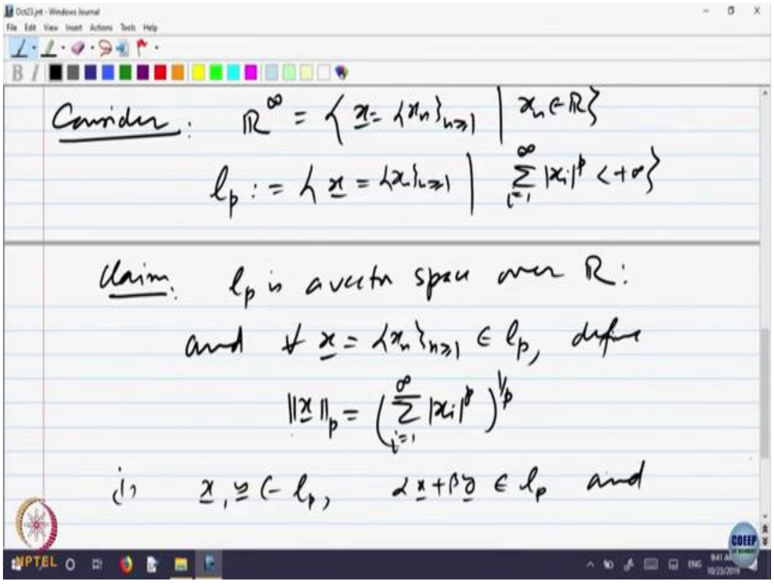

So, let us just put the claim which we had already loed at, but anyway let us just prove it again that l p l p is a vector space over reals, and for every sequence x n belonging to l p l p define the norm p equal to, p i equal to 1 to infinity raise to power 1. So we are just copying the norm of Rn, the P norm in Rn. And of course we have to restrict it to all sequences, which are pth power. This is, this is a series of non-negative numbers, which is convergent and it should be finite. So we will also lo at series soon, convergence of series, but for time being. There are saying there is a partial sums converge. So this is finite, so the claim is, is a vector space over R, so that means what if we define this then 1 for x, y, belonging to l p.

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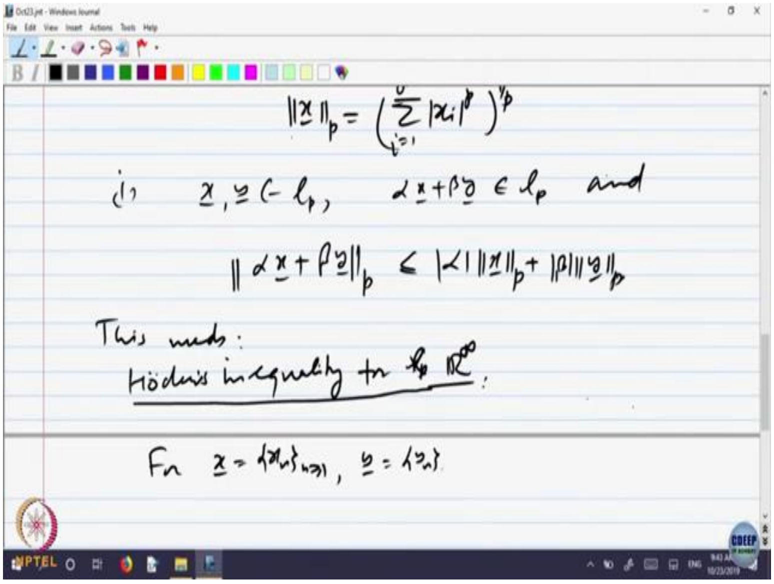

