A Basic Course in Number Theory
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Lecture 49
Beyond sums of squares -II

Welcome back. In the last lecture, we proved Lagrange's theorem which said that every natural number is a sum of at most 4 squares and then we saw two directions in which we could generalize this result. So, those directions were the first one was given by Waring's conjecture which generalised squares to cubes biquadrates and so on.

And then the second direction was given by the geometrical figures, I must tell you that the explicit computations in both the directions are not yet done completely the higher gk is in the Waring's conjecture which are mentioned in the last lecture, those are not all computed and similarly, for figure 8 numbers or the polygonal numbers not much is known.

However, there is something interesting that I must mention, which is that Fermat, the French mathematician had mentioned in a letter written to Pascal, that he could prove that every natural number is a sum of at most k k gonal numbers. So, this works well, because Gauss proved that every natural number is a sum of at most three triangular numbers (())(01:38) equal to delta plus data plus delta.

Lagrange also had proved already that every natural number is a sum of at most 4 squares and a square is nothing but a square. So, that represents to 4 edges or 4 points. So, it is a 4 gone and similarly, Fermat said that he has a proof that every number is the sum of k k gonal numbers, but no one has seen this proof.

Fermat is perhaps famous more for his marginal note on what is now known as the Fermat's Last Theorem. But this another thing about Fermat is not so well known. So, you should know about that also. Now, we want to go to another direction by which we can generalize Lagrange's theorem. Remember Lagrange's theorem says that every natural number is represented by the form which we refer to as the form in the last lecture which was x square plus y square plus z square plus w square, we are now going to allow some coefficients here.

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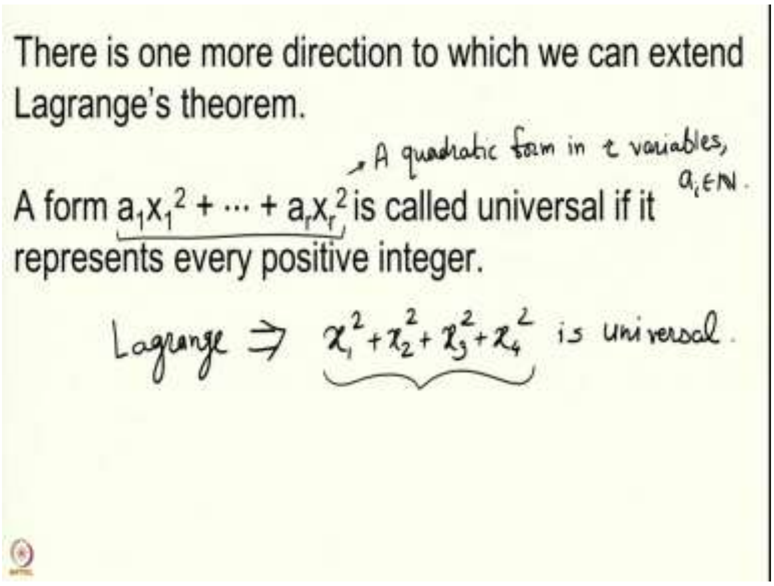

So, let me make the notations clear, we consider a form, a quadratic form in r variables a1x1 square plus a2x2 square plus dot, dot, dot plus ar xr square. So, this is the form in r variables, quadratic form in r variables and we also take ai to be natural numbers and we call such a form to be universal if it represents every positive integer.

So, we would have Lagrange has proved that x1 square plus x2 square plus x3 square plus x4 square is universal. This is what is Lagrange's result in another language, once we have introduced the notion of universal positive definite forms, but we will consider let us not go, positive definitive is ofcourse something that you know, but you know about it for binary forms, the notion of a positive definite form makes sense form in general r variables. But let us not go into that definition right now.

So, Lagrange proved that every this form this particular form is universal and now, the question is what are the integers that you can have here, a1, a2, ar such that the form that you get is universal. So, there are some very basic constraints that we have on the integers. For instance, or if all the ais were to be bigger than 1, then there is no way you can write 1 as represented by the form.

So, if you write a1 less equal a2 less equal a3 and so on then a1 has to be equal to 1. Then a2 has to be 2 or more, if a 2 is 3 and onwards then 2 is not returned by the form. So, there are some such natural constraints and therefore, whenever you fix an r you have a finite number of these forms.

