Basic Real Analysis
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Lecture 9
The LUB Property and Consequences - Part III

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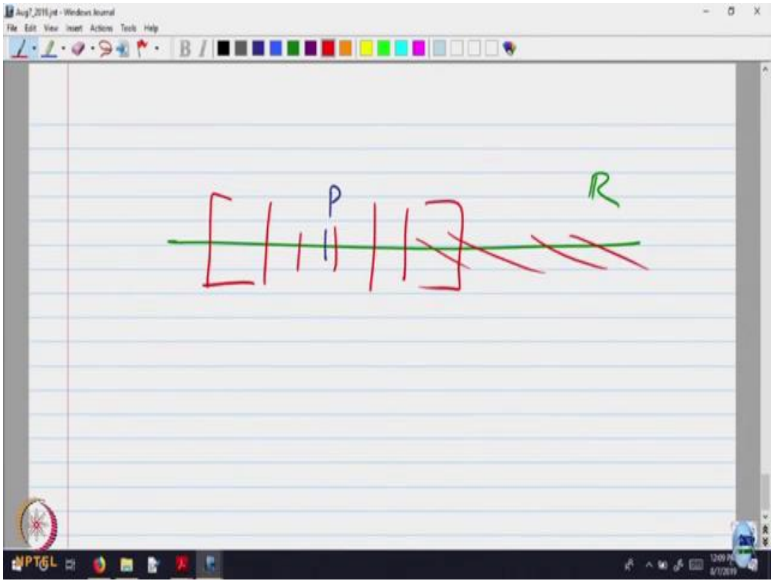

So, intervals. Now, here is a property of intervals which is very important, so let us take a, I think he said do not use red, because it looks bad in the, okay so let us take a green one, so this is a line a real line. So, let us take a point on the line, so this is a point p, right? I want to locate that point p, where is that point? p is a point on the geometric object namely the line, okay?

I want to pin point where is that point, imagine as if there is a big road, right? Going from one into another, right? In a desert, there is no sign or anything anywhere around, and there is somebody stranded on the road and he sends a message that, "locate me", "find me", "help me". So, how will you locate that point? So, where is the possibility, right?

Let us draw some point on the line, some object, right? Some city, so try to see whether the person is on the left, so the person is able to answer you, yes or no. Are you on the left side of that city or the right side of it, then if it is not on this side, then our search becomes only on that part. So, I can go on doing that, right?

A better way would be, let us take a interval, are you between this city and that city? Have you crossed that city? Have you reached that city? So, no, so you are in between that city, right? Now, I want to narrow down the search, so I will make it smaller and make it smaller, so search, search window smaller and smaller, so eventually I should be able to pin point that point, where is that person, is it okay? So this we want to formulize mathematically, okay? So, let us write down the statement and then try to prove it.

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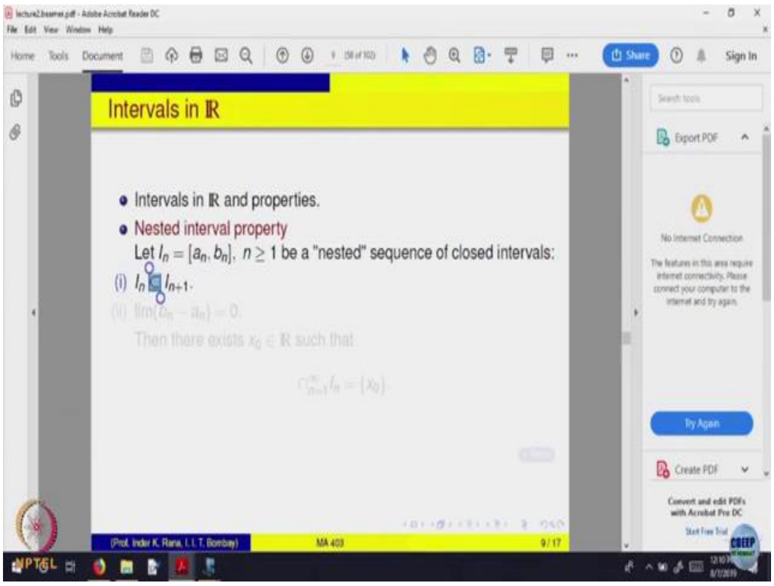

So, here is, let us take a sequence In of intervals, so keep that picture in mind now, at the nth stage you have got the interval An to Bn, okay? And we say it is nested, if the next one is inside this, we want to make it shorter, right? The next interval is inside locating.

