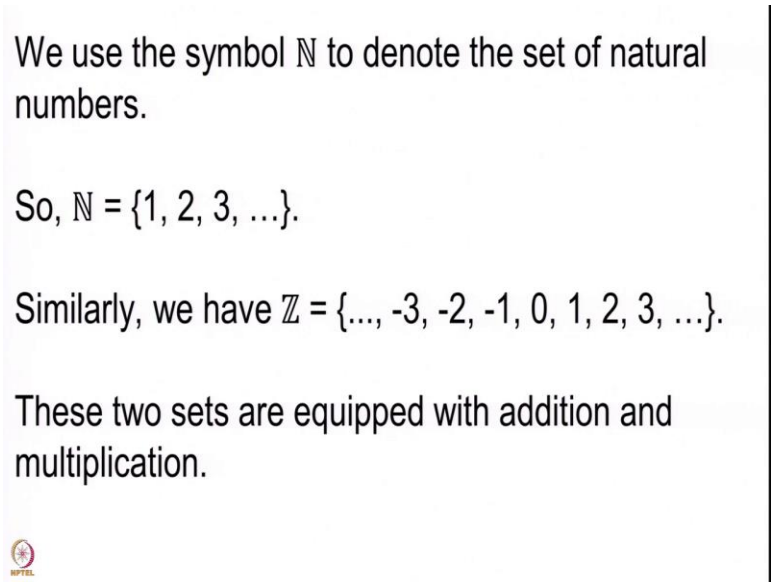
A Basic Course in Number Theory
Professor. Shripad Garge
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture No. 1
Integers

Welcome to this course. As the title of the course says this is a basic course on number theory. So, number theory is really the study of positive integers, or what are also called as natural numbers so, these are the numbers 1, 2, 3, 4 and so on. The number theory deals with properties of these natural numbers and you will perhaps not trust me. But, some of these properties can be so difficult that we need to use techniques various other parts of mathematics.

However, this course is not going to deal with any of those techniques. We will really see some of the very basic things in number theory. So, before we begin we need to fix the notations and the natural numbers set is the most important set for us. So, let us fix our notation for this. One very natural notation that we should use for this set is capital N. But, capital N is also used for many other things. So, we devise a new notation.

(Refer Slide Time: 1:35)



And this is the symbol that we use to denote the set of natural numbers. This is the slightly decorated N, you know the slanted line comes with some small width, and there is a gap between the two slanted lines. So, this is what we will always use to denote the set of natural numbers.

Now, mathematics is all about sets, and you know what we have said here can also be written in mathematical way in the following way.

That N is the set consisting of numbers 1, 2, 3 dot dot dot. These dots that we have here represent that the numbers go all the way up to infinity. So, we are counting all the natural numbers in this set. This is what it means. Now, you may wonder what has happened to the number 0, and also there are negative numbers. So, there is indeed a set consisting of all natural numbers, their negatives and also the zero.

And since we have fixed a notation for the set of natural numbers, we should use some other symbol to denote that set that traditionally is used by this symbol capital Z. So, again we see that there is a slight decoration there for the capital Z, and N as it we know stands for natural numbers. What the Z stands for? So, in German whole numbers or complete numbers those which are not fractions, they are called Zahlen, the spelling is Zahlen. That is the plural.

So, Z comes from that Zahlen. So, we have 2 sets now here. We have the set of natural numbers, which is 1, 2, 3 onwards, and we have the set of whole integers or the all integers Z, which comes from negative infinity, then you have minus 3, minus 2, minus 1, 0, 1, 2, 3 and then you go on. So, it goes to infinity in both the directions. Now, mathematics is really the study of sets with various structures put on them. So, what these structures that we can put on these two sets?