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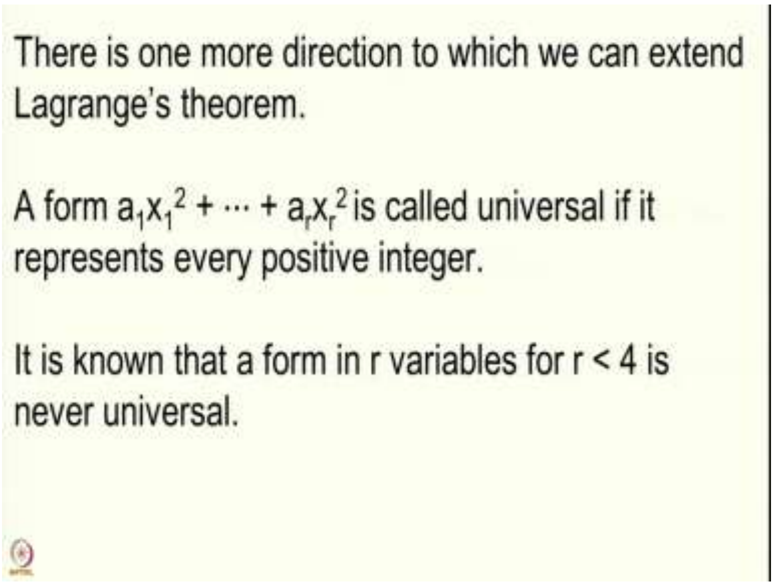

So, Lagrange it is also known that if you take a form in r variables for r less than 4, then such a form is never going to be universal, such a form can never be universal. So, you need at least 4 variables to have a form to be possibly to be universal and our very own Ramanujan worked on this after studying Lagrange theorem and he wanted to answer, what are all universal forms.

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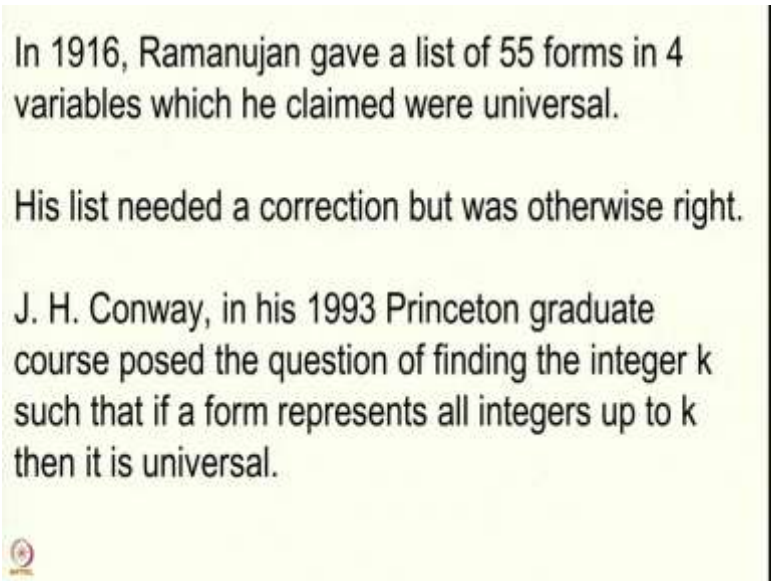

So, in 1916, remember this was the same time Hilbert had proved Waring's problem, Waring's conjecture and (())(6:09) had proved it and so on and around similar time Hardy-Littlewood also had given their proof of Hilbert's theorem. So, Ramanujan in 1916, he actually listed a set of 55 forms in 4 variables and he claimed that they were all universal.

So, he had a list of such forms of 4 variables and he claimed that they were all universals universal forms. His list needed a small correction, there was one form which had to be deleted from the list, but otherwise it was right. However, when a person like Ramanujan makes a mistake like this, there is a natural question that how are you going to determine that some form is universal? Already, Lagrange's theorem took some considerable effort to be proved.

So, one form is universal that form given by Lagrange is universal that took some quite some effort to be proved. So, we will have to prove that for all these remaining 54 forms. How do you prove that they are universal? That is not easy and one is prone to making mistakes in this. So, this is where a brilliant mathematician by the name, J. H. Conway, came up with an idea.