So with the property that In is a subset of In plus 1, right? Is it okay? This is saying In is inside, In is bigger, In plus 1 is, is it right or wrong? Which way, have I written it rightly or wrongly? In plus 1 should be smaller?

Students: Yeah.

Professor: So, this is written wrong, right? Is there typo in this? Yes, so this one should be In plus 1 should be inside In, I want to say that, right? That picture In In plus 1, smaller part of it, so this is wrong here.

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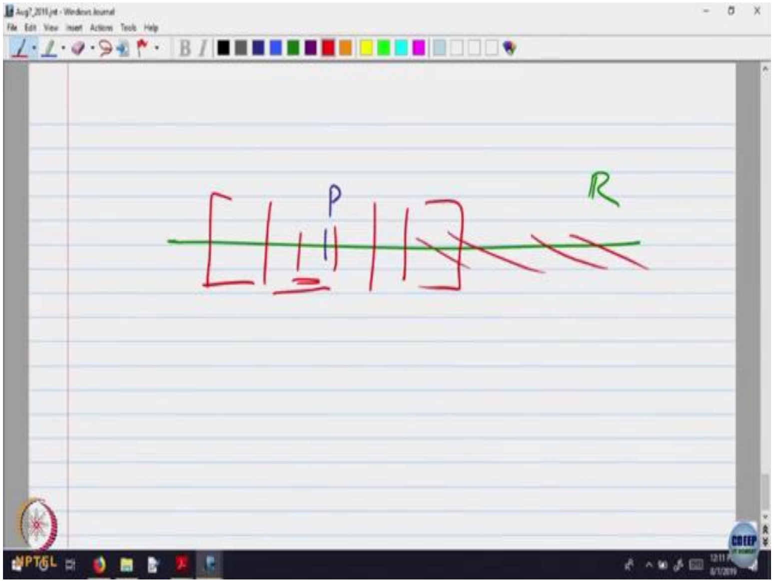

And what should happen, in that picture I want to make it smaller and smaller. So, how do you say that really they are becoming smaller and smaller? Eventually they should capture that point, that means the sequence An that is a right hand points, left hand points Bn, the difference of the two should go to 0, the length of the interval, right? Should go to 0, what is the length of the In? Bn minus An, that should go to 0.

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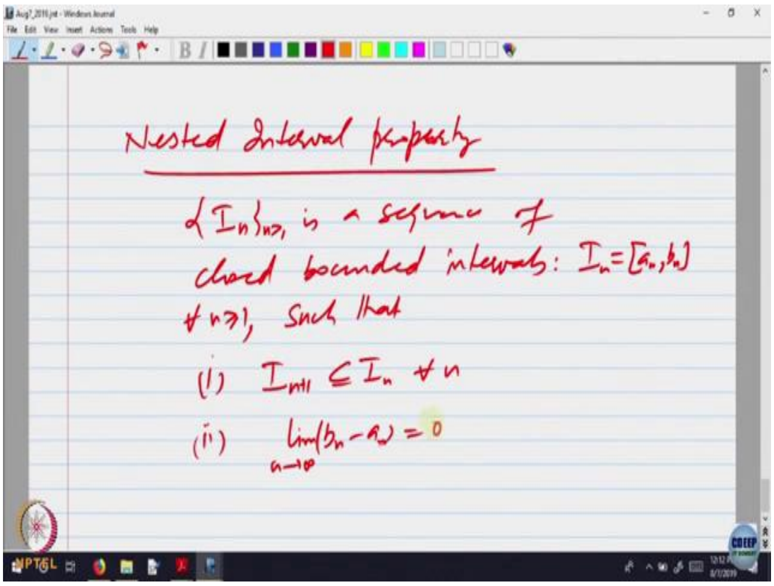

