A Basic Course in Number Theory
Professor Shripad Garge
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 50
Continued fractions basic results

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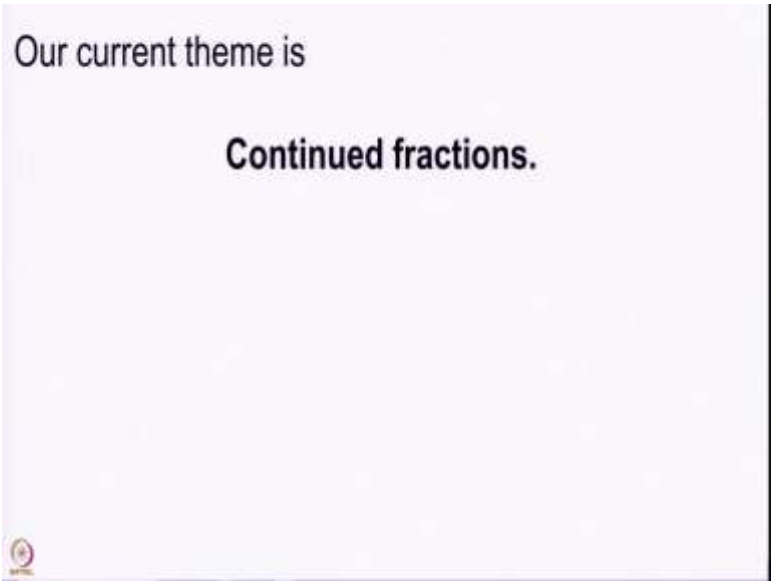

Welcome back, we are in the final theme of our lecture course. The theme is entitled continued fractions. So, these are basically fractions, which are continued in some way.

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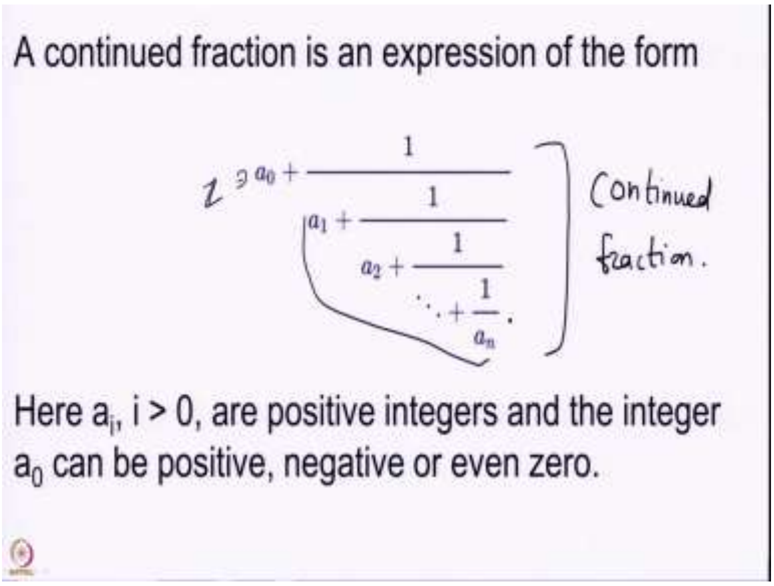

More precisely a continued fraction is an expression of the following form here, these integers ai they are positive integer from a1 onward. So, these onwards these are all positive and a0 is just an integer it can be 0 it can be positive or it can even be a negative integer, we have such an expression and this expression observe that it involves only finitely many integers, such an expression is called a continued fraction. It is being called like this, because it involves fractions and they are continued in some sense. So, we call it continued fraction.

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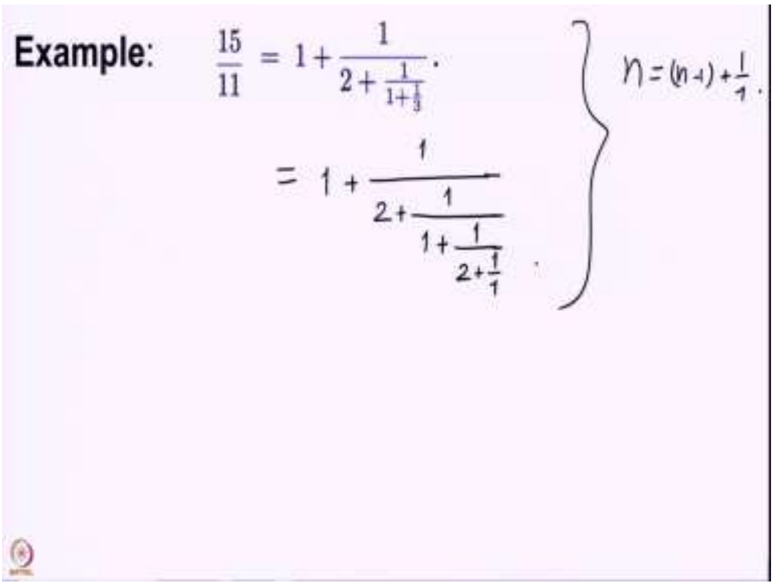