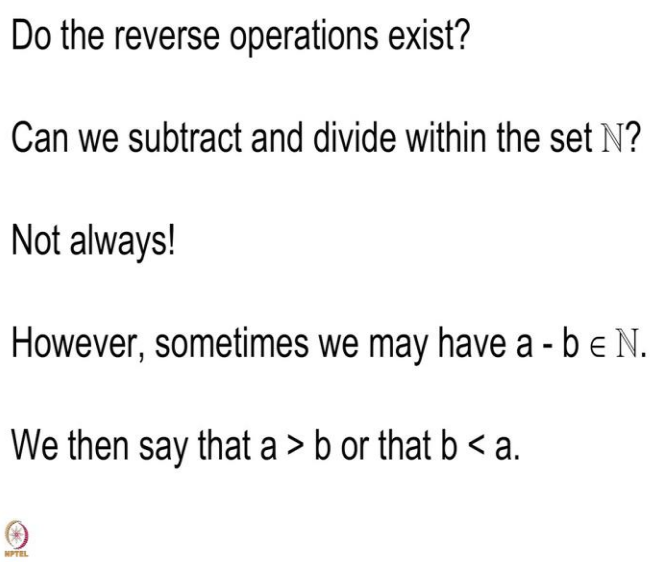
You will observe quite easily that we have addition defined on both these sets. You may take two natural numbers, add them, you get one more natural number. If you have 3 apples and your friend has 2 apples; then in all you have 5 apples. Or if your sister has 3 bananas and you have 2 oranges, in all you are going to have 5 fruits. So, 3 plus 2 is 5 that is the addition on natural numbers.

There is a similar addition on the set Z. If you add 0 to any natural number or to any negative of a natural number, you get that number either that natural number or its negative. If you add a negative number to a natural number, then you get the difference of those two natural numbers. So, if you have a and you add minus b; where b is another natural number. You are going to get a minus b, which could be positive; it could be negative or it could also be 0.

But, you have these operations that you can add 2 elements of N, get one more element of N; or you add 2 elements of Z, and get it another element of Z. There is also the product structure. You can take 2 natural numbers, multiply them and get one more natural number. If your class has 20 students and each of them is given 30 rupees; then the whole class will have 20 into 30 rupees. I will let you figure out that product.

Similarly, we have the product structure on the set Z, so what it means is the following line. These two sets N and Z are equipped with addition and multiplication. One nice thing about Z is that you can subtract. If I wanted to subtract 3 from 2, I will get minus 1, which is yet another element in Z. Can we do that for N? That is something that would be the most natural question that we would have.

(Refer Slide Time: 6:49)



Do the reverse operations exist? Or more precisely, can we subtract and divide within the set N? What I mean to say, is that if I take two natural numbers a and b. Do I get a minus b, which we know is a number in Z? Do I get a minus b in N? And if I have a number a, a natural number a and another natural number be, is a by b a natural number? So, there are 2 questions here really. Let us tackle to them one by one.

We first go to a minus b. Can we do that? Not always. In fact the answer ‘not always’ holds for both of these questions. However, you can still salvage the situations, sometimes it is possible. However, it is sometimes possible to get the difference a minus b in N. For instance, if you take

the current gregorian year 2020 and subtract the year of your birth; say, 1999. That will give you 21.

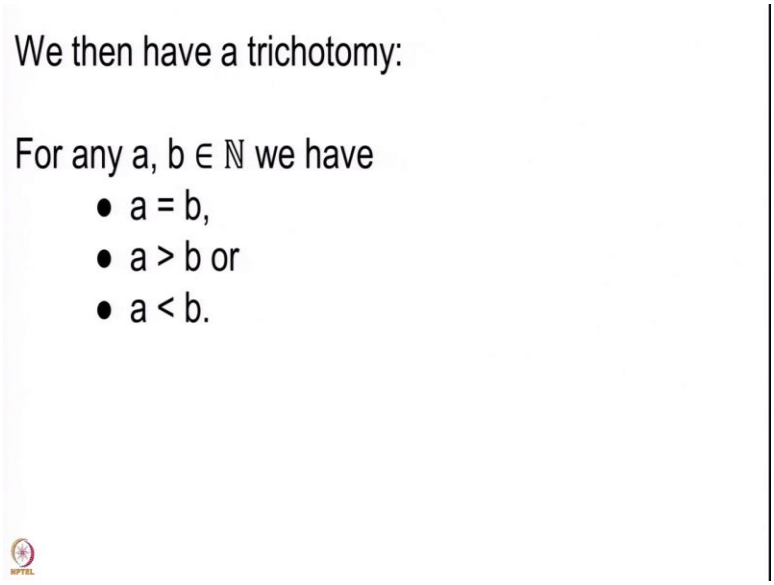
So, 2020 which is the natural number minus 1999, which is another natural number, their difference is yet another natural number. So, it is indeed sometimes possible to have a natural number a and another natural number b such that a minus b is also a natural number. So, you would say of course this is quite natural because 2020 is bigger than 1999, or you will say that 1999 is smaller than 2020.