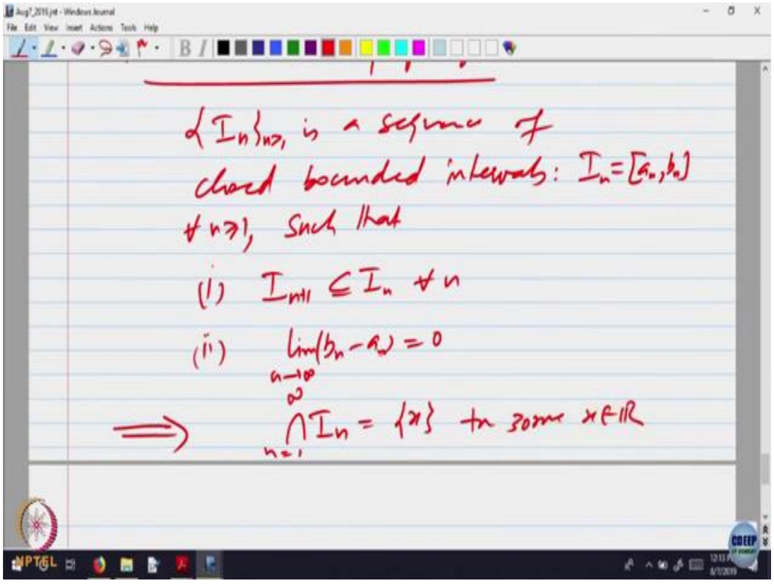

So, I want to prove, so let us write here this is called nested interval property. So, In is a sequence of closed bounded intervals, In equal to An, comma Bn, for every n bigger than or equal to, such that 1 In plus 1 is a subset of In, for every n, right? Second, limit Bn minus An, n going to infinity is equal to 0, right? The intervals are becoming smaller and smaller that is captured by saying, this is 0, so this implies, eventually what do you expect? I should capture that point and only that point, right?

So, it says intersection of In's 1 to infinity is a singleton point x for some x belonging to R, is a nested sequence means next one is inside the previous one, right? That is why we said it is nested interval property, is a sequence of intervals which is nested, right? Next one is inside the previous one and they are becoming smaller and smaller, so saying that the length is going to 0.

And one more property very important that, they are each In is a closed bounded interval, otherwise you cannot say Bn minus An is going to 0 anyway, anyone other In is unbounded, then you cannot say that, right? Length will be infinity; it will be bigger than anything, so it cannot go.

So, either I should prove, okay let us prove this first, and then say why all these conditions are required, okay? See, whenever you prove a theorem, under some conditions, you try to see, whether any one of the conditions can be dropped or not, you can remove that condition from the theorem or not, because if you can remove it, then your result you have got a much stronger result at your hands without that condition also, right? So, In is a sequence, so let us try to understand the proof.

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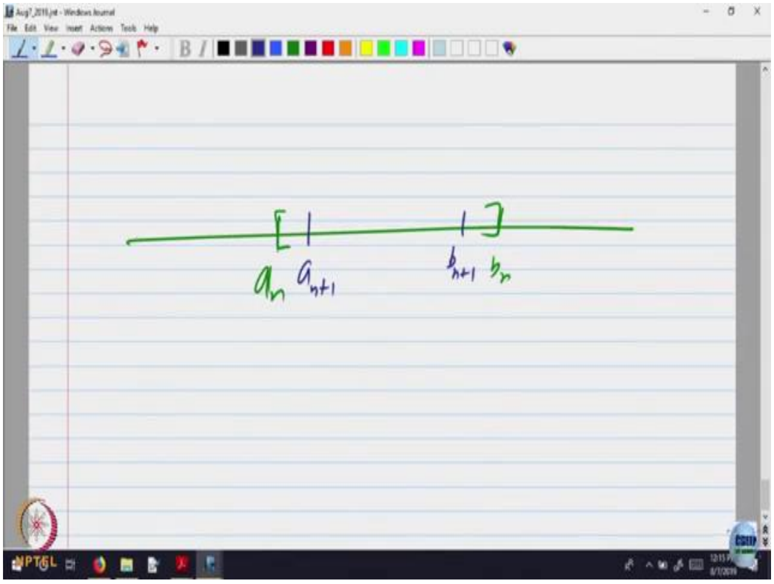

So, this is An and this is Bn, here is, right? So, where is n plus 1? So, here will be An plus 1 and possibly here will be Bn plus 1 because it is nested, it is the nested sequence, right? So, now look at the left hand points, An's, it is a sequence of points, what can you say about that sequence?