So, J. H. Conway, this is a mathematician who worked in Princeton, he unfortunately passed away not long time back because of the Chinese virus, COVID-19. But this is the story about some more than 20 years back. So, when he was teaching his graduate course, graduate course meaning it is a course for PhD students in Princeton, he suggested that let us try to find a number k with the property that if you have a form, if you have a quadratic form which represents all those k integers, then it is going to be universal.

He suggested this as a research problem to his PhD class in 1993 in Princeton. Once again, the question that he proposed is that he said that let us prove this result. What is the result? He proved that there is an, he conjectured or he formulated this question, that there is an integer k, such that any positive definite quadratic form represents every integer of upto k, then it is universal.

So, if this result was proved, then checking Ramanujan's list would be a very easy thing, because suppose that number k is some number say it is 100, for the sake of discussion. Then you have to check that all those 55 forms that Ramanujan had already listed or the 54 of them which were turned, which turned out to be correct.

All you have to do is that those 54 forms represent each of the first hundred numbers and you do not have to check it for everything because once 1 is represented, 4 is going to be represented, 9 is going to be represented because once you have a representation for 1 as a sum of squares as per that coefficients given in the form, you simply multiply the x y z w by appropriate multiple, so that square multiples of 1 get represented. If 2 is represented 8, 18 all these numbers are represented.

So you ofcourse, you do not to check for all hundred integers. So the actual problem of determining whether some form is universal or not, will turn out to be a finite problem once you prove this result. So, this was a very interesting idea that Conway proposed in his class.

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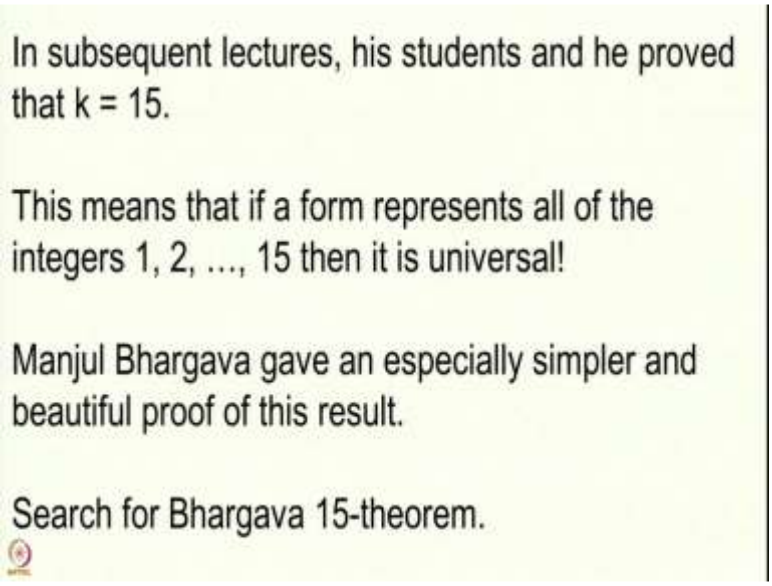

And remarkably, in subsequent lectures, he and his students proved that k is 15, they proved that if you can show that a form, a positive definite form represents all integers from 1 to 15, then it is universal. So for this computation you do not even need computers. If the k was 100 then perhaps we would need computer to show that 97 is represented by the given form.

But if you have to check only for 15, these are squares up to 15 and their multiples. So it is a very finite problem. It is a problem doable by pen and paper, once you are given a form, it is a very remarkable theorem and this is called Conway–Schneeberger theorem. So, this is a theorem which says that if you have a positive definite form, which represents every integer from 1 to 15, then it represents all integers then the form is universal.

But there was a twist in the tail, because the positive definite forms that we had taken had a certain constraint on the coefficients, the Ramanujan forms, the forms that Ramanathan considered, were what are called diagonal forms. So, they were a1x1 square plus a2x2 square plus a3x3 square plus so on. The terms x1x2, x1x3 they were not there.

So, Conway said that, let us introduce those terms also and now let us try to find the number k and Manjul Bhargava, a recent fields medallist was a student in Princeton then and Conway thought that if he could induce Manjul Bhargava to think about this more general problem, then it would be a nice result to have. So, Conway tried to induce Munjul Bhargava to study this slightly general problem where you are having positive definite forms, but those are not only diagonal, you have the cross terms also x1x2, x1x3, x1x4, x3x4 and these also then try to determine the corresponding key.

