Basic Real Analysis
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Lecture 08
The LUB Property and Consequences – Part 2

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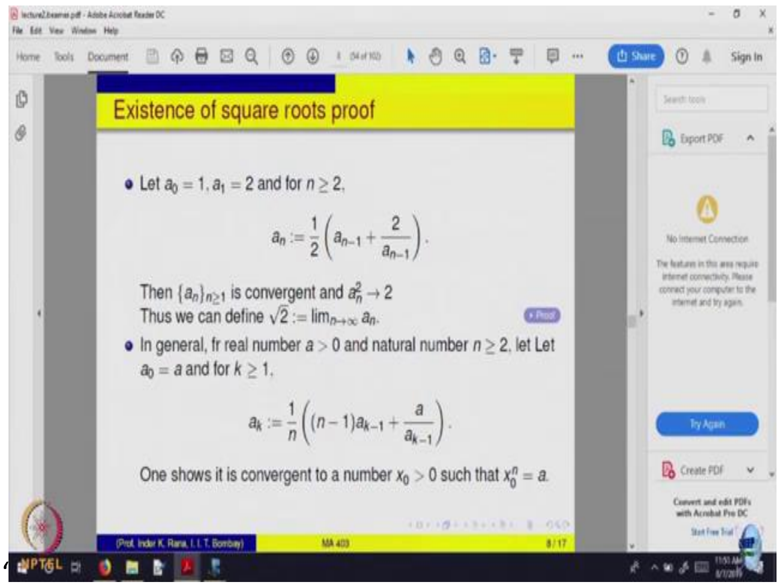

So, let us define a sequence a0 to be 1 and a0 to be 1 a1 to be equal to 2, inductively for n bigger than or equal to 2. So, what is an is half of an minus 1 plus 2 by an minus 1. You will wonder from where this equation is coming, I do not know weather you will have a course in where you will have Newton Raphson method of finding 0s of a function, so it comes from there actually. So, there is a Newton method of doing it things. So, anyway probably in the calculus if we do something of that type we will probably indicate.

So, the claim is that this sequence an is convergent, this sequence an is convergent is a sequence of real numbers, claim it is convergent and an square converges to 2 that is what we will show. Now look if we are able to proof this each an is a rational number. What you are doing you are adding, dividing, multiplying rational numbers only, a1 is rational, ans all are rational numbers, all ans are rational, yes, by the property that rationals form a field. You can add, multiply, divide rationals and not 0.

So, an is a sequence of rational numbers such that an square converges to 2. So, where will an converge? It will converge to a number which is not a rational number. So, this is Cauchy sequence of rational numbers which does not converge to a rational number, it converges to a what we now call as irrational number normally square root of 2. So, the proof is not very difficult, we will prove it (())(2:31) induction.

So, the idea is that try to show that this is a monotonically increasing or decreasing, you can try to show it yourself, if you want you can note it down or when I say give you the slides, try to proof that this, here is the application of at every monotonically increasing or decreasing sequence which is bounded above or bounded below must converge. So, using that property one shows that this sequence an must converge.

If the sequence an converge, what will be a limit, can you guess the limit from here from this formula, an converges. So, l the limit must be equal to half of l plus 2 by l that gives you a quadratic n l and that says is a quadratic, l must be equal to square root of 2. So, that is a idea of the proof. So, try to do it yourself, I will not ask you in the exam or such things, but it is nice to have a sequence which is rationals, which is monotonically increasing, bounded and hence, convergent but the limit is not a rational number. So, that you can think as the need for constructing real numbers.

