Basic Real Analysis
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Lecture- 64
Pointwise and Uniform Convergence - Part III

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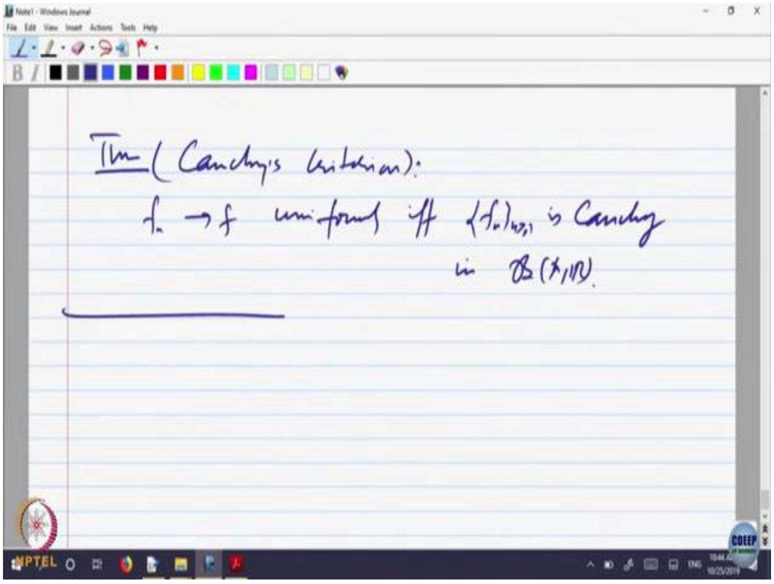

This goes by then theorem Cauchy Criterion for so this is a theorem, let us write it as theorem, Cauchy's Criterion, fn converges to f uniformly if and only if, fn is Cauchy, Cauchy in B X, R. So, that is, so we have seen Uniform Convergence, we have seen when a sequence does not converge uniformly a criterion for that. We have seen now a criterion which says, when does a sequence converge uniformly, namely it should be converging in the B X, R and that is same as saying it should be Cauchy there. So, we know we can test sequences being converging uniformly or not.

So, let us finally look at whether this serves our purpose of saying that, when pointwise convergence limit was not continuous functions, converging pointwise the limit was not continuous convergence uniform, differentiability did not imply the derivatives converge. And integral also we had an example that, fn's are integrable converging pointwise did not imply that limit is integrable. And these are important things as far as the convergence problems are concerned.

We were important problems in analysis because, whenever you want to go beyond algebra, algebra is 1, 2 and 3, only finite number of operations, when you, you can look at only polynomials. To generate functions which are nonpolynomial, nonrational functions, you have to go to analysis that is why we had to do all that exponential logarithm, trigonometric all those functions were cannot be obtained using algebra alone, you have to go to limits. So, things that are preserved.

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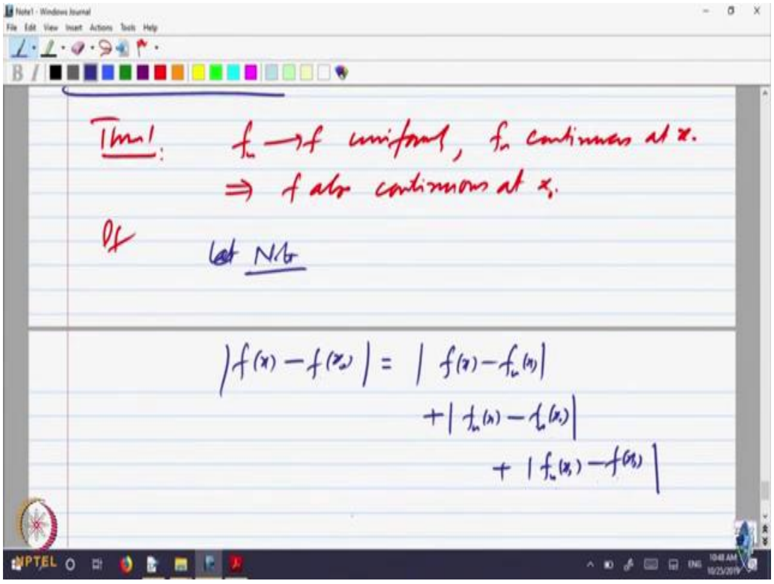