It is as it happens, Manjul was not a member of Conway's class when this theorem was proved, k equal to 15 was proved, so Manjul was not there. And Manjul thought to himself that if he has to prove the slightly generalised result, he should at least come up with a proof for this result, and so Manjul re-proved the theorem for himself and his proof was so simple and so elegant that Conway and the students Schneeberger ultimately decided to not publish the proof.

So, there is a paper by Manjul Bhargava which is called on the Conway-Schneeberger 15 theorem and before that, there is an introduction to this result. So ultimately, this result was not published in a journal but in a conference proceedings on quadratic forms and the introduction to the proceedings is given by Conway, where he lists this whole event of introducing this problem to the class and subsequently the solution by his students and him and then inducing Manjul Bhargava to solve this problem and then ultimately Manjul Bhargava solving this problem.

So, if you want to read more on this, then you should go and search for Bhargava 15-theorem, Bhargava 15-theorem. The number for the generalised forms is 290. So, if you allow the cross terms like x1, x2, x3, x4 and so on, then you have to notice that everything up to 290 is represented, then the form is universal. So, this is the third direction by which we can generalise Lagrange's theorem and here we have a very satisfactory answer and we also see that there is a very important contribution from an Indian born, Indian origin mathematician Manjul Bhargava.

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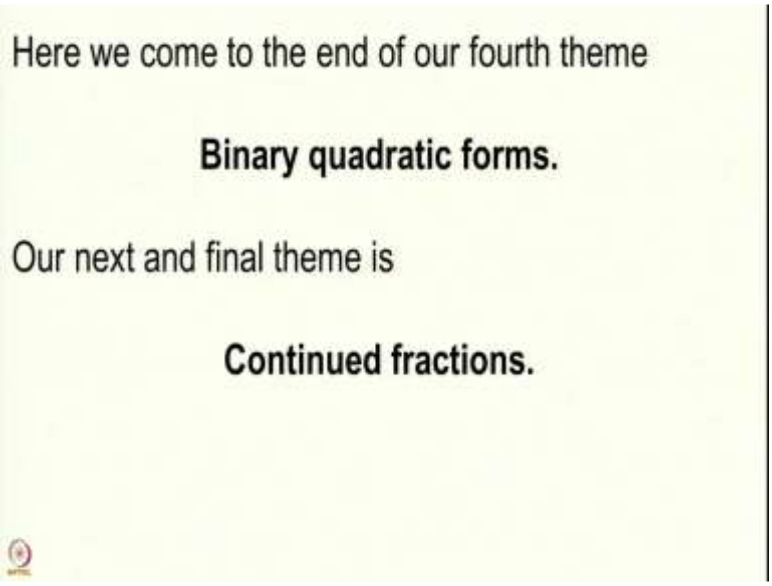

So, we now come to the end of our fourth theme, which is on binary quadratic forms and we now go to our last theme. Our next and final theme is continued fractions. So, what are continued fractions? Clearly, as the name suggests, what were binary quadratic forms? These were quadratic forms which were binary and what is a quadratic form, it is a form which is quadratic, what is a form? Form is a homogeneous polynomial.

So, binary quadratic form were forms meaning homogeneous polynomials, which are quadratic that means of degree 2 which are binary. So in 2 variables, continued fractions are fractions, which are continued, but this is not a very well-known term continued and so, it is better to see an example here is an example.

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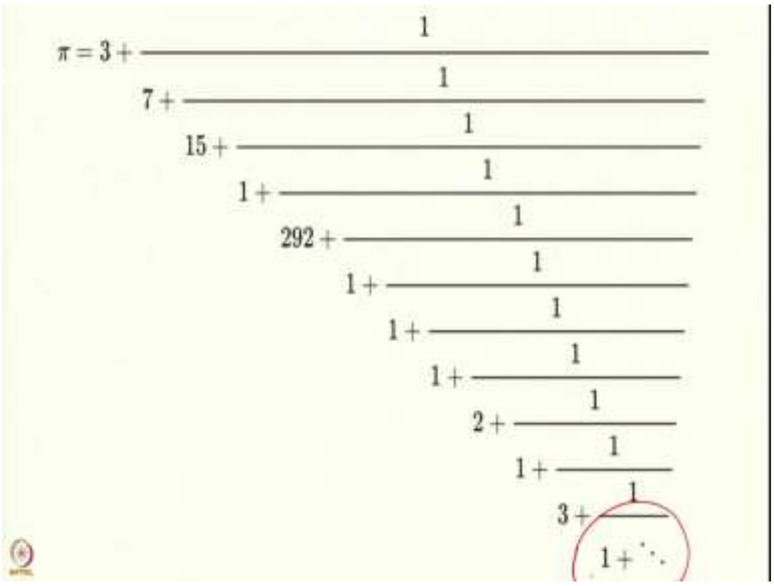

This is a continued fraction expansion, you notice that there is dot, dot, dot in the end. So, this dot, dot, dot say that this fraction expands, this fraction goes on towards infinity, these kind of representations for a real number are called continued fractions expansion for the real number, let me give you the explicit definition for continued fraction.

