A basic Course in Number Theory
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Lecture 40
Discriminant of a binary quadratic form

Welcome back. We are studying integral binary quadratic forms and we introduced some transformations which are essentially change of variables. And we also introduced a concept of discriminant of the quadratic form.

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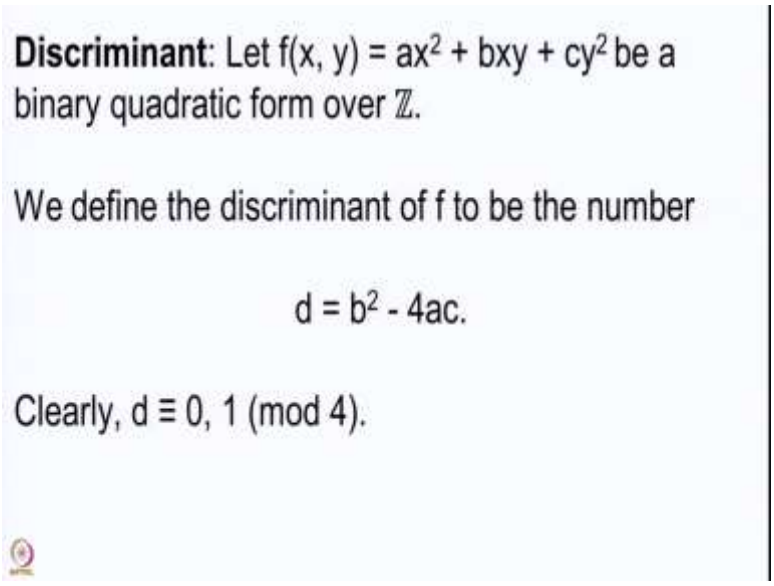

So, the definition is here in front of you in the slide, if you are form is ax squared plus bx y plus cy square then the discriminant is b square minus 4 ac and we also noticed that since b square is always 0 or 1 modulo 4 the discriminant is always going to be congruent to 0 or 1 modulo 4. Now, we asked several questions in the end of last lecture and let us go by answering them starting with the simplest one.

So, the questions were to remind you once again, what use do we have for the discriminants? So, that is a difficult question we will answer it later. That will be the last question that we answer there was another question whether the discriminants are invariant under the transformations. That is a second difficult question not quite the simplest question. So, the answer would also be second not the first answer.

The simplest question is whether every d which is congruent to 0 or 1 modulo 4 is discriminant of some form because we have defined the discriminant and we would then like to classify the integral binary quadratic forms by means of discriminant. Actually this is a short answer to what use do we have for discriminates but this answer would also require a proof. So, we will see it later in detail.

But it is of interest to know whether every d which is congruent to 0 or 1 mod 4 is discriminant of some quadratic form. Now how do the elements which are 0 mod 4 look like? These are multiples of 4 and the elements which are 1 mod 4 they look like multiples of 4 plus 1.

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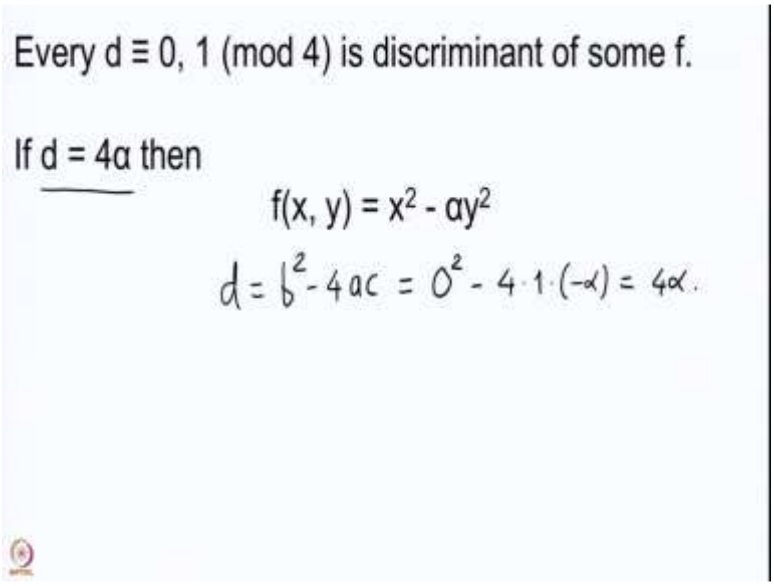

So, starting with these we will prove that every d which is 0 or 1 modulo 4 is discriminant of some form f. So, the answer is in front of you when you have d equal to 4 times alpha where alpha is some integer then f x y equal to x squared minus alpha y square will give you the discriminant equal to d. Here if you to compute the discriminant, discriminant is remember the formula for d is b square minus 4 ac here b is 0.