But, bigger or smaller that is not a well-defined concept for us. Yet, we have not talked about what it means to say that, a natural number is bigger than another natural number. What we have done until now, if you have been following the lecture from the beginning. Then we had N and Z; we talked about 2 sets, and we talked about the addition and the multiplication on them. And now, we have this possibility that given a natural number a and given another natural number be, you may have that a minus b is also a natural number.

Then, let us use this and define our relation of something being bigger than or smaller than another number. Let us use this very definition. So, if you have a and b such that a minus b is natural, then we say that a is bigger than be, or you may say that b is less than a. So, let us have this as our definition. If you can subtract a, a natural number which we have taken here to be b from another natural number a and remain in the set of natural numbers. Then we say that a is bigger than b, or we will say that b is less than a.

So, using this definition we see easily that 1999 is smaller than 2020 or 2020 is bigger than 1999. So, this is a relation on the natural numbers. This is called order relation, and this order relation is the fantastic relation. The thing is, given any 2 natural numbers, there is always some relation.

(Refer Slide Time: 11:09)

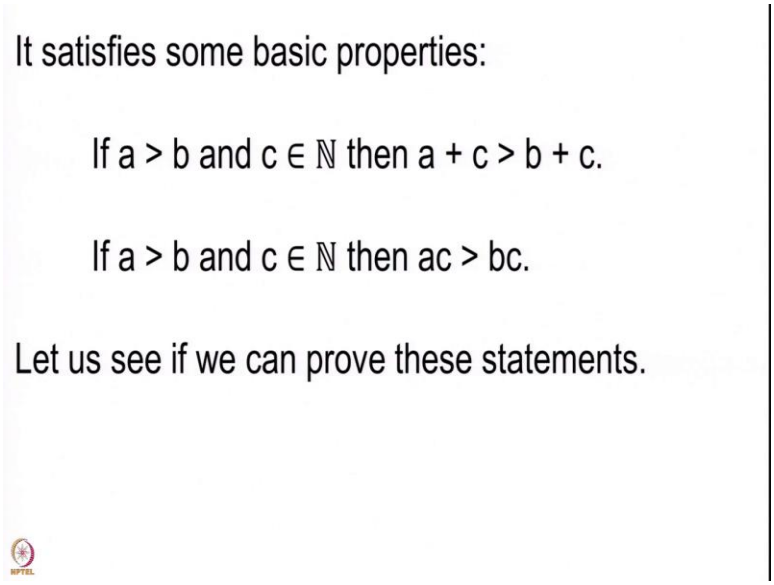


This is what is called trichotomy. If you have any 2 natural numbers a, b; you will either have that a is equal to b; because remember Z is equipped with subtraction. You can subtract any element of Z, from any other element of Z which means that, if you take 2 elements from N, 2 natural numbers, then, their difference can either be a natural number. It can be negative of natural number or it can be 0. It may happen that you take 2 elements to be the same; you may take 600, and you may subtract 20 into 30 from 600. So, this is a trichotomy.

Trichotomy says that, there are only these 3 possibilities. These 3 relations, these 3 types of relations exhaust all pairs of natural numbers. So, let us read it again, whenever we have a and b any 2 natural numbers, you will either have that a is equal to b, a is bigger than b or a is smaller than b. Again you may think that this is quite obvious. Indeed it is obvious, by what we have developed the tings so far.

So, the order relation satisfies some very basic properties. You know, we have the addition and we also have the product on the set of natural numbers. And now we have the order relation. Is there any correlation between these concepts that we have defined? There are and those are very easy to state, and indeed also very easy to prove.

(Refer Slide Time: 13:02)



So, this relation, order relations satisfies some very basic properties. What does the first one say? It says that, whenever you have a natural number a bigger than another natural number b, and you have a third natural number c. If you add this third number to both a and be, then the sum is preserved the sum preserves the relation. The sum respects the relation. The addition in N does not disturb this relation.