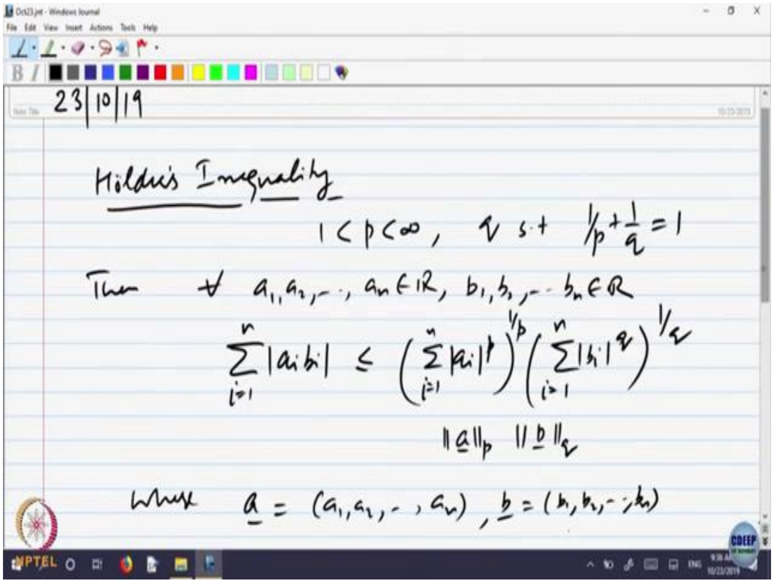

So, what does it mean we want to say that l p Alpha x plus Beta y also belongs to l p l p and the triangle inequality holds namely al pha x plus beta y p is less than mod alpha times norm of x plus beta times norm of y the pth. So, basically the idea of the proof goes as in the case of Rn, first step should be to extend Holders inequality from Rn summation 1 to n to 1 to infinity, and then using that proof, Minkowski's inequality because that proof does not require anything else other than the Holders inequality. So let us just proof Holders inequality, so this needs the Holders inequality for l pl p. One should say, not l p l p for R infinity let us write. So, what does it mean? That means for sequences x x n, y equal to y n, so what does Holder's inequality in R infinity will mean is the following.

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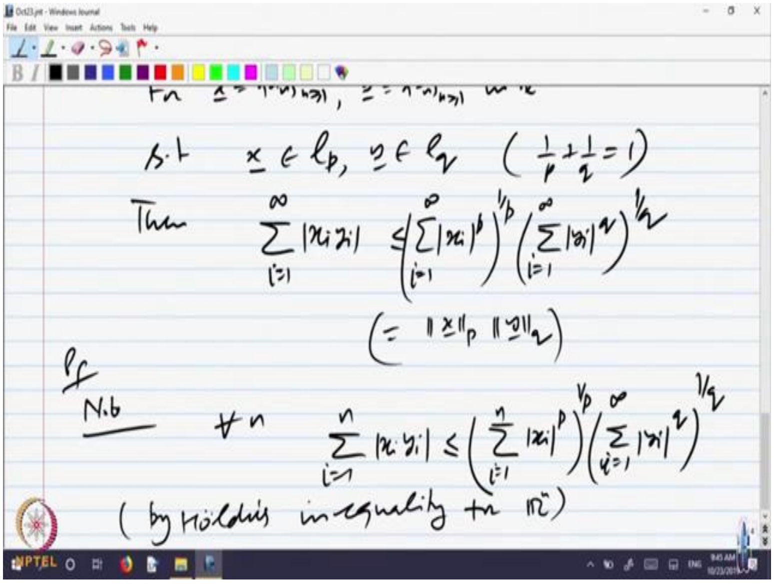

In R infinity given two elements, such that this x belongs to l p l p and y belongs to l q 1 over p plus 1 over q equal to 1, then sigma mod x i y i, i equals 1 to infinity is less than or equal to the corresponding norm so that is mod of x i to the power p sigma i equal to 1 to infinity, raise to the power 1 by P and sigma i equal to 1 to infinity mod y i raise to power q raise to power 1 over q. So that is perfect generalization namely, so this is x to the power p and y to the power q, pth norm and the qth norm, so this is same as this.

So the idea is, how do you extend that inequality, for Rn we already have it, so let us note. So note, so proof of this, for every n, if we just take the sum from 1 to n mod x i y i, then this is less than or equal to by the Holder's inequality on Rn, this is less than or equal to, mod y i raise to power q raise to power 1, so, this is by Holder's inequality for Rn, when the sums are finite up to n and this holds.