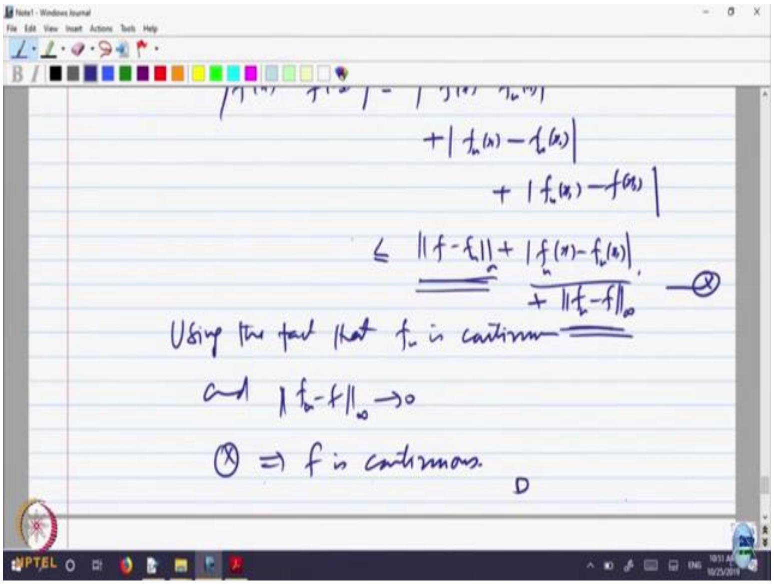

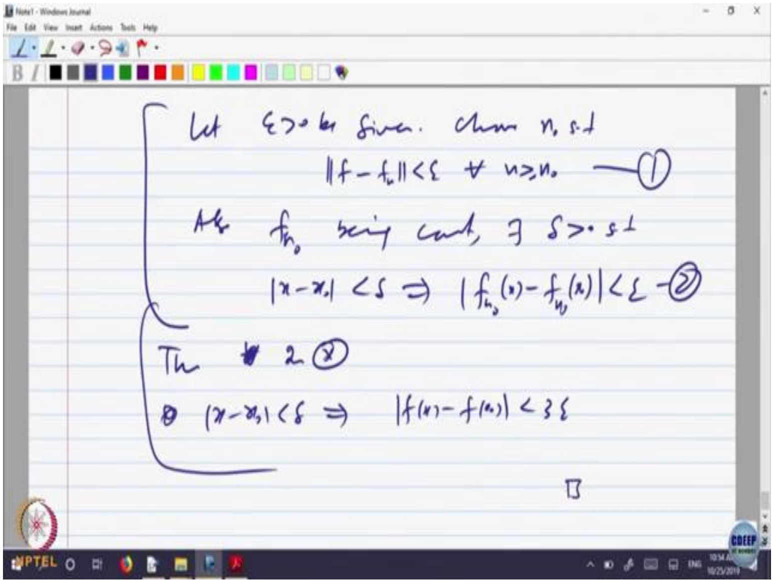

So, let us write theorem 1, fn converges to f uniformly implies and each fn continuous at some point x not implies f also continuous at the point x naught. So, let us write a proof, that continuity is preserved in uniform convergence. So let, we want to analyse so for convergence what we have to analyse, f of x minus f of x naught, this difference right. We want to show f is continuous, we have to make this thing small whenever x is close to x naught. So, fn converges to f uniformly, so let us how to somehow bring fn inside it right.

Because something is given about fn so let us bring in, so this is fx minus fn of x plus, I have subtracted so let me write fn of x minus fn of x naught plus fn of x naught minus f of x naught. I am adding and subtracting, what I am f of x, I subtracted fn of x, I add fn and then I subtract fn at x naught add fn at x naught and the final term is there as it is. So, I have added and subtracted use triangle inequality.

Why I am doing that, I am forced because something is given to me about fn to use that fact I have to bring fn, then only you can use it. And now, if I look at this quantity, the first one is less than or equal to norm of f minus fn, the first term, this term. And this term, let us keep it at it is if you like, so it is mod of fn x minus fn at x naught plus and this quantity again is a same point. So, less than or equal to fn minus f norm infinity.

And now, I know that this quantity is small because fn is converging uniformly to f, both these quantities are small this and this. And what is this quantity middle one, that is fn of x minus fn at x naught. Same function at the point x naught but fn is given to be continuous, fn is given to be continuous. So, this quantity is also small so we can just simply write, so using the fact that each fn is continuous, using the fact that each fn is continuous. And fn minus f goes to 0, so this equation star implies f is continuous. I am not writing that epsilon every time kind of a thing. So, if you want to write all that thing, you can write given epsilon bigger than 0.