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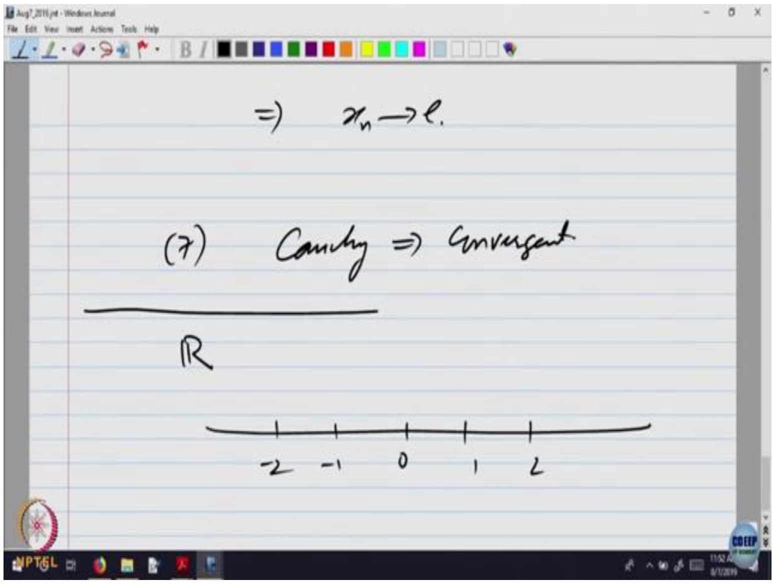

So, let us, now next thing that I want to do is I want to talk about some subsets of real numbers. So, we have got the set R which is a complete ordered field various ways of analyzing convergence of sequences and geometrically this is… let us assume we have got this 1 to 1 correspondence that real line can be realized geometrically as set of all points on the line. Every point represents a number if a point, 2 points x and y, y is in the right side of x then it is bigger that is order. As you go from left to right your numbers are increasing and these are the milestone 0, 1, 2, 1 so on.

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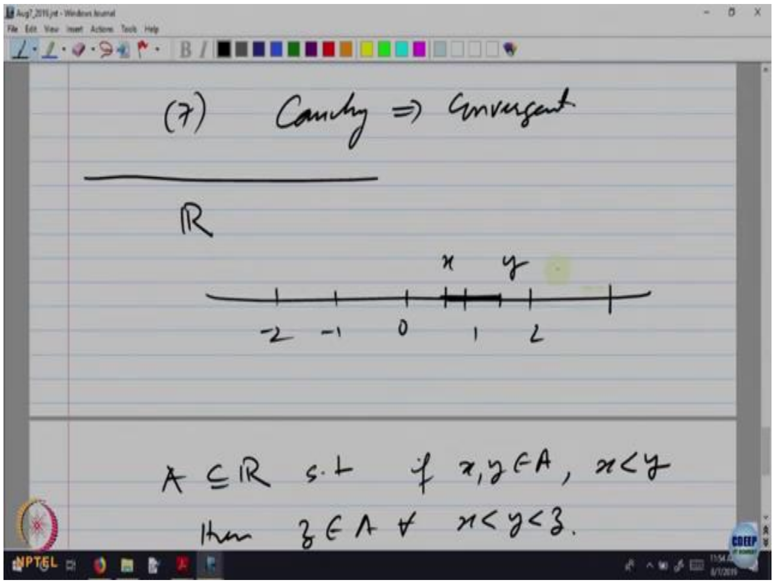