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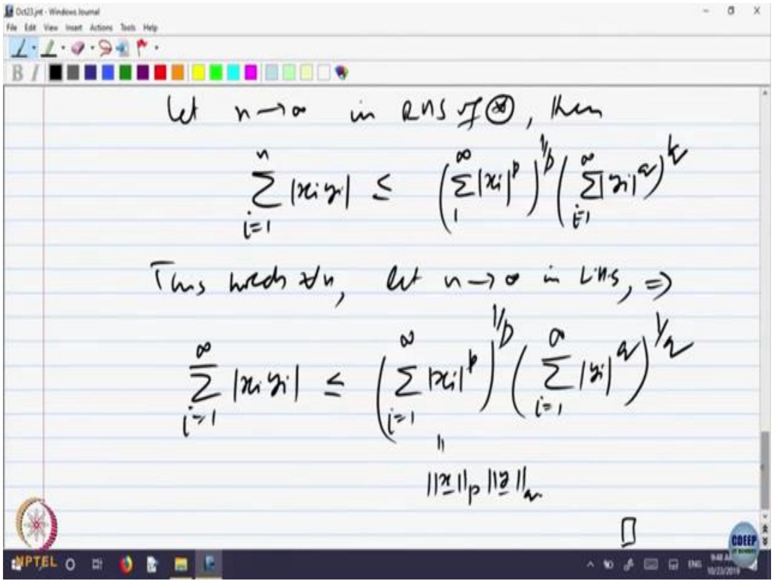

Now, this is something which is done very often in analysis, the left hand side here is less than or equal to the hand side for every n. So, on the hand side, let n go to infinity, keep the left hand side n as it is and let, so let n go to infinity in hand side of star, that means that then sigma i equal to 1 to n that essentially means, this is less than or equal to sigma 1 to infinity to the power p, raise to power 1 over p, y i raise to power q raise to power 1 by q. Essentially it means, it is adding up non-negative numbers, we increase n that will be less than or equal to the hand side. So, sums will increase and they all for every n it will be less than or equal to this quantity.

And now this holds for every n, now let n go to infinity in the left hand side of n inequality. So on this side, left hand side let n go to infinity implying that sigma i equal to 1 to infinity mod x i y i is less than or equal to this quantity, so sigma i equal to 1 the hand side which is as it is before. So, the idea is that essentially we are letting n go to infinity on this inequality, ay Holder's inequality for Rn, but the justification comes from the fact that we can let the n go to infinity on the hand side first and these quantities are finite by the given hypothesis.

So, for every n this holds, and we can let n go to infinity so, that proves Holder's inequality. So, this is equal to, so that proves Holder's inequality for R infinity, in the sense that if you have got sequences such that the sequence x is pth power summable and y is qth power summable, where 1 over p plus 1 over q is equal to 1, then the corresponding result for Holder's inequality holds.

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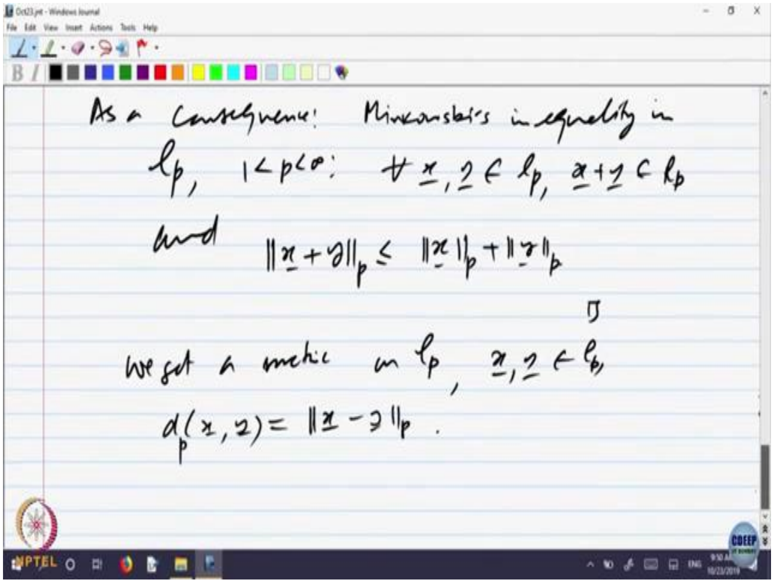