So, we get 0 square minus 4 a is 1 and C is minus alpha. So, we get 4 alpha which is what we had here. So, whenever you have a number which is a multiple of 4, a number which is divisible by 4 in the integers then any such number is discriminant of this very simple looking form. The form is very simple it does not have the middle term its coefficient for x square is 1 and the coefficient for y square is then the number it has to be because you want the discriminant to be equal to 4 alpha.

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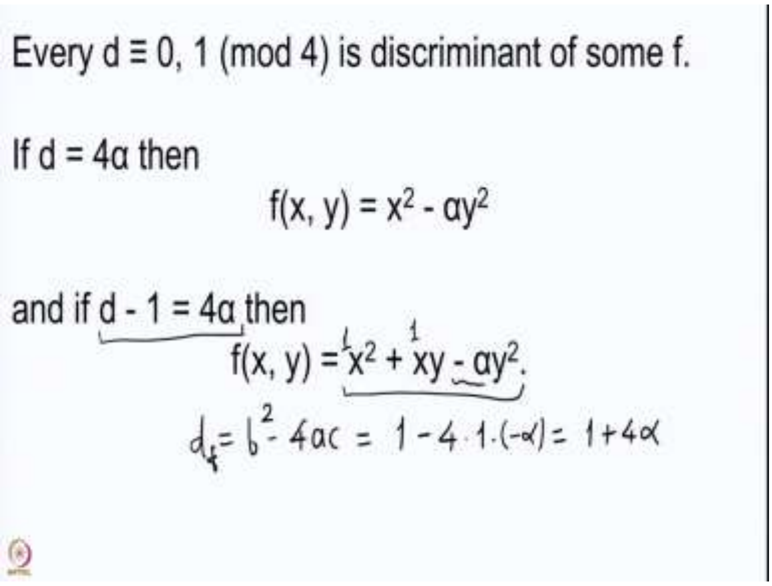

Let us go to the next case when you have that d is 1 modulo 4 or when you have that d is 4 alpha plus 1 d minus 1 is 4 alpha. Then the form is x square plus xy minus y square. Let us again compute the discriminant b square minus 4 ac here b is 1, a is 1, C is minus alpha. So, we get this number to be 1 plus 4 alpha which is indeed the d we started with.

So, to be more precise I should call this df for determining it as the discriminant of the form f. So, we have seen very easily that the numbers which are 0 or 1 modulo 4 will always occur as discriminants of some form. Moreover these two examples of forms that we have given are the simplest looking form. Even the second one here the coefficient of x square is 1, this is 1, this is 1 and then this is the number that it has to have because you ultimately want the discriminant to be the number 1 plus 4 alpha. So, these are the simplest form. So, these are the forms we would associate naturally to these discriminants. And so they have some names.

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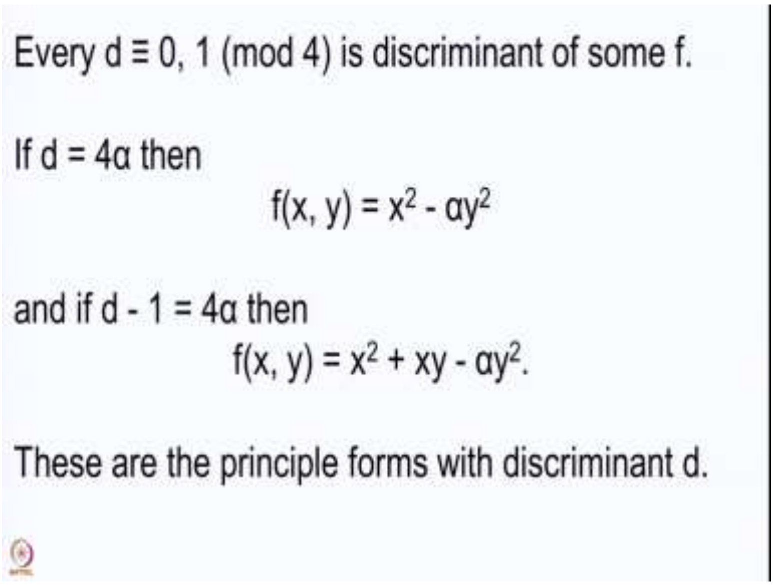

These are called principle forms with discriminant d. So, whenever we have a discriminant and we talked about the principal forms Associated to d. You should immediately see whether d is 0 mod 4 or 1 mod 4. If it is 0 more for you write down the corresponding form. If it is 1 mod 4 you write down the corresponding form, those are the principal form.

So, every discriminant occurs every number 0 or 1 modulo 4 occurs as discriminant of some form. Moreover you can choose this form to be simplest looking forms, and they are called the principle forms of discriminant d. Let us go ahead and do some examples.