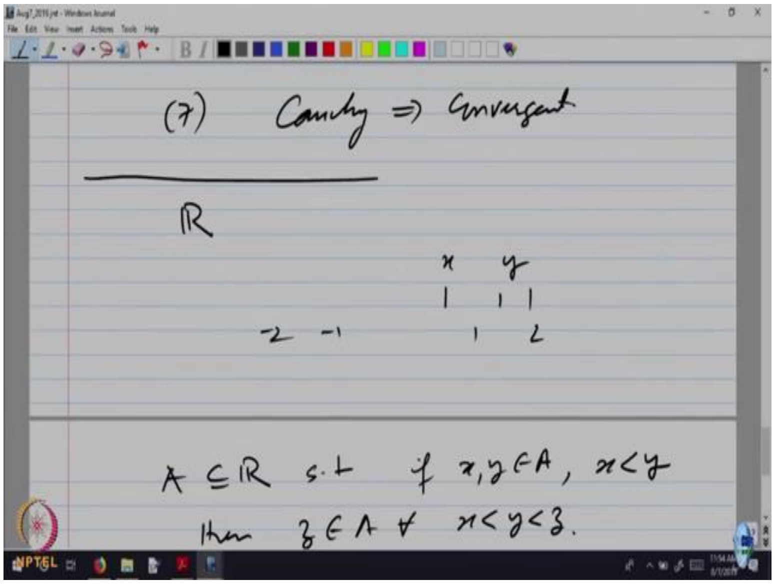

Let us look at subset A of real line with the property such that, if x and y belong to A let us say x is less than y, then z belongs to A for every x less than y less than z. We are looking at those subsets of real line which have the property. If there are two points x and y in A then either x will be less than y or y will be less than x. Then look at all points z which are in between x and y, they should also belong to that set.

So, A is a set with that property. So, such a set is called, what should we call it, it is like a new baby being born in our class, in the mathematics class, we want to now give a name and in mathematics names are given which signify the property of that object. So, let us look at geometrically here is x, here is y if these 2 points are in a then everything in between must be, if this is another point which is inside A then the whole of this must be inside it. It looks like it is part of the line it is a segment of the line.

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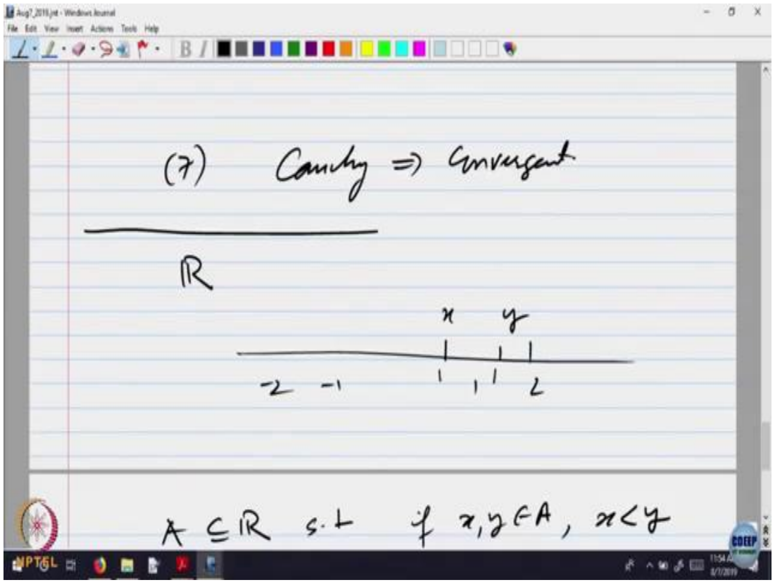

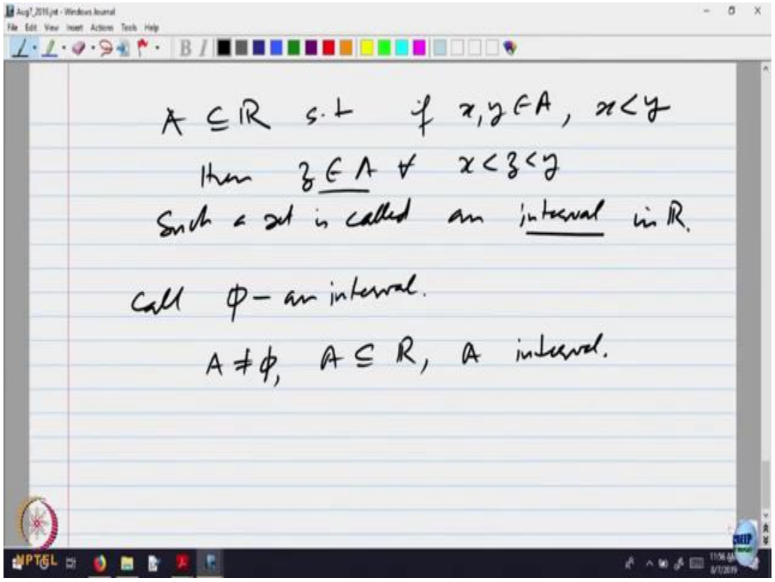

So, it is a interval of the line. So, we call such a subset as interval is called an interval, is called as a interval in R.