And using this so as a consequence of this, one proves Minkowski's inequality in lp, 1 less than p less than infinity, prove is same, there is no change at al other than writing Holder's inequality at appropriate place,. So we will not repeat the proof and that says for every x, y belonging to lp, x plus y also belongs to lp and is less than or equal to. So, we get on lp a metric, so we get on lp namely for every x belonging to lp, you have x and y belonging to we want to define a matrix, so let us write for x, y belonging to lp. Define d of x y to be equal to minus y. So Minkowski's inequality says this is precisely is a metric, it has triangle inequality property. So, what I am trying to show is that whatever you do in R 2, you can do the same thing in Rn and same thing as in R infinity,

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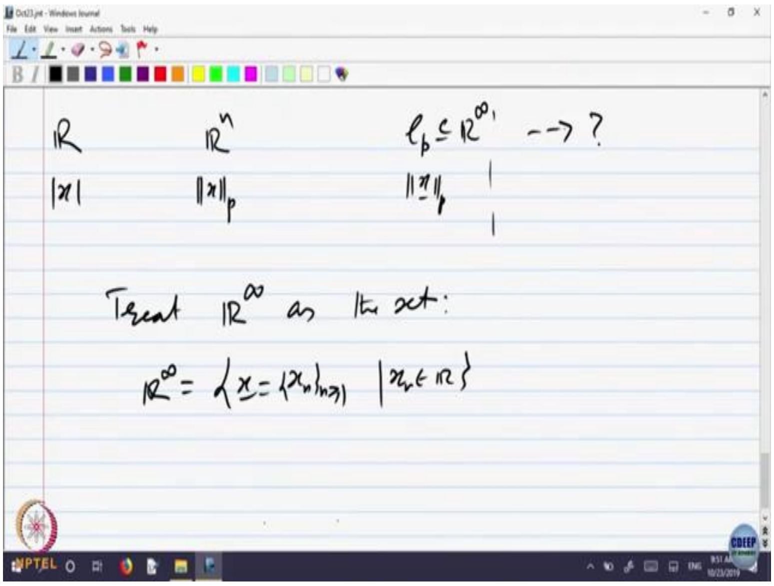

It is a quite interesting mathematically to ask the question. So we had the real line, we had Rn and we had lp which is a subset of R infinity. So, we had the notion of absolute value, we had the notion of norm of x to the power P and we had also, we had the norm of x power p basically, here is 1 to n and here it is 1 to infinity. The interesting thing is one can go beyond this and what should be. So, the idea is this R infinity treat so here is the R infinity as the set, so x, what is R infinity? That is a set of all sequences.

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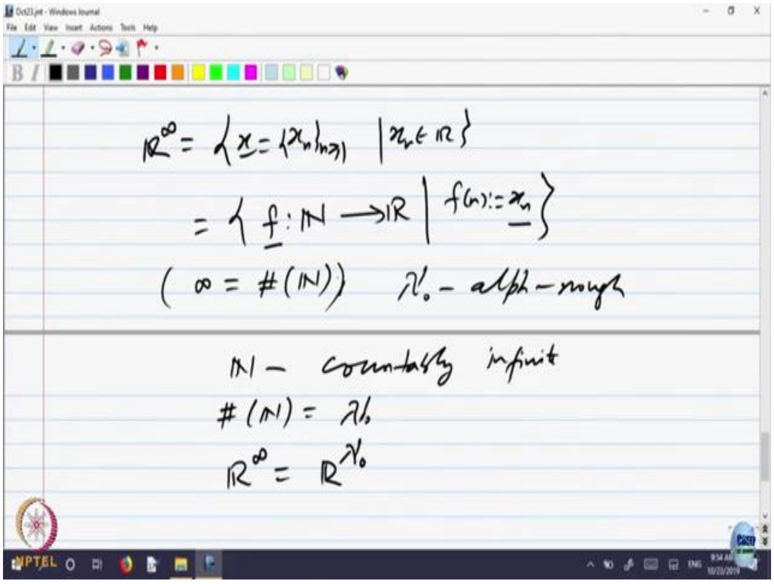