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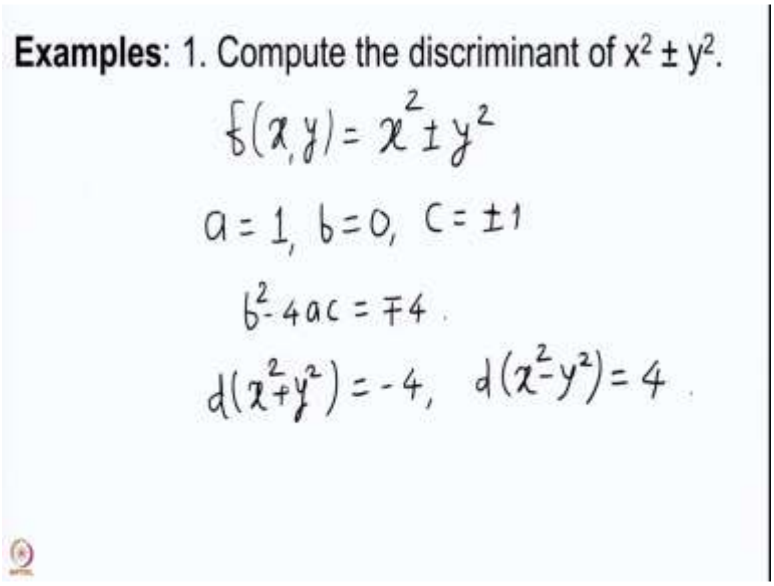

So, here we have the discriminant to be computed of the forms x square plus or minus y square. So, f of x y is x square plus minus y square a here is 1 b is 0 c is plus minus 1. So, we actually we have two forms here depending on whether c is 1 or minus 1 and if you remember these are the forms that we have already seen these are the principle forms. So, b square minus 4 ac, 0 square is 0 and then you have minus 4.

So, the answer is minus or plus 4 that means the discriminant of x square plus y square is 4 and the discriminant of x square minus y square the discriminant of x square plus y square is minus 4 and the discriminant of x square minus y square is 4. This is very funny because when we had plus sign in the form the discriminant was negative and when we had negative in the form, the discriminant is actually positive.

You will see later that this is something which is very significant whether the discriminant is negative or positive is something which is very significant. It is going to give you a very important information about the form, but we will come to that. Let us do one more example.

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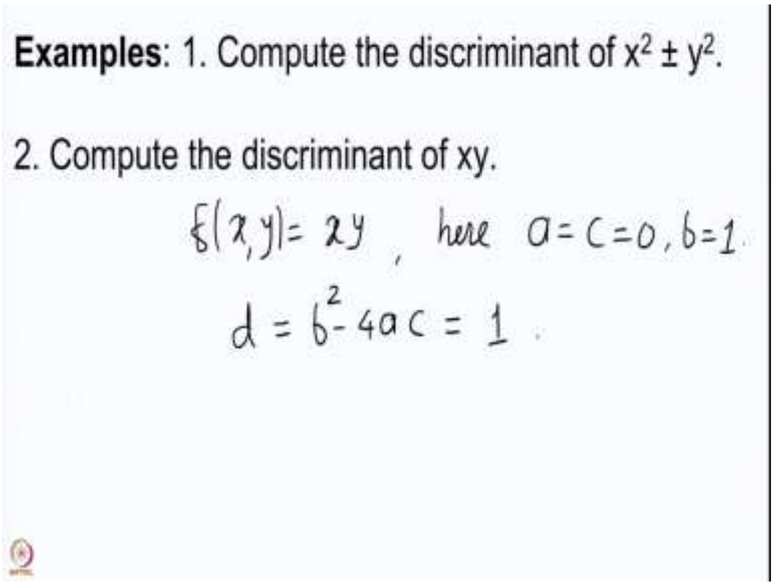

Compute the discriminant of x y. I have chosen this example because here both a and c are 0. Here a equal to c equal to 0. You do not have x square coordinate, you do not have y square and b is 1. So, our d is b square minus 4 ac and this is simply 1. So, the form is the form x y and its discriminant is positive that discriminant is equal to 1.