Student is answering: Increasing.

Professor: It is a monotonically increasing, what about sequence Bn's?

Students is answering: (())(8:18).

Professor: Are both bounded? So, they much converge. So, first observation, the left hand points much converge, the right hand points must converge. So, limit An n going to infinity equal A exists, that is the first step because An is a monotonically increasing sequence which is bounded above. And limit n going to infinity Bn equal to B exists because of the property every monotonically monotone sequence which is bounded must converge. What is the relation between A and B?

Students is answering: (())(9:08).

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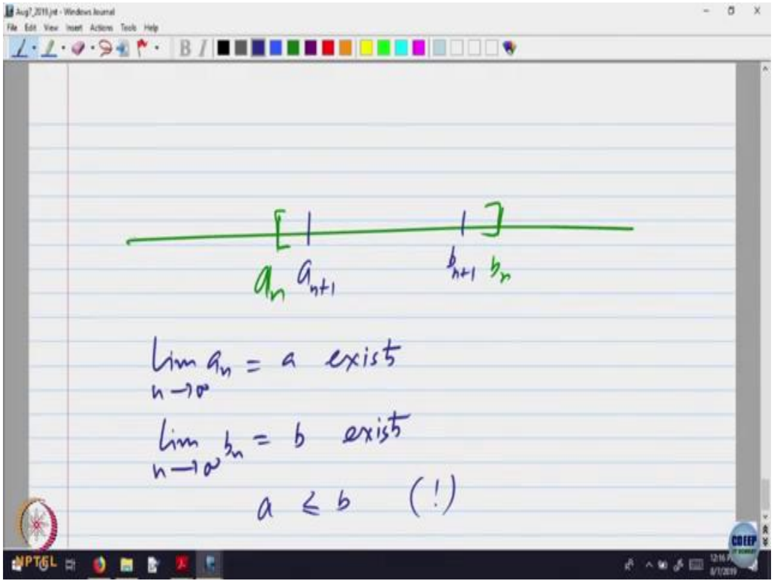

Claim A is less than or equal to B, is that okay? A is less than or equal to B, is that okay for everybody? Because An is less than Bn, right? So, or look at each An is less than Bn and Bn is decreasing, so Bn's are upper bounds for sequence An, right? So, the limit of that, that is A so A must be less than or equal to B, okay? Right.

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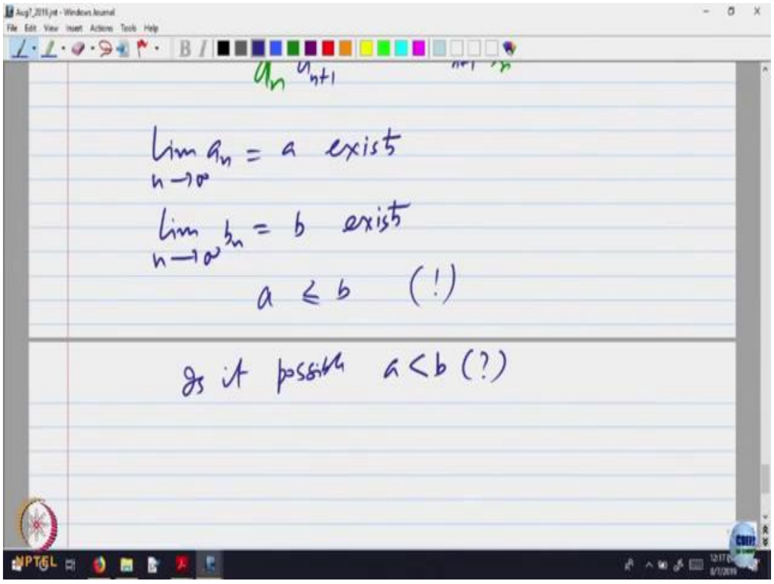

Now the question is can I say, is it possible A is strictly less than B? Is it possible that A is strictly less than B? See, saying that limit of An exist, limit of Bn exist we have used the property, that is sequence is nested, that property has been used once nested. If you want to say that if A is strictly less than B, that means what? If A is strictly less than B then what?