Let us interpret each sequence slightly differently and you will see how this change of interpretation. So let us lo at all functions defined on natural numbers taking values in R and what is f of n is denoted as x n. So, a sequence is treated as a function from the set of natural numbers to real numbers. So, every n goes to a point in Rn that you call as x n. So, when is a function known completely, if you know its images, so knowing a function f is same as knowing x n for every n.

So, that is an interpretation for sequences, creating a sequence of real numbers as a function on the set of natural numbers and here, this infinity so infinity is equal to how many elements are there in n natural numbers, they are countably infinite so, that is the infinity. I hope you all know what is called countable infinite and the set of natural numbers you want to say how many are there you, assign a number to it, which is called Alph naught and it is denoted by the symbol, this is called Alph naught. Alph is a Greek letter and naught is.

So, this is in some sense the first infinity you count 1, 2, 3, 4, n, go on counting and you reach a something, visualizes something infinity so that is counting and going on, not stopping. So, this is called so N is countably infinite. One says L is countably infinite and cardinality of it or the number of elements is Alph naught, so this is we can think as infinity. So, instead of R infinity, we like to write it as, it is better to write as Alph naught, so this is a better notation for R infinity. And this Alph naught is the cardinality of the set of natural numbers that is an indexing set and that is a domain here coming.

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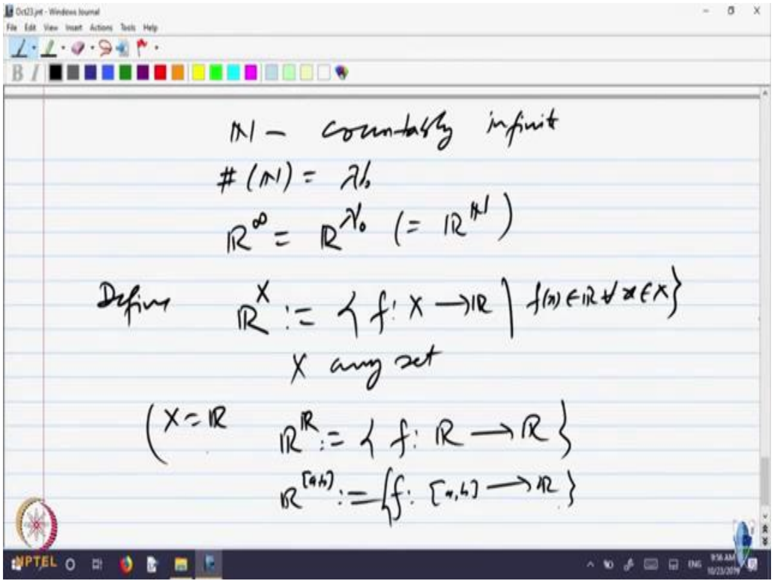

So, this gives us a way of extending, so define instead of natural numbers, let us put something any set x and what is this? So, this is all functions x to R, f of x belongs to R for every x so we just R to the power infinity, sorry Alph naught or this is same as R to the power n natural numbers if you want to write it as a set, so generalize it, just replace. So, this is the definition what is R to the power x, where x is any set. So one can interpret it that way. For example, if x is equal to R so what is R to the power R? That will be the all functions from real line to real line. What is R to the power a b that is all functions f from interval a b to R. So, that is another way of saying what is this object.