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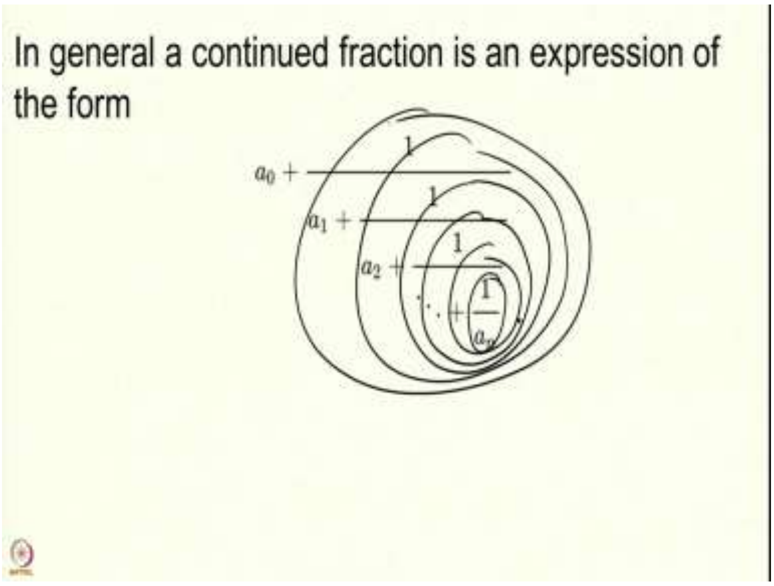

So, in general, a continued fraction is an expression of the form a0 plus 1 upon a1 plus 1 upon a2 plus 1 upon dot, dot, dot 1 upon an. So, we stop at a finite stage after n such things we stop, this is what is called a continued fraction. We see here that each of these is a fraction and we are continuing these fractions in some way. This is why this is called continued fraction.

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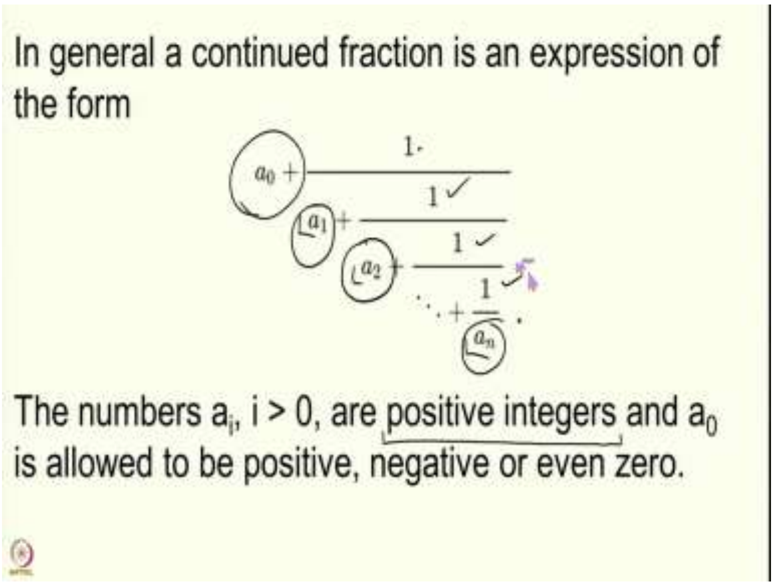

You may ask what are ais? So, the numbers ai for i bigger than 0 that means from 1 onwards that means these numbers are onwards, we are sure that these are positive integers and this first integer is a0 this is also an integer, but this is allowed to be positive negative or even 0. So, the first one a0 that can be negative 0 or positive if you are in good luck, but all the others are assumed to be positive. This is called the continued fractions expansion. We have these 1s here in the denominator.