Student: (())(07:02)

Professor: x is less then y

Student: (())(07:14) x is less than y

Professor: This one for every z less then y. So, let me right correct it, you are right. This is what I had in mind then for every x less than z less than y, the z must belong. Anything that is in between, caught in between 2 elements of the set or must be inside the set. That means there should not be any gap, there should be continuous of points. So, a such a thing is called an interval.

So, let us declare call empty set an interval, if you like you can call an empty set interval or empty set by definition is an interval because to check a set is interval, what we have to check. If there are 2 points but a set has no points. So, vacuously our statement is true, this is what you said vacuously our statement is true. So, or if you do not like such kind of arguments you can say let us declare empty set to be a interval.

Now A let us assume it is not empty and is A subset of R, A interval. I am trying to now visualize what a interval should be. So, it is set which is a non-empty and it is an interval. So, A must have lub property, if A is bounded. Every, lub property says what, every non empty subset which is bounded above will have least upper bound, if it is bounded below it will have greatest lower bound.

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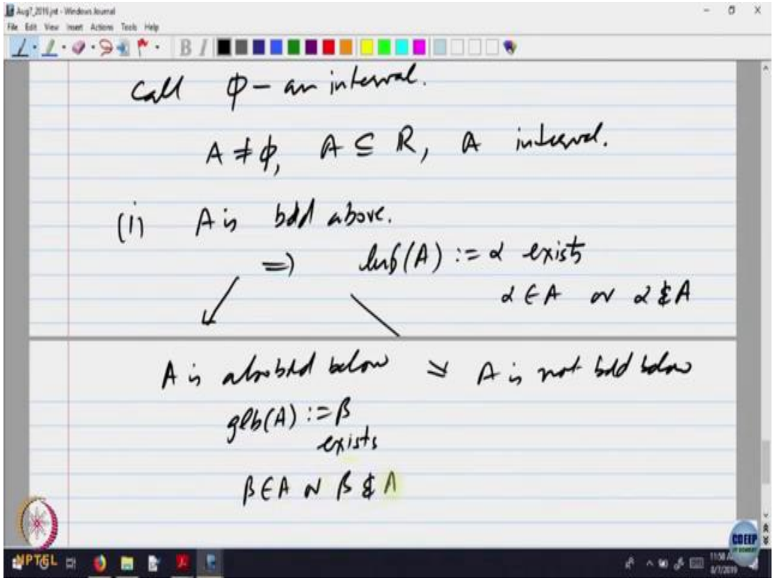

So, possibility is first possibility A is bounded above. If A is bounded above then what must happen implies lub of A call it as alpha exists, lub of A namely alpha exists. Now what about below it may be bounded below it may not be bounded below. So, let us look at 1 subcase A is also bounded below. Suppose it is also bounded below other possibility will be A is not bounded below.

Sub case is, if it is bounded below then greatest lower bound of A call it as beta exists. It is not bounded below that means given anything I can find something smaller it goes on. Now another possibility comes, this alpha which is a least upper bound may belong to the set may not belong to the set. So, alpha belongs to A or alpha does not belong to A. Similarly, this beta may belong to A or beta does not belong to A.