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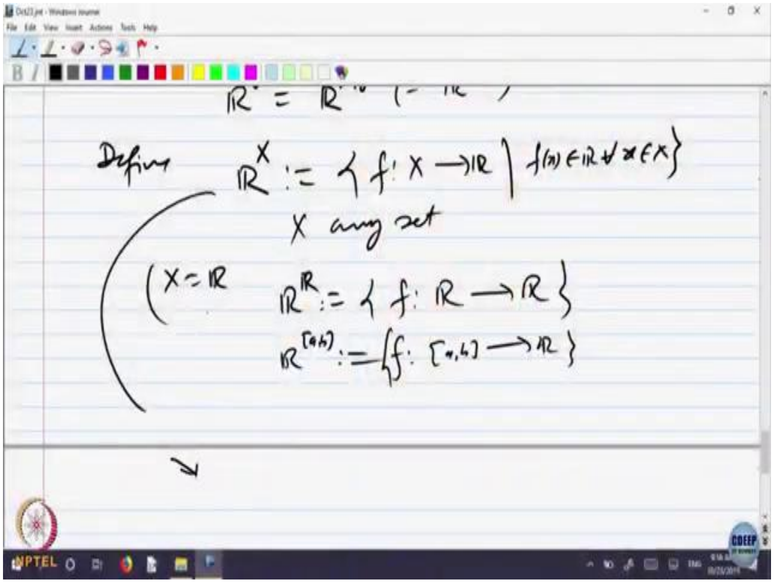

So, now let us look at this R x, so consider I am just trying to illustrate the way mathematicians think and generalize. So R infinity we had and we had defined R to the power x, so these all functions f from x to R. I want to copy that idea lp l2, I want to copy that ideas. We had the notion of 1 1, we had l 2 that is the ordinary Euclidean distance, we had l infinity that is the supremum, we can try to copy all of them on this set now, so let us try to copy this.

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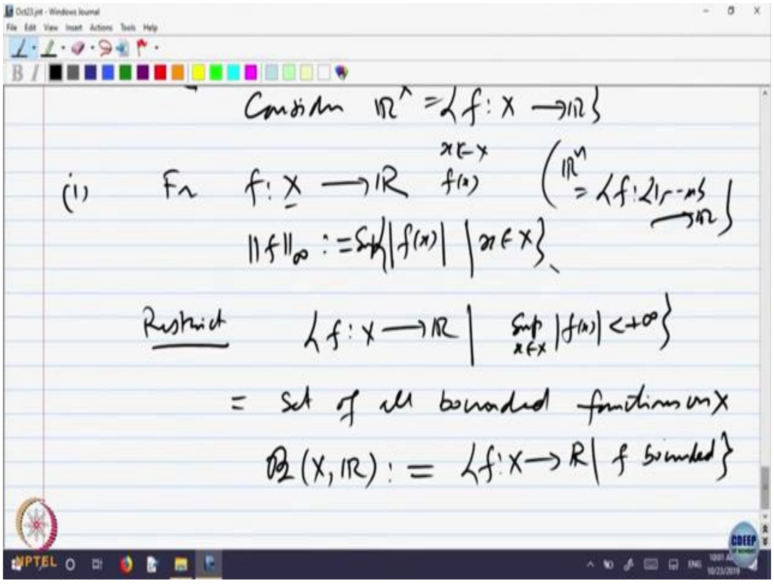

So first one, for a function f x to R, I want to define what should be this thing for x n for the function defined on Rn. So, what is Rn? That is same as f defined on 1, 2 up to n to R, n components. And what was this thing define for a function, for a vector we looked at the supremum of the components. For a function what are the components, they are infinite, they are as many as. So, you can treat for every x belonging to x, f of x as its component, xth component you can think it of.

So, if you think it a function as a vector with as many components as the number of elements in the set, then for every x, f of x the value is the component that is what is happening in sequences, that is what is happening in vectors. So, look at the xth component, look at the mod of that and what is our supremum, so let us take the supremum of this, here x belongs to x. So, copying that supremum thing, but the problem comes, this supremum may not exist, because we know the completeness property of real number says, every non-empty subset of real numbers which is bounded above will have a supremum.