Students is answering: (())(10:27).

All that does not help actually, that means the limit of An and limit of Bn has got a length in between, is that possible? Because length of Bn minus An, so limit of Bn minus limit of An, what is that limit? Limit of An is A, limit of Bn is B and that is going to 0, the property says limit of, so this property says A cannot be strictly less than B, right?

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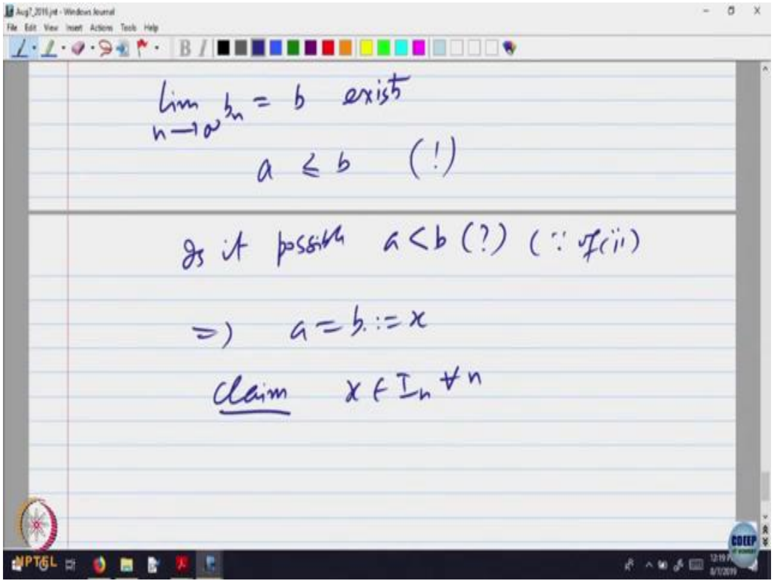