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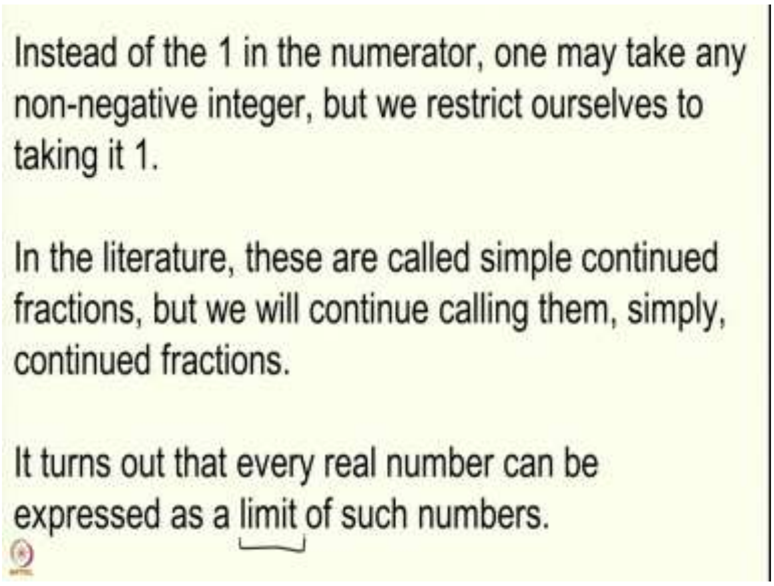

You may actually take some another non-negative integers also, some people have the convention that when you are considering the continued fractions in the numerator you take any non-negative integer somebody may take 2, someone may take 20, 25, 317 any such interior. But, we are going to restrict ourselves to taking it to be 1.

So, by our note convention our continued fractions will look like what I have shown you in the last slide. But, in the literature, when you have to deal with the continued fractions, which have these general numerators our continued fractions the one which have only 1 on the numerator, they are called simple continued fractions and the other ones where you allow any non-negative integer they are called generalised continued fractions.

But since we are not going to talk about those generalised continued fractions, we are not going to introduce the word simple for our term, we will just call them continuous fractions. So, these are the continued fractions for us, you understand that you have a0 plus 1 upon some bunch, which is a1 plus 1 upon some another bunch and so on.

So, it turns out that every real number in fact, has a continued fractions expansion which means that every real number is a limit of such numbers. You remember that for the expansion for Pi had the dot, dot, dot. So that means that you are not going to stop at a finite level. But you are going to continue after that after every n and that is important, because if you stop at any level, then what you get is, in fact a rational number.

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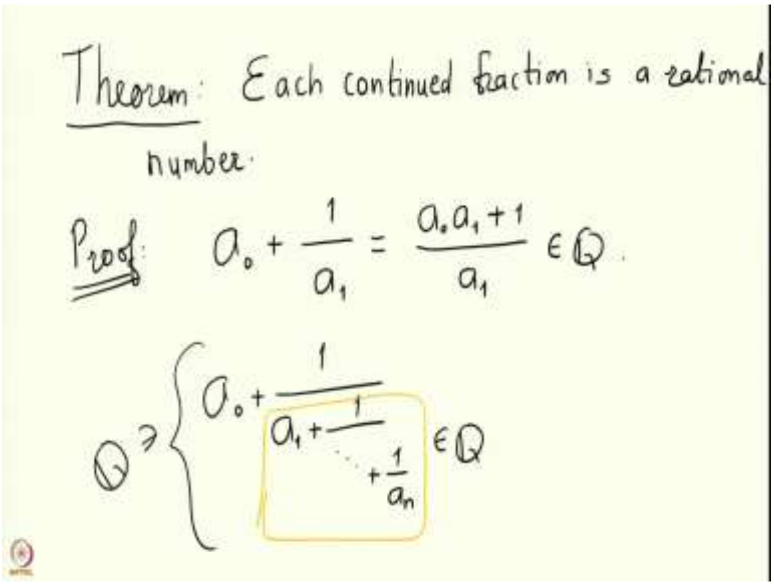

So we prove this result each continued fraction is a rational number. Proof, so ofcourse if you have a0 plus 1 upon a1 then this is nothing but a0, a1 plus 1 upon a1 this is a rational number. So, if your n is 1 then you get it to be a rational number. Now, we go to the general case where we have a0 plus 1 upon dot, dot, dot and finally we have one upon an, then we would look at the number which is here.