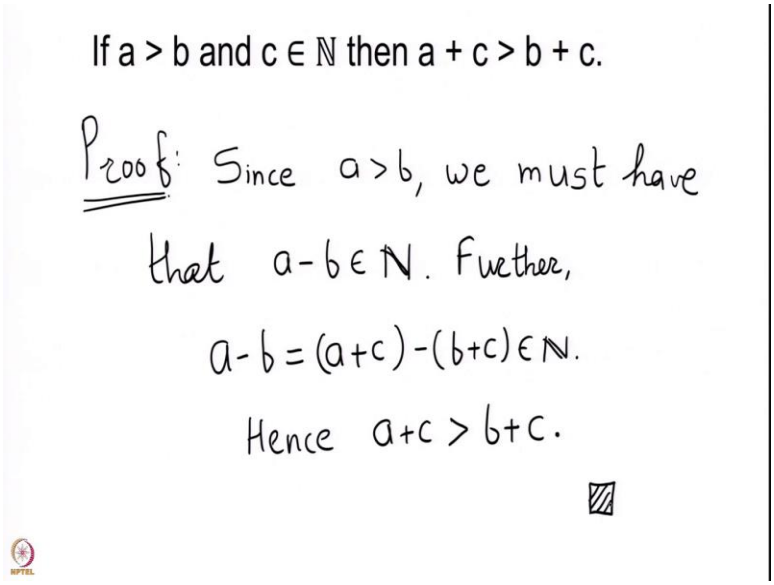
If you have a bigger than b and you take a natural number c, a plus c is bigger than b plus c. This is quite evident but it needs a proof. We will see whether we can improve this. Let us go to another such property, which will connect the relation with respect to the product. Just like the sum, product also respects the relation. If I take a bigger than b and I multiply both a and b by a natural number, then a, c is going to be bigger than b, c.

Note: if you multiply by a negative number, then the inequality will change its order. If you multiply by a negative number to a and be, the inequality or the order will not be preserved. Here, we are within natural numbers and therefore the order relation is preserved. Now, these are 2 statements, which are given as properties. These are not definitions until now we have seen only definitions. We saw the addition definition, we saw the multiplication definition which I did not really spell out but we know what we are talking about.

And using the addition or the reverse operation of that, which is the subtraction, we defined the order, and now we have 2 properties. So, as a warm up exercise let us see whether we can prove

these. So, what I will do is that, I will write the first property in the next slide, and I will give you a minute to think about this.

(Refer Slide Time: 15:29)



What I want you to do is to think about the definition of order that we have learned until now, the properties of natural numbers that we have seen until now, and come up with a proof.

The slide will be there in front of you for a minute. We will see a proof later. But, perhaps your proof might be different; if it is different please write to me, and let me know your proof. So, we wait here for a minute. So, now that you have had your time playing with the proof. Let me give you one proof. So, I will write this proof on this slide, so we begin with the proof. What do we have to prove? We want to prove that, whenever a is bigger than b and c is a natural number, then a plus c is bigger than b plus c. This is what we want to prove.

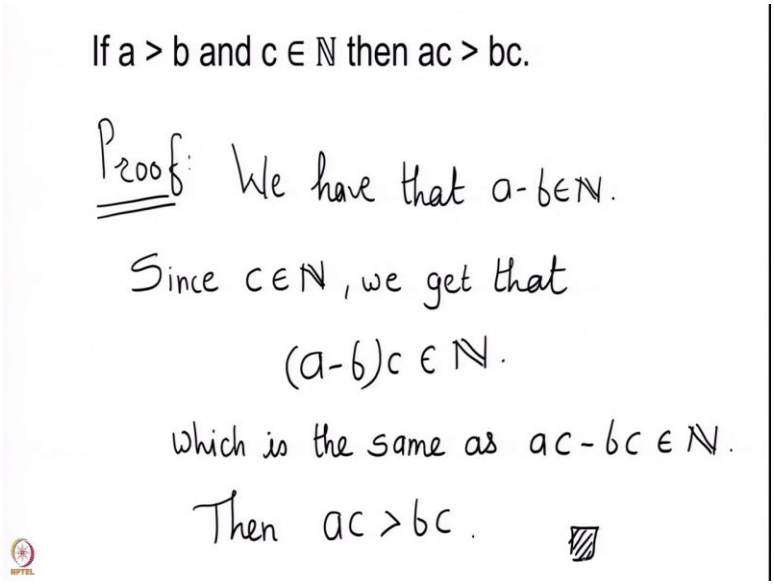
So, since a is bigger than b, we must have that our definition of order will tell you that, a minus b is in N. If you have a bigger than b, then a minus b should be a natural number this is our definition of the order. Further, a minus b is same as a plus c minus b plus c. This is where we are using some properties of the addition that a minus b is same as, a minus b plus c minus c and the first c which is come which comes with the positive sign can be put inside next to a. And the next c which is with the minus sign can be put with the b.