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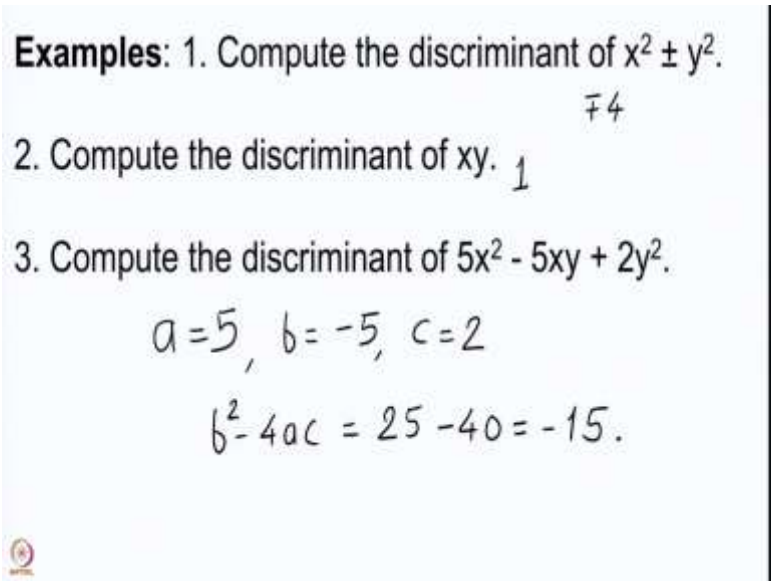

Let us go to one more example compute the discriminant of 5 x square minus 5 x y plus 2 y square. This is a slightly complicated example. So, we get that b square minus 4 ac is 25 minus the product ac is 10 and you multiplied by 4, so we get 40 and so the discriminant is minus 15. So, we have completed these various discriminants to recall the answer here was minus or plus 4.

The answer here was 1 and the answer here is minus 15. I would like to tell you one more thing. The form in the second example, which is xy has the property that its value set is the whole set of integers because we can take x to be the integer 1 and then we can vary y over the whole set of integers. So, the value set for the set x y is the set Z. Here if you look at example 1 where we have x square plus or minus y square.

So, the example x square plus y square this is a very important example. And in fact, we would be interested in computing various numbers, which can be written as sums of two squares. But the very first thing we should notice is that all these values are going to be bigger than or equal to 0 a square is always positive unless it is 0 and sum of two positive numbers is also positive.

So, the values of x square plus y square is bigger equal 0, although the discriminant is negative. It was minus 4 the values for the second example x y is the whole set and the discriminant is equal to 1. Let us look at the example one with a negative sign you have x square minus y square and we have learned in our high school that x plus y into x minus y is x square minus y square.

So, x plus y and x minus y these can be chosen to be two factors and you can have any sign coming from x plus y and x minus 5, ofcourse you can have x square minus y square to be a positive number or a negative number or equal to 0. So, there are all possible signs of numbers that are possible for x square plus y square you had the signs were always positive and finally the discriminant of 5 x square minus 5 xy plus 2 y square.

Now before even determining the value sets can we at least determine the sign of this quadratic form!? That seems to be very difficult what are all possible values of this quadratic form, do you have all possible signs coming in? Would the signs be only positive, would it be only negative? This is captured in the description of the discriminant. So, let us go ahead and study that.

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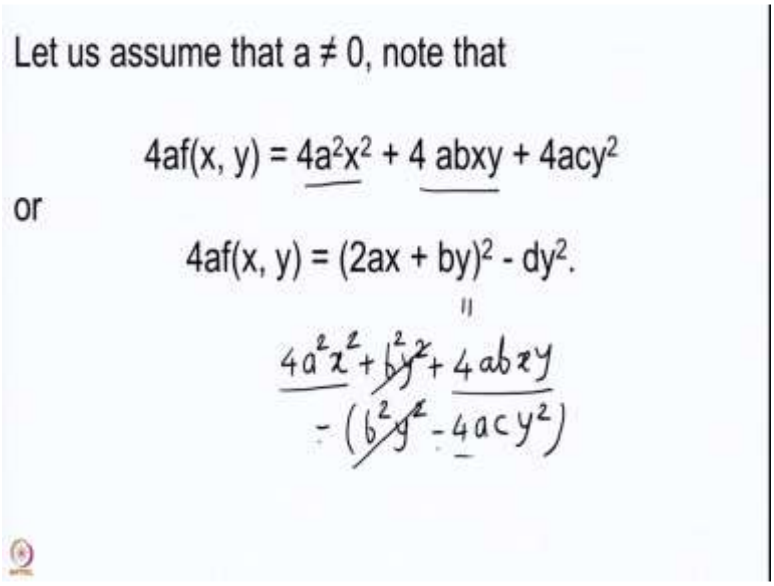

We begin with an integral binary form quadratic form at assume for the moment that a is not 0. So, our form is ax square plus bx y plus c y square and we are assuming that the coefficient of x square is a non-zero number. Once we assume that consider the multiplication to your form by 4 a. So, this is something which is very easy to compute. LHS left hand side is 4a into a x square plus b x y plus c y square and indeed what you get is this 4a square x square plus 4ab x y plus 4 ac y square this is what you get.