This has one less element than our earlier expansion. So, by induction argument if we assume that this is a rational number, it follows that the whole thing is also a rational number. So, we prove whenever you have a0 and a1 then it is a rational number that is proved. When you have just a0, of course, it is an integer every integer is a rational number.

If you have a0, a1 up to an, you ignore a0 for the moment and write a1, a2, an as b0, b1 up to bn minus 1. Now you have an expression of length 1 less and by induction hypothesis, we will assume that for such an expression you have, what the number you get is a rational number, then by simple arithmetic, it will follow that the whole expression you started with is also a rational number. So, every continued fraction is a rational number and now we want to say whether every rational number has a continued fractions expansion. So let us do an example.

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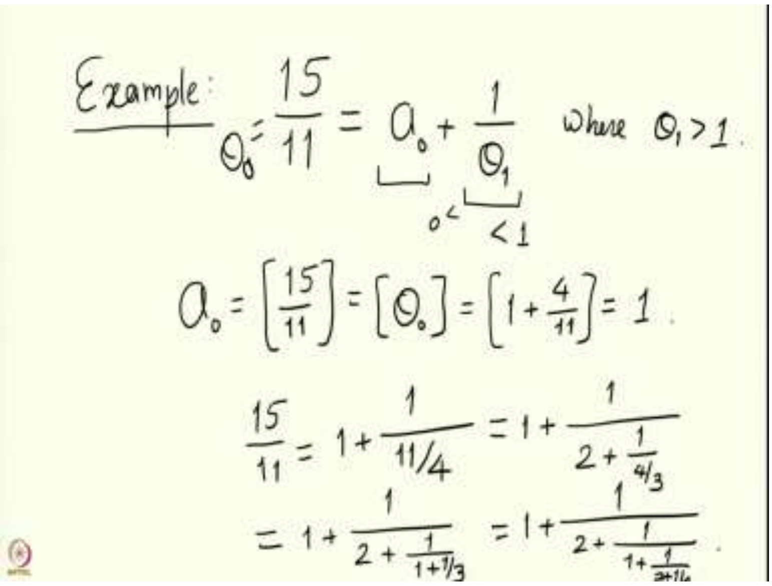

Suppose we want to write 15 by 11 as a continued fraction, so we start with writing it as a0 plus 1 upon theta 1, where theta 1 is bigger than 1. When theta 1 is bigger than 1, its reciprocal is going to be less than 1. So, a0 has the property that a0 is an integer and the difference between 15 upon 11 and a0 is less than 1, we have to find an integer whose difference with 15 by 11 is a positive number. We want this to be bigger than 1, so this fraction is also positive less than 1 but it is a positive number.

So, a0 is the largest integer smaller than or equal to 15 by 11. This is a notation that we use to denote the largest integer smaller than or equal to a given real number. So, if I call 15 by 11, as theta 0, then we have defined a0 by the notation square bracket theta 0. We note that 15 by 11 is 1 plus 4 by 11 and therefore it is integral part the largest integer smaller than or equal to 15 by 11 is 1. So a0 is 1.

Therefore we have written 15 by 11 as 1 plus 1 upon 11 by 4, it was 1 plus 4 by 11, but after taking the reciprocal, it has become 1 by 11 by 4, we repeat the same procedure. Note that 11 by 4 has integral part to be equal to 2. So this is 2 plus 1 upon 4 by 3. Notice that our denominators are decreasing. Earlier we had denominator to be 11, then we got our denominator to be 4 and now the denominator is 3.

So, we get it to be 1 2 1 and then simply 1 by 3, the 1 by 3 is the reciprocal of 3. So, you actually just have it to be 1 by 3 or you may further expand it by adding a 1 in the end, but that is where the fraction continued fraction expansion for 15 by 11 stops. So, this example tells us that possibly every rational number has a continued fraction representation, we have that every continued fraction is a rational number necessarily.

And now, we will prove in the next lecture that every rational number is also a continued fraction and we will ultimately prove that every real number is approximated by the continued fractions. This is our last and final theme on, in this course on basic number theory, this theme is going to be very interesting, we are going to see many applications of this theme. We will see some things related to transcendental number approximations to real numbers by continued fractions and so on. Stick around. See you in the next lecture. Thank you very much.