So, this set may not be bounded above, so one has to restrict now, instead of R x so, restrict. So, look at all functions x to R such that supremum x belonging to x mod f x is finite. You see, automatically those similar conditions we had put earlier when sigma mod x i square is finite, x i to the power p is finite. So, for functions we should put this condition. So, what are such functions, if a function f x to R whose supremum exists that means it is a bounded function that is same as this is equal to set of all bounded functions on x. So one just writes, M, X, R. You can write any notation, you can write B here to indicate, let us write B instead of m, let us write B ZX that may look like ball of radius something, so let us write some funny B called script B, how do we write script B, script B, X, R all functions f x to R f bounded,.

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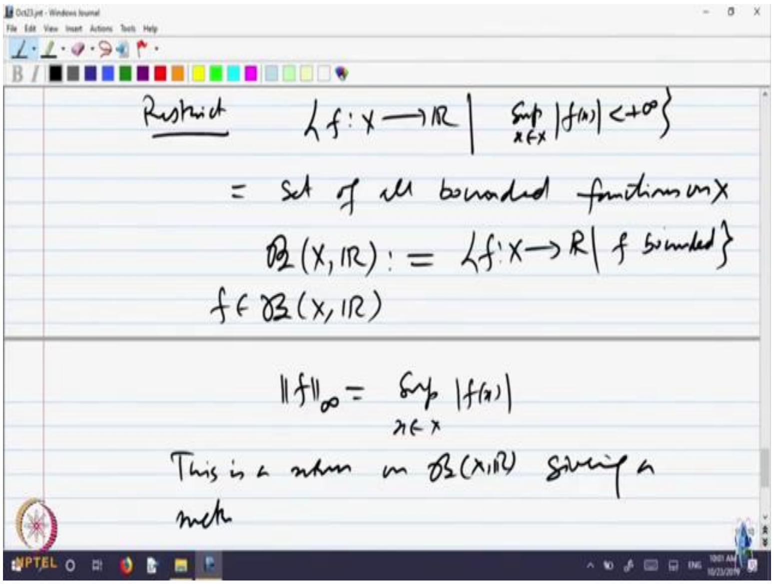

And for any function f belonging to B, X, R, we can define to be equal to supremum x belonging to x mod of f x. And this becomes this is a norm on B, X, R giving a metric. So it gives a metric, so what is a metric?

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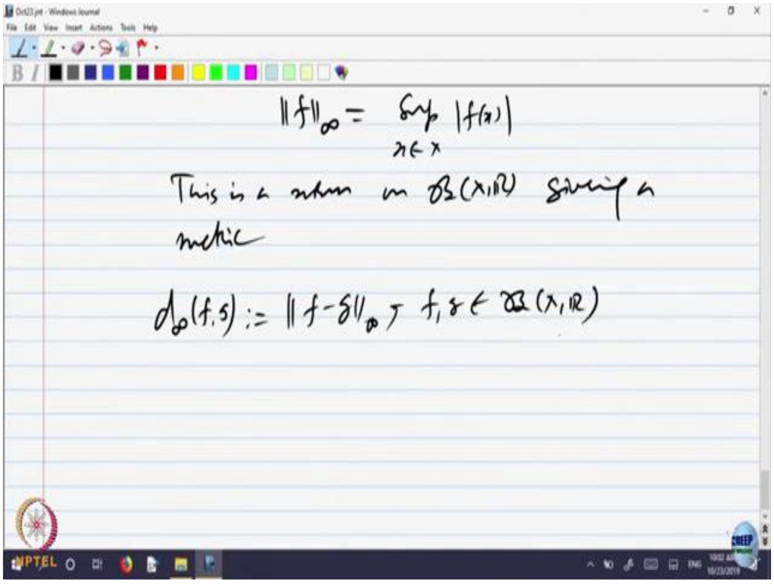

As once norm magnitude is defined, the metric is so d infinity f-g is equal to norm of f and g belonging to B, X, R. So, basic idea is defining a norm absolute value for.