Next thing that we observe from this is that this can be written as 2 a x plus b y whole square minus d y square. Let us write this thing down let us expand this whole square that would give you 4a square x square plus you get b square y square from that squaring of the second term and now you need to multiply both the terms and multiply by 2. That will give you 4ab x y and now here we have minus d.

Remember d is b square minus 4 ac. So, this b square y square and b square y square gets cancelled and indeed we get 4a square x square plus 4ab x y minus-minus will become plus so 4ac y square so we have written our form multiply by 4a. But 4a times the value is a square minus d times a square alright.

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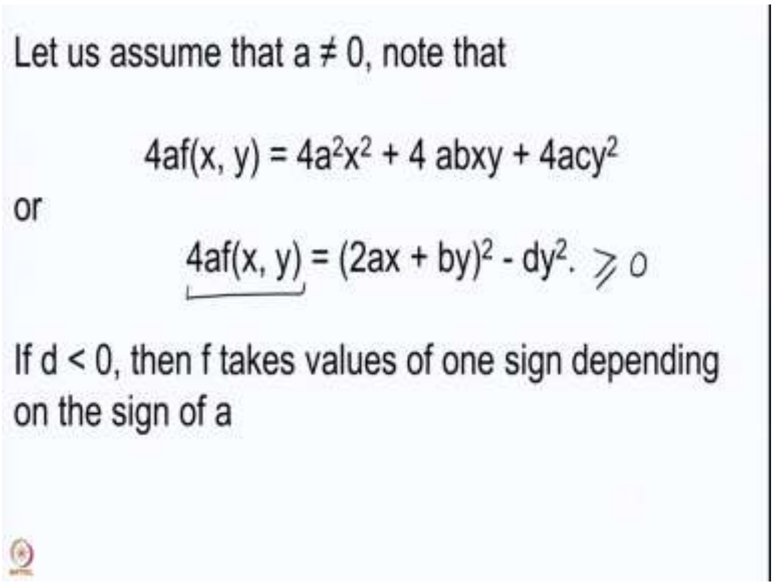

So, if you are d happens to take negative value, if the d of the form, the discriminant of the form is a negative number then on the right hand side you had a square minus d times of square. So, you get d square plus a square into a positive number. So, the right hand side is always bigger than or equal to 0 whenever you are d is negative, this side is bigger than or equal to 0 and here we have 4 into a into f.

So, that means forget 4, it is a positive quantity anyway. So a into f will always have positive values. So, if a is positive f has to be always positive if a is negative f has to be always negative. So, whenever d is negative the value set of the quadratic form f will have only one sign and that sign is the same as the sign of a. Remember we are taking a to be non-zero.

So, a has to have some sign it can be positive or it can be negative. If it is positive the value set will have numbers which are bigger than or equal to 0. If a is negative the value set will have the numbers which are less than or equal to 0. So, the behavior of the sign of the value set a is definite whenever d is negative.

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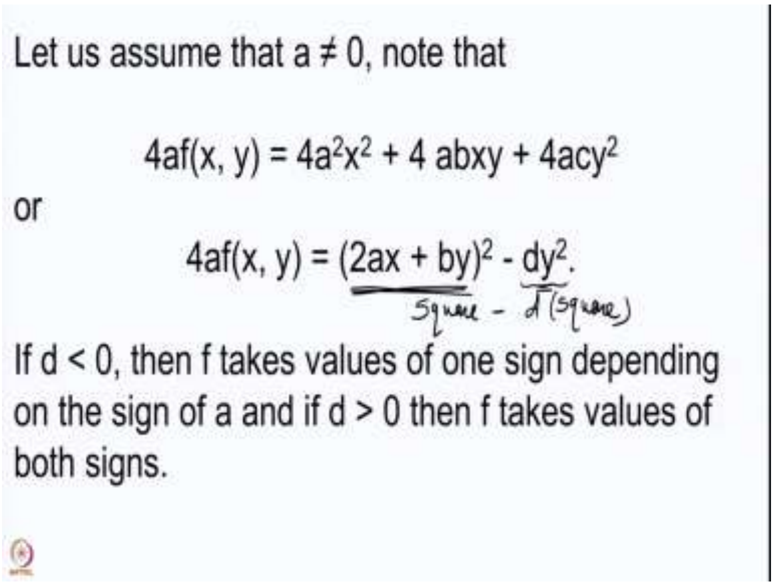

And if d is positive on the other hand, then your right hand side now becomes square minus a positive square. So, this is the square minus positive thing times a square so positive d times of square and moreover you can make y large enough and keep x small. So, that this thing becomes very small, but this quantity is big.