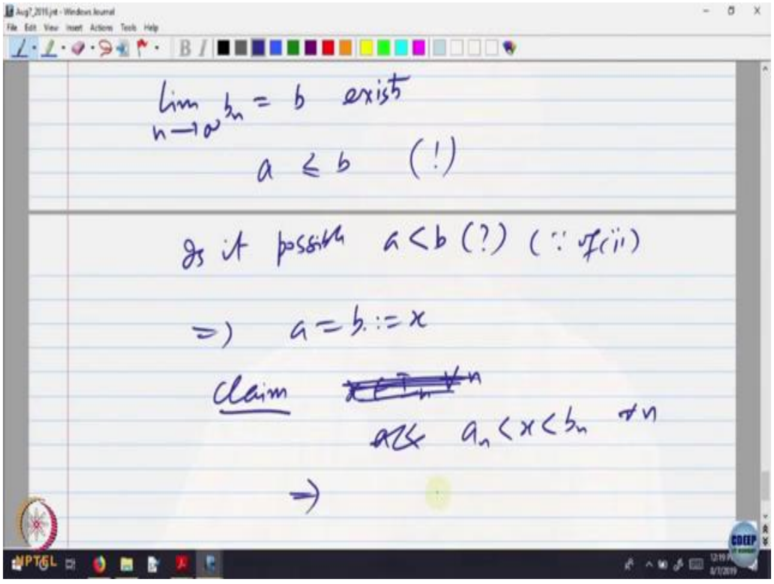

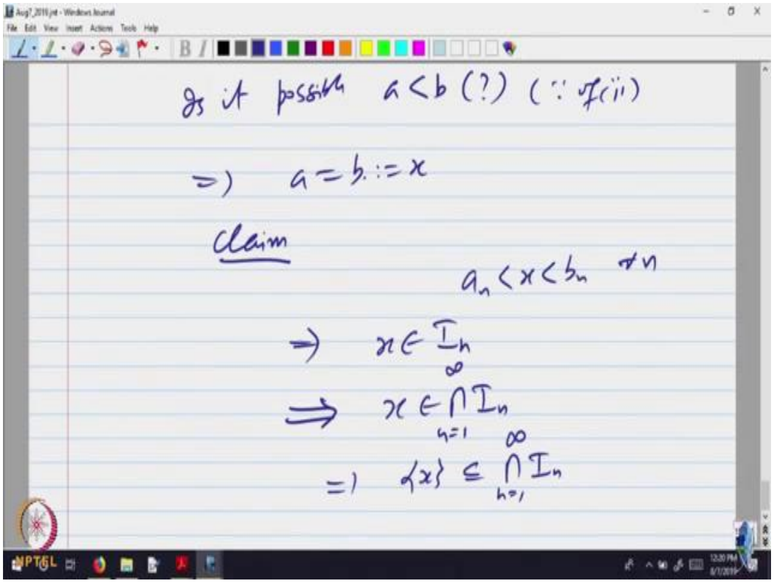

So A is equal to B, so that much is because of 2, so implies, so where is so A implies A is equal to B. Now, can I say this number, so claim can I say this number call it, if you want to call, call it as x now, A is equal to B call it as x or keep it A and B that this number x belongs to In for every n that belongs to each one of the intervals because, where is x? It is less than, it is bigger than An less than Bn, is it okay?

Because it is a limit of An's and limit of Bn's, so this must be inside A In, A less than, is it okay? x is An less than x less than Bn for every n, claim this is okay. So, this implies, is that okay? I am just writing a weaker statement because that is true, so implies x belongs to x belongs to In. Claim, can I say x belongs to the intersection In's? It belongs to each In, so it must belong to intersection. So, implies singleton x is a subset of the intersection In.

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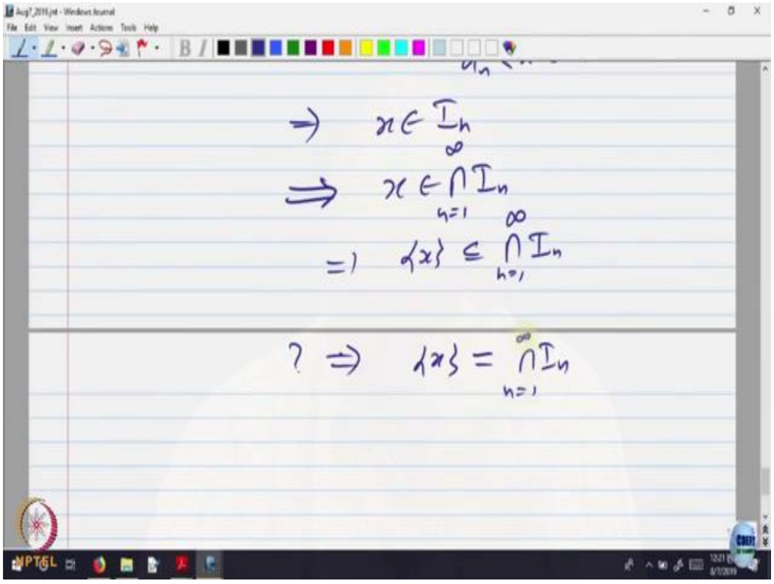

Now, question is, can I say it is equal to, I want to say it is equal to, how can I say that this is equal to intersection In? If not, the simplest way of doing this, if not suppose there are two points, x and y both in, then what will happen? So, length of In will be bigger than y minus x, both are inside In, so length cannot go to 0 then, if x and y x not equal to y are inside In, then what will be the length of In? It will be bigger than or equal y minus x which cannot go to 0, it is a number.

So, that is not possible, again because length goes to 0, so this is okay. Now, my question is we have used nested property, we have used the property that, what else we have used? Nested property we have used, and then property 2, length going to 0 we have used, where have we used the fact that they are closed intervals? Where did we use the fact that each In is a closed interval?