I can find some stage n, so that this quantity is small, this quantity is small less than epsilon. And by continuity, for fn after that stage, for that stage this will be less than epsilon, by continuity. So, I can find a delta such that you want me to write everything, let me write so that you feel happy. So, what does this mean, so here is what I am saying. Let, epsilon greater than 0 be given. So, there is how formal proof will be written. Choose n naught such that, mod of f minus fn is less than epsilon for every n bigger than n naught.

That is by using the fact that, fn is converging to f uniformly, also for, so let us take this fn n bigger than n naught. Let us take n naught itself, fn naught being continuous, there exist a delta bigger than 0 such that, mod x minus x naught less than delta implies fn naught x minus fn naught of x naught is less than epsilon. So, this is 1, this is 2 then in this equation, star then in star put this values. So, this is less than epsilon, this is less than epsilon, this is less than epsilon and this is less than epsilon, when x is closed to x naught by delta.

So, you get, so in star x minus x naught less than delta implies fx minus fx naught will be less than 3 epsilon, does not matter. If you want it very nice you get off cut down everything in, so that is an idea of the proof. So, you should understand what you are doing saying this thing I can manage, I can make it small by using the fact that fn converges to f uniformly. I can make this thing is small, using the fact that fn is continuous at the point x naught and I can make this small again by uniform continuity.

So, the middle term is small whenever x is closed to x naught by continuity, this will be small. So, that is what we have written, given epsilon there is a delta and so on, so that proves the thing. So, this is, that means continuity is preserved in uniform convergence.

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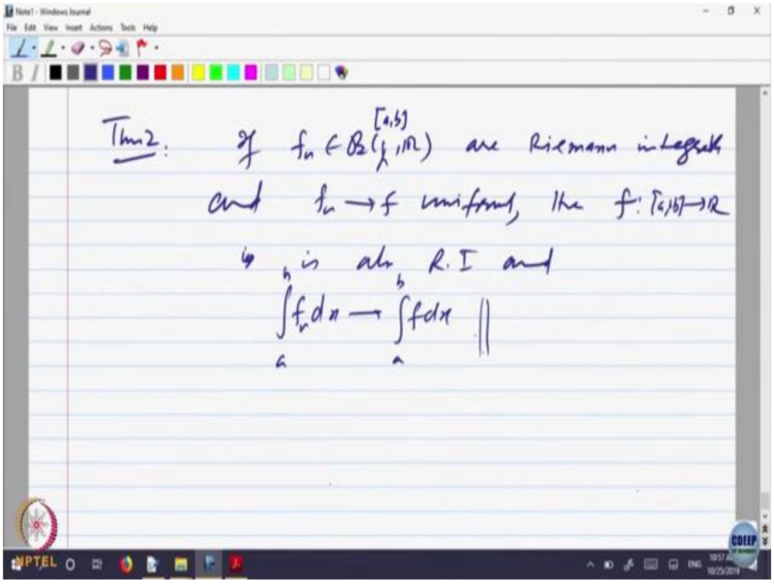

So, let us prove that, so theorem 2 says, if fn belonging to B X, R are Riemann integrable and fn converges to f uniformly, then f, I should Riemann integration so I should not write X now. Because Riemann integration is only for some interval, so we are taking X is equal to the interval a, b. Integrability of functions is defined Riemann integrable only for closed bounded intervals. So, here x is taken as, so we are looking at all bounded functions on a, b to R, which are Riemann integrable.

And fn converges to f uniformly, f is a function on a, b to R, then is also Riemann integrable and integral of fn a to b dx converges to integral f dx, a to b. So, one has to put a slightly stronger condition, pointwise we saw that even the function may not be integrable at all, limit may not be integrable. In fact, you can give examples of functions where fn's are Riemann integrable, fn's converge pointwise and the derivative function is integrable but the integrals do not converge. So, many such examples are possible we are not going into all those things.

What we are saying is uniform convergence at least is a one criteria which allows you to pass on under limits, integrability is preserved. I think will prove it next time because the proof requires a bit of more partitions and so on. And we will also analyse what happens to differentiability, will show that differentiability, also is not preserved in the pointwise convergence. And in fact, one has to put very strong conditions for differentiability also, even for uniform convergence. So, these are slightly differ theorems. And this, this thing that vendors integral of fn converge to integral of f for Riemann integrable functions.

As I had mentioned is a beginning of the story of Liebig integration. In general, it does not happen, it happens for uniform convergence. What other class can happen that is, so we will stop here.