So that you have a positive square minus D, which is positive into a big Square you can have such a thing which will give you negative values. And ofcourse you can keep y equal to 0 and then you simply the square in the first term. So, here the signs are both signs are possible. You may have positive signs and you may have negative sign.

So, whatever is the sign of a that is immaterial. The values of the quadratic form f will have both possible signs. It can be positive and it can be negative. So, this is an example that we have already seen. We looked at the form xy where the discriminant was equal to 1, the discriminant was positive and then we saw that xy actually takes the whole set of integers every integer is of the form x into y where you can take x to be 1 and y to be the integer that you want.

So, whenever the discriminant is positive the value set behavior as far as the sign is considered is not definite. Therefore this is called indefinite whenever d is negative. We have a definite behavior of the sign of the value set and whenever d is positive we do not have definite behavior. So, this is called indefinite behavior.

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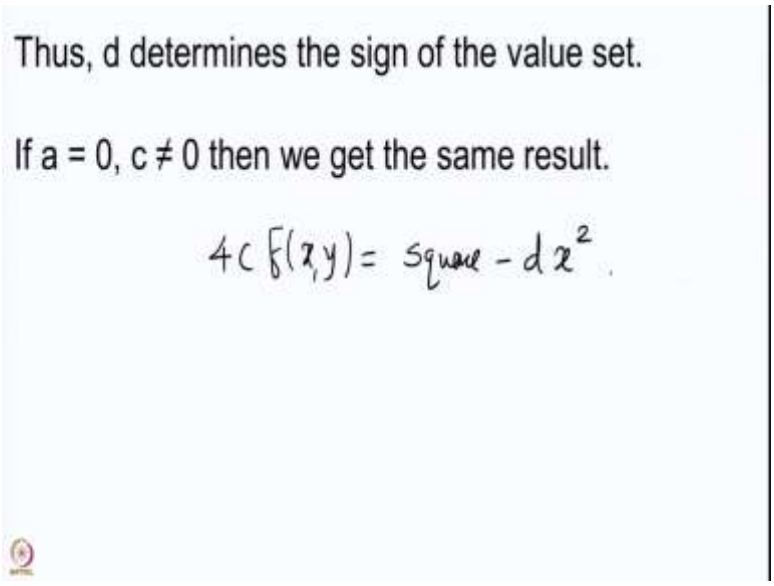

So, to write this observation in one line, we have that d determines the sign of the value set. But remember that we had one assumption in the last slide that a is not 0. If you are a happens to be 0 assume that c is non-zero. So, in the case that a is 0 and C is non-zero we get the same result instead of multiplying the value set f of xy by 4a we would multiply it by 4c.

And we would get 4c into f of xy to be square minus d times x square. If you remember in the last slide we had 4a f xy was square minus d y square here because we are multiplying by the coefficient of y. The thing that remains here outside is x square. So, once again depending on the sign of d we have the same behavior. And now ofcourse we have to consider the last case where you may have a equal to 0 and c equal to 0.

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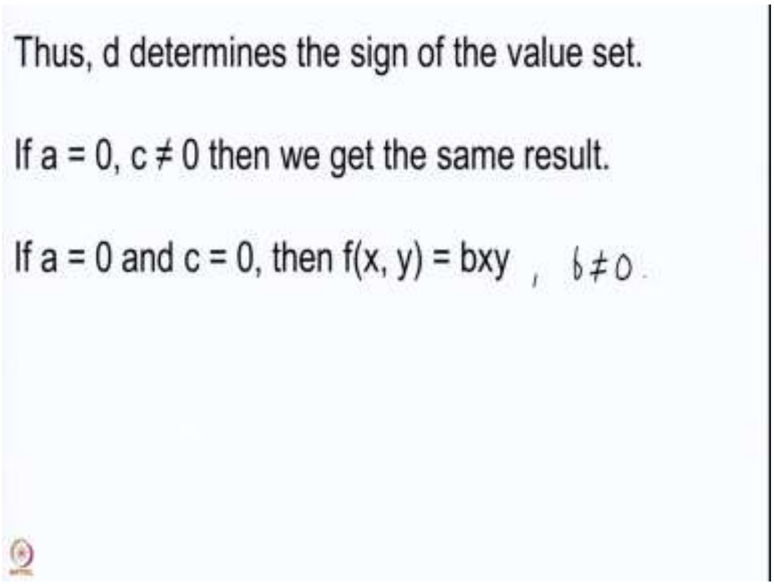

So, if you have a equal to 0 and c equal to 0 you are f of x y is simply b x y. We should also have that b is non-zero. So, whenever b is non-zero this form as we have already seen in the last example that it will take both possible values. It will take positive values and negative values.