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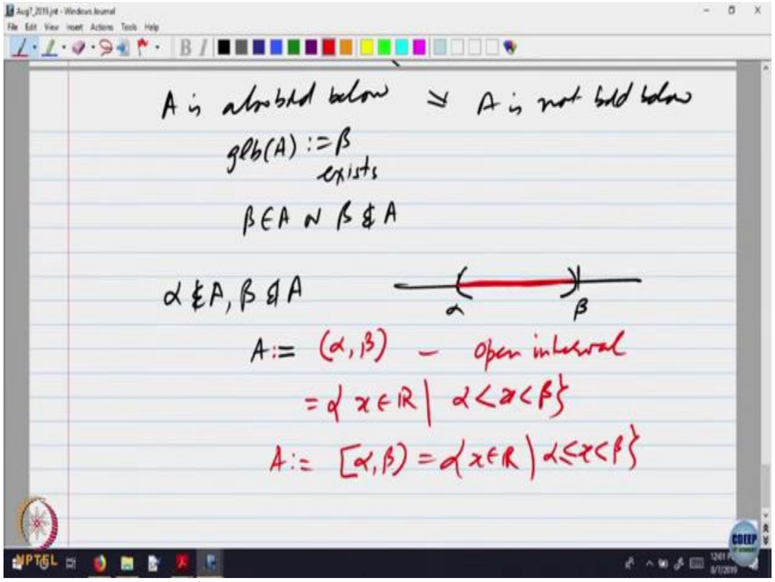

So, the case is alpha does not belong to A, beta does not belong to A. Let us analyze that case. So, here is alpha here is beta both of them do not belong, is it okay? Then what portion of the line is A. Anything that is bigger than in alpha that is in A because alpha is the greatest lower bound, anything which is smaller than beta also is in A. So, it looks like it should be this part of the line. So, we write alpha comma beta. So, we write A equal to alpha comma beta. Is it okay the notation now?

Because anything bigger than alpha is going to be in A because alpha is a greatest lower bound and beta is the least upper bound. So, anything smaller has to be inside A. So, this portion of the line must look like the set A. So, this is what is called the open interval. So, a interval, we call it as an open interval. So, what does it look like? So, this as a set is all x belonging to R such that alpha less than x less than beta. Is that okay?

So, other possibilities now I think you can try to write it yourself. If alpha belongs to A, if alpha belongs to A but beta does not belong, that is another possibility. So, then you will write A equal to alpha belongs and beta does not. So, that is equal to x belonging to R, such that alpha less than or equal to x less than beta. Similarly, you will have all other possibilities this is for the bounded once. If it is not bounded above so let us look at the case the set is bound.

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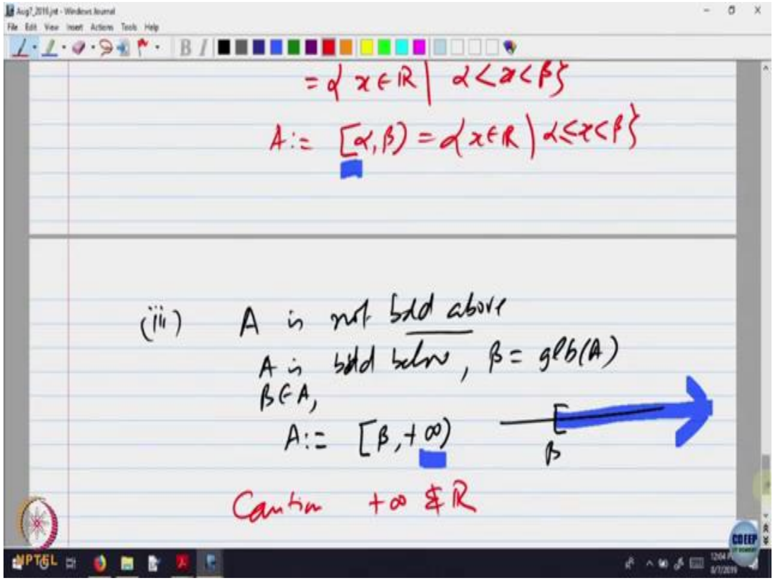

So, just for the sake of 1, A is not bounded above, A is bounded below and bounded below what was given name something, anyway does not matter. If it is bounded below, above alpha below beta bounded below by beta, beta is equal to greatest lower bound of A. Then what should A be signified as? Depending on weather beta is part of A or not. So, let us say beat belongs to A that is also given say, then it should start at beta. Anything bigger than beta is part of A and it is not bounded above.