So, we are using the commutativity of the addition. But, now you have that a plus c minus b plus c is a natural number and hence a plus c is bigger than b plus c. This completes our proof. So, let us read it again. If we are given, there were 2 things that we are given to us that a is bigger than b and c is a natural number, then what we did was to write a minus b, which was a natural number because a is bigger than b.

So, we wrote a minus b as a plus c minus b plus c and that told us that a plus c has to be bigger than b plus c. Now, you may ask that we have not used c being a natural number here that is true we have actually used that c is an integer. Whatever we have done the order relation is preserved by addition of all elements in Z. But, there is a slight problem here which is that a plus c where a, is a natural number and c is coming from the bigger set of Z. Then a plus c need not again be a natural number that is the only thing.

So, either you define the order relation for the whole set of integers, or the whole Z. And then we can replace this statement by c being in N by c being in Z indeed. So, that is done, good. We now go to the next statement.

(Refer Slide Time: 20:54)



If a is bigger than b and c is a natural number, then we need to show that ac is bigger than bc. Once again I will give you a minute and after this minute, I will give you my proof. And again if you have a different proof for this statement, then please write to me, and let me know your

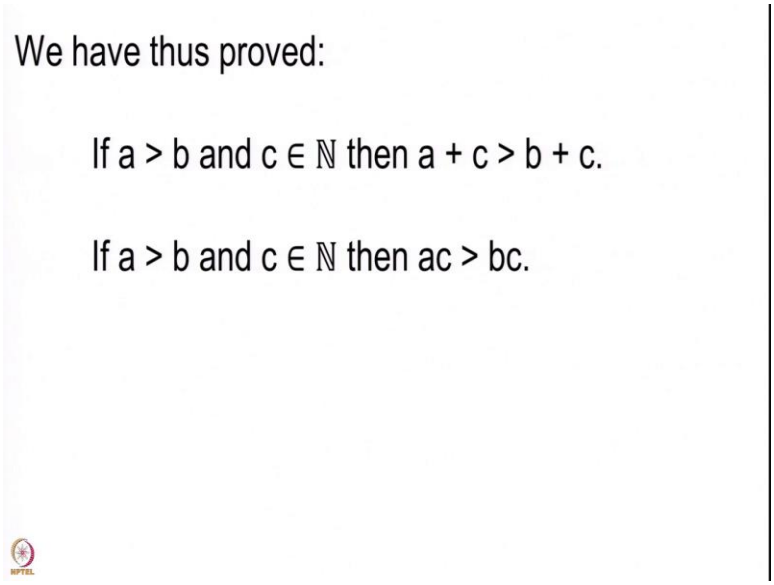
proof. So, we will now begin proving this statement. I will give you my proof. The proof is as follows.

Remember what we did in the earlier proof. We had a bigger than b and so a minus b has to be a natural number. Now, c is another natural number and natural numbers come equipped with product. So, since c is in N, we get that a minus b into c is an element in N. But, a minus b into c which is the same as ac minus bc. But, if you have ac minus bc, a natural number, then we get ac bigger than bc. So, you say this proof was also not quite difficult. All we needed to do was use the properties of natural numbers.

So, if you have a minus b, a natural number and another natural number c, then the product is a natural number that is the only thing we have used here. If you multiply a minus b by a non-natural integer that means a negative integer or 0, then you are no more within the set capital N. You go outside the set capital N and therefore the inequality may be reverse. If you multiply by a negative number, or the inequality may turn into equality if you multiply by 0. You know you can have 2020 and the another number can be 1.

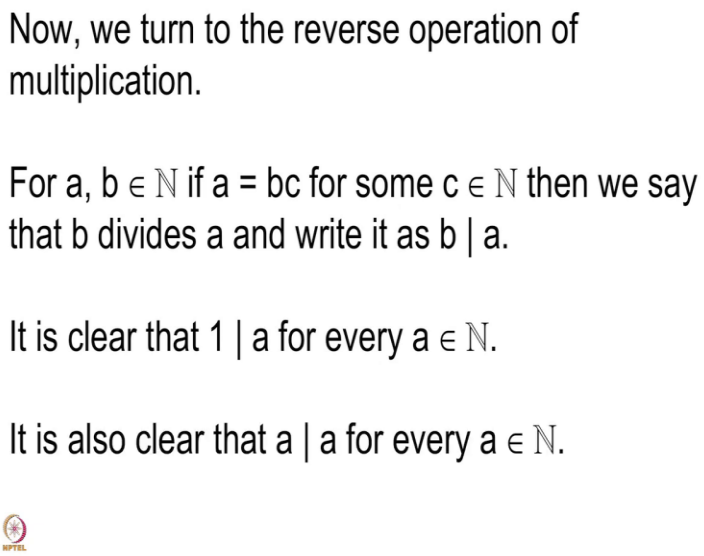
2020 is much bigger than 1. 2020 is certainly bigger than 1. But, if you multiply both the numbers by 0, you get only 0. So, when we go outside the set of natural numbers, then we have to be careful with respect to the order relation. So, let us just repeat.