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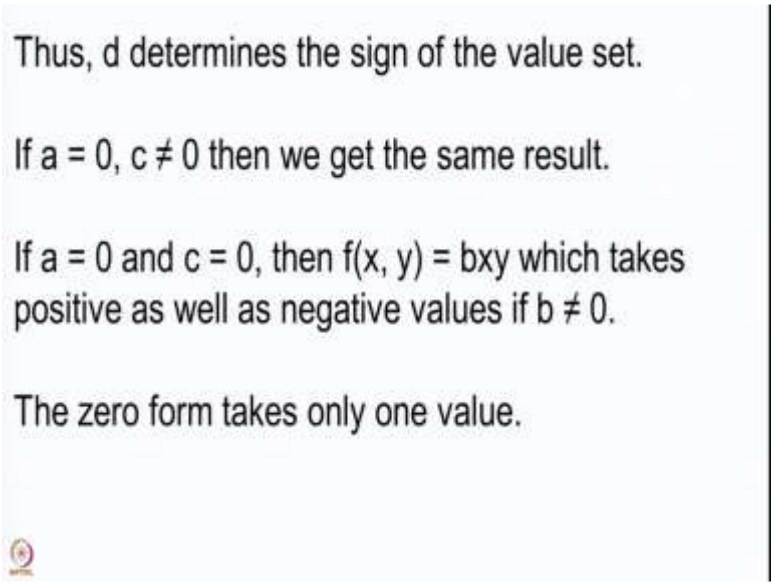

And ofcourse the discriminant here is positive. The discriminant is b square. So, it agrees with our earlier observation that discriminant is positive both signs are possible. Discriminant is negative then only one sign is possible and that will depend on whether a is positive or negative or equivalently whether c is positive or negative. So, here ofcourse we have to assume that b is non-zero. If you have all a b c equal to 0 then your form is 0 and if you have the 0 form, then it can take only one value and then there is nothing much that we need to do.

So, from our next this discussions we are not going to consider this form because its values set is singleton 0 it has no sign. It really has no much interesting properties. So, let me know phrase whatever we have discussed about the behavior of the sign of the discriminant in the next slide.

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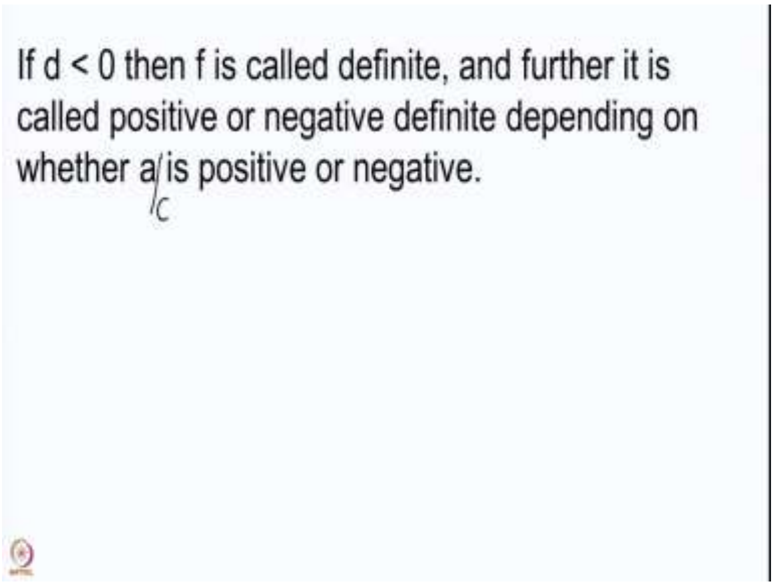

So, whenever discriminant is negative we say that our form is definite. Further we will call the form to be positive definite or negative definite depending on whether a is positive or negative or equivalently I should say whether c is positive or negative, or you should also see this in the following way that whenever you are form has negative discriminant.

We know that the values are going to be bigger than or equal to 0 if you are going to have only positive values then we see that the form is positive definite if the values are negative then we say that the form is negative definite. So, this behavior is quite well understood whenever d is negative.

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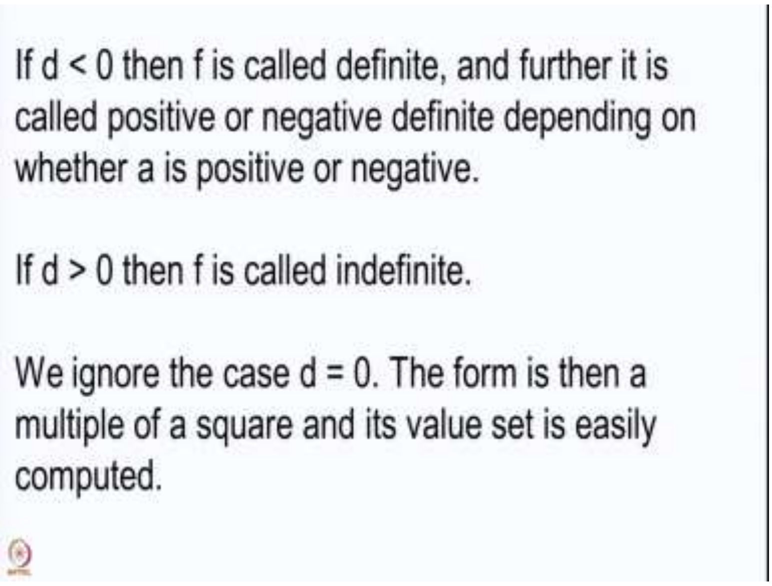