In this proof, can u spot somewhere some line, which I should have justified, I did not? There must have been somewhere, otherwise I can remove that condition that In’s are closed intervals, I could just say In’s are nested sequence, length going to 0 should be happy, A and B are.

Students is answering: Equal.

Professor: What are, A and B are?

Students is answering: If A and B are equal.

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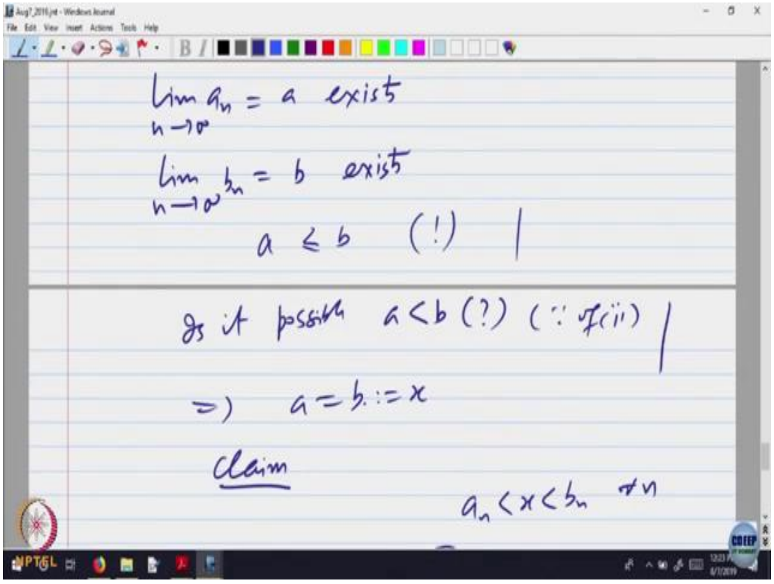

Professor: So, you intend to say that, here but I used property 2 here, namely the length is going to 0. Yes, in this proof where I have used the fact the closed, what you are trying to say is, I can give you a counter example where for open it will not be true, but I am saying in this proof, see that is how we should try to understand in what line of this arguments did I use the fact that In's are closed intervals, till now saying An's are increasing, that is nested property Bn's are decreasing that is nested property.

Bounded, intervals are bounded, so both sequences are bounded so convergent, no problem. And because An is less than or equal to Bn, so limit A will be less than or equal to B, that is okay that is property of limits. So, we are saying, if possible A is less than B, that is okay because, if A is less than B, then what happens? We said x will be between An and Bn, that is okay, because An's are increasing Bn's are decreasing, so x will be inside.

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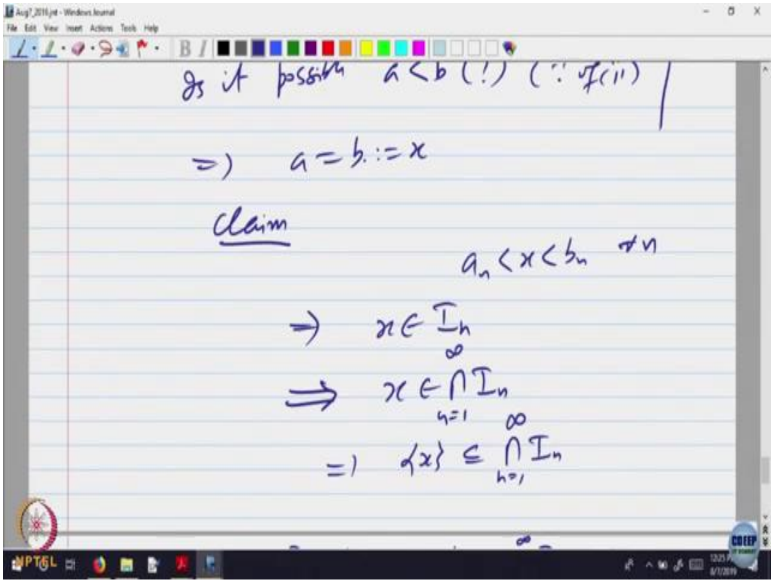

But does it imply that x has to be in the interval In, Bn closed interval, if it is not closed, would you say that x has to be in the interval An, Bn? So, something to think about. So, try to go through the proof yourself and try to understand, this is how you will understand and then remember the conditions, what are the conditions, what are required, where in the proof we used the fact that In's are closed intervals.