So, everything bigger than beta is part of A. So, we should write it as something which goes on and that is denoted by this symbol called plus infinity. So, this is, it is like here is beta and you look at all something that is going on that kind of a thing. So, you generate all kind of intervals that you have been probably familiar with. When there is a square bracket that means that is a part of the interval, when it is open that means it is not part of the interval.

So, we will have both sides. So, that is called a closed and point inside that we saw is, that side is closed. Alpha belongs, alpha is greatest over bound so square bracket will come we will say A is closed on the left, open on the right. In words if you want to say that but keep in mind this thing is not a number. So, I think caution I should write because caution plus infinity is not a number. It is only a symbol to indicate that you are not stopping anywhere on the right side.

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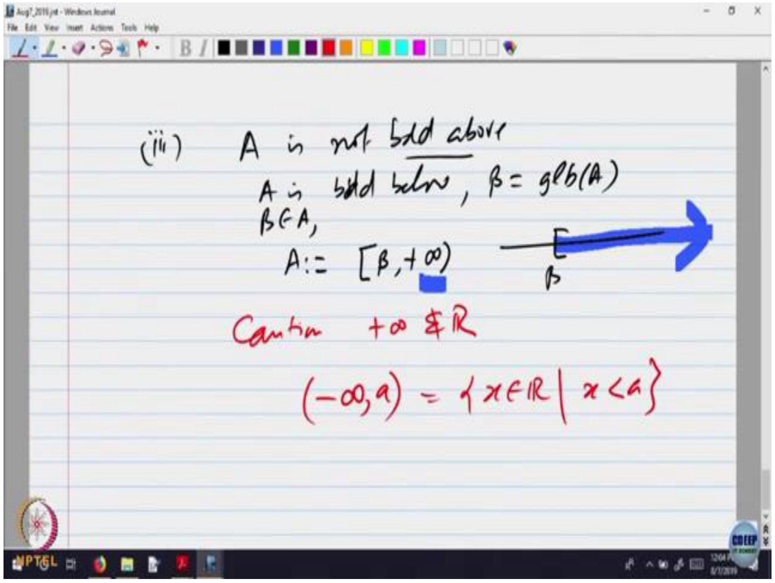

Similarly, you will have something like say minus infinity to a. So, that will indicate it is all real numbers which are strictly less than a and a is not a part of that interval. So, this is same as all x belonging to R such that x less than a. Can we include plus infinity and minus infinity as part of the number system? You understand what I am saying, we have got set of real numbers, we want to introduce to more elements int the number system, real numbers, call them as minus infinity and plus infinity.

But if you want to introduce to lose symbols in that we should tell how are they going to behave with respect to addition, how are they going to behave with respect to multiplication, how are they going to behave with respect to the order structure on the set of real numbers. Because already there is a order, there is a structure on real numbers addition, multiplication and order. If 2 new elements are thrown in, they should be better informed how will you interact with addition, how will you interact with multiplication, how will you interact with order.

Otherwise the system will become unstable. It is like in a house, some people are already staying and they have some rules and regulations of the house. We will not do this, we will do this, we will sleep at this time, wake up at this time and so on and 2 new guest come. So, you would like to inform the guest that "apni chappal bahar utar kar aana ghar mein nahi lana. Hum aath bajay khana kha lete hai."

So, that your house is not disturbed. Same way real number system has got some structure some properties when you add 2 new elements, you have to specify the rules and regulations how will they behave with respect to those structures. This is called extended real numbers. So, let me stop here saying that you one can do that but one has to specify the rules and regulations. So, let us stop that.