If d is positive then we call the form to be indefinite because both the signs are possible you may have positive signs as well as negative signs in the value sets. And there is one more case that your d might be 0. So, this case we are going to ignore for the reason that we had written our form multiplied by 4a as a square minus d times the square of d. If d is 0 the form is a multiple of a square and then it is easy to compute the value set. So, again this problem is not a very difficult problem.

So, when the discriminant is 0 that is the case that we are going to ignore e are only going to consider these two cases: d negative, definite either positive definite or negative definite depending on the values and d positive indefinite. So, this is the definition of a form we call a form to be definite if the discriminant is negative. We call it to be positive definite if the discriminant is negative and one there is a positive value represented by f.

We call a form to be negative definite if the discriminant is negative and there is a negative value represented by f. Similarly we call an integral binary quadratic form to be indefinite if its discriminant is bigger than 0. So, these are the properties. So, you already see that discriminant is very useful because once you have the sign of the discriminant it will tell you what sign you are value set may have.

So, this is a great use of the discriminant. Now we go and try to see whether a discriminant is invariant under the change of variables that we have studied.

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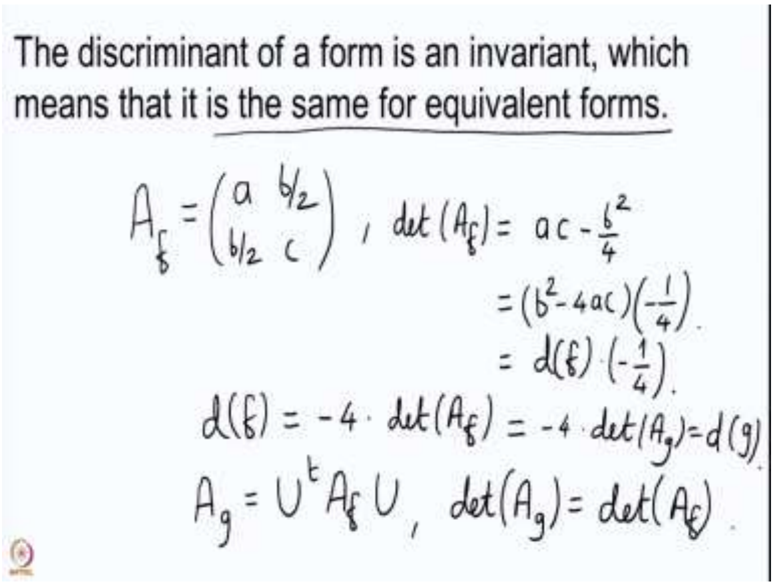

So, actually the discriminant is an invariant whenever you change the variables using the transformations that are allowed the discriminant does not change the discriminant remains the same for equivalent forms. So, for that let us observe one thing we know that our matrix for the form f is given in this way let us compute its determinant.

So we know the formula for the determinant. This is ac minus b square upon 4. So, this is nothing but b square minus 4 ac into 2 minus 1 by 4. So, the determinant of matrix associated to our binary quadratic form is a multiple by minus 1 by 4 of the discriminant or in other words the discriminant is minus 4 into the determinant of our integral binary form.

Now when I change my form f to formed g the change happening for the matrices I hope you remember that Ag was U transpose Af U and here we can easily compute the determinant Ag is nothing but determinant Af. This is because U has determinant 1. So, there is no change in the computation in the values of the determinants of Af and Ag.

And once there is no change in the values of the determinants we have that these values are also going to be same. So, we have answered all the questions that we had raised in the last lecture. We had asked whether every number which is 0 or 1 mod 4 is discriminant of some form and we gave a answer in the affirmative by explicitly writing down two forms to very simple-looking forms. We call them principle forms of those discriminants.

Later we also asked whether the discriminant is an invariant and we have proved it just now that whenever f is equivalent to g the discriminants remain the same and finally we ask what is the use of the discriminant and we saw one small application of the discriminant. We will see the some more applications of discriminants in the coming lecture. Stay tuned for more interesting things. Thank you.