So, let me give you some example, and then probably that will help you to understand, so what we are saying is, so this nested interval property is no longer true if In's are open intervals, okay.

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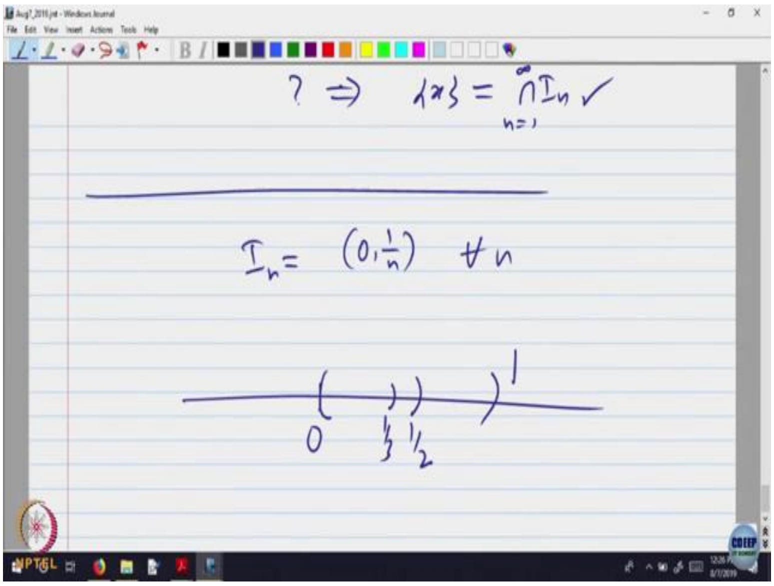

So, let us look at an example of that type, so let us look at In to be equal to, so where is the first one, 0 and 1, where is the next one? 1 by 2, 1 by 3. So, what is the length of In? Length of In is 1 over n that goes to 0, so this is a nested sequence of intervals In's length going to, what is the intersection of In's? What do you think is the intersection of In's?

Students is answering: 5.

Professor: If you say it is 5, that means I have got a counter example indicating that, in the theorem of nested interval property, you see when in the proof when you are taking limit, An is A limit Bn is B, then A and B need not be part of the, that is what is happening here, of the intersection A and B need not be part of the intersection that is what is happening.

So to say that A and B, when you want to say x is in A In for every n, that is a limit of An and B and you want to say that belongs to In, then you have to have closed intervals, otherwise this may not be true, so that is the place there you have to use, is that okay? So, we have proved what is called the nested interval property, that means and there is a consequence of basically, LUB property can say because, LUB property implies every monotonically increasing bounded sequence is convergent and we did not prove that, but one can prove that actually that is equal and to LUB property.

So, LUB property, monotone increasing and bounded convergent, Cauchyness all are equivalent properties, if you assume one you can prove the other, but we are doing only LUB implies monotonically increasing and bounded is convergent, monotonically and LUB property implies, Cauchy and we also proved Cauchy implies LUB property in the way that we did not prove Cauchy implies LUB, we only proved that every Cauchy sequence, a sequence is convergent if and only if it is Cauchy, one can prove all these are equal and we will not go.

We also proved as a consequence of this, what is called nested interval properties locating a point on the line, one can prove that this is also equivalent to LUB property, converse also if you assume this property on a field, then it has to have LUB property, these are all slightly deep theorems in construction of real numbers we will not go into that, but as a consequence LUB properties, these properties are true.

So, what we have done today is nothing much actually, what we have done is we have defined one a Cauchyness as is equivalent to convergent one, second we defined what is an interval, we try to characterize what are intervals left open, right close in the familiar geometric form and then we proved the nested interval property, so let us stop, thank you.