(Refer Slide Time: 25:06)



We have thus proved that, when you have a and b 2 natural numbers. And a is bigger than b, then this order relation is preserved. It is respected by both addition and product within the set of natural numbers. This was about the subtraction which is the reverse operation of addition. Now, we go to the another operation that we had on N namely the product, and let us ask whether the reverse operation exists.

(Refer Slide Time: 25:45)



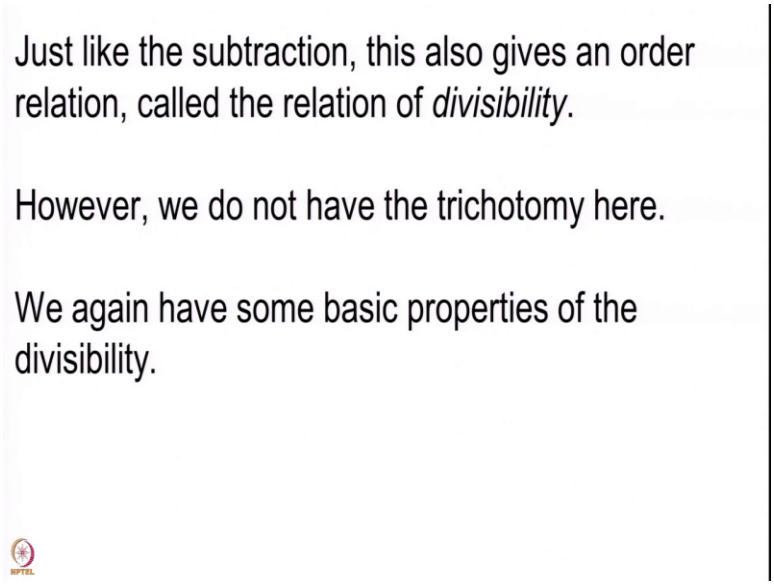
So, we now turn to the reverse operation of 600 hundred rupees to use and there are 20 students in the class. How many rupees should be used per student? That would be 600 divided by 20, which is a number reasonable number 30. You can use it for every student. But, you may have say 485 rupees, and you have 20 students. How would you do it now? You cannot of course convert everything to paise, the rupees. You know, 1 rupees 100 paise. You cannot convert everything to paise, and then use the division.

So, we want to be within the natural numbers and try to see when some natural number divides another natural number. So, let us see what we get with respect to this. Suppose, we have a natural number a and another natural number b with the property that, a is b into c for some third natural number c. If you have this, then we say that b divides a. Note it again, that we are giving a definition of a natural number a divisible by another natural number b. We are saying here, if you have a equal to bc for some third natural number, then b divides a.

This is our definition of a natural number dividing another natural number. We will use our mathematical short form to write this as, b vertical bar a. But, while reading it we will not read it as b vertical bar a we will simply read it as b divides a. What we have seen until now will tell you that 1 divides every natural number; of course you can write a as, a into 1. So, it follows that 1 divides a, for every natural number a.

But, when you write a as, a into 1, and use that to say that one divides a. You can also use the same equality to say that a divides a because you have written a, as a into some another natural number. So, we have seen that 1 divides a, and a divides a.

(Refer Slide Time: 28:46)



This also gives you an order that two natural numbers will be said to be related by a divisibility; if one natural number divides the another natural number. Now, for the order relation that, we had seen earlier of less than or bigger than we had a trichotomy. Do we get a trichotomy here? No. We do not have the trichotomy given any two natural numbers. We may not always have that a is equal to b or a divides b or b divides a, that may not always be the case.

You can have two natural numbers, which are simply not related to each other. But, that is what makes this order relation, the divisibility relation very interesting. So, there are again some very basic properties of this divisibility relation. We will see these properties and their proofs in the next lecture. Thank you.