We also saw an example in the last lecture that 15 by 11 can be written as a continued fraction in the following way. Moreover, this is not quite unique such expression because the last integer 3 can also be written as 1 by 2 plus 1 by 1. So, we have some sort of non-uniqueness whenever there are rational numbers involved, but the non-uniqueness would only mean that the last term where you will have an n that can also be written as n minus 1 plus 1 by 1, this is the only way by which we will have non-uniqueness for the continued fraction for rationals.

Otherwise, it is essentially unique. So, before saying all this things, we should also give a way to construct a continued fraction representation for a rational number and so on. We will do all these things as and when the time comes.

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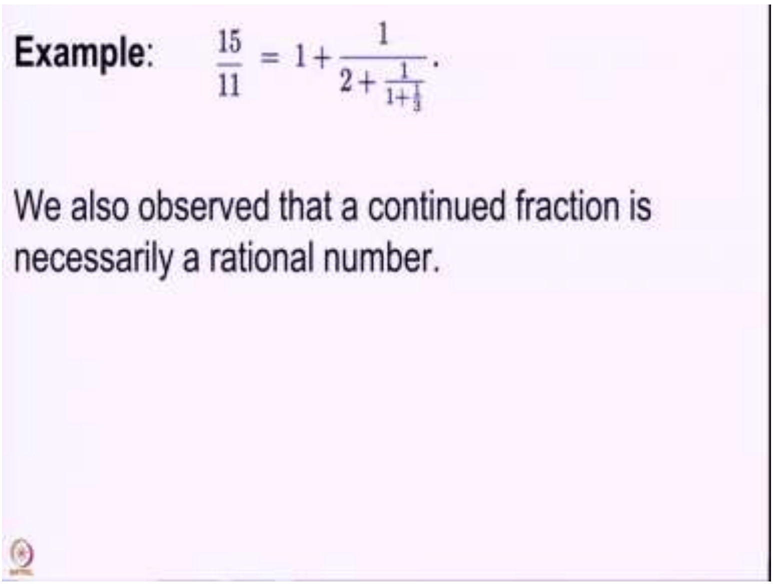

But we also noticed another thing that if you have a continued fraction, then it is necessarily a rational number, observe that a continued fraction is obtained by starting with an integer and then you write it as 1 upon a. So, you have a0 plus 1 and then you have a big horizontal line, a long horizontal line below that, you write a1 plus 1 upon again another longer, slightly shorter, but a long horizontal line, then a2 and so on and we stop at an only finite data. So, we observed it in the last lecture also that this is necessarily a rational number, let us see the proof once again.

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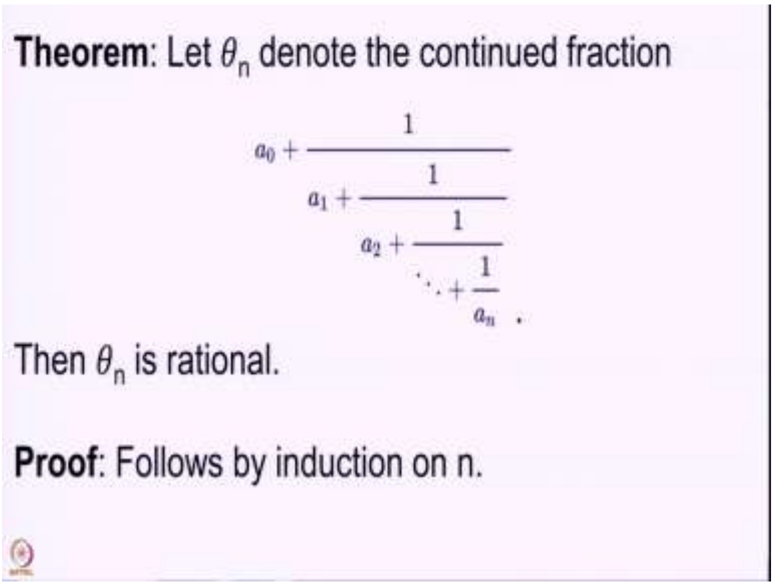

So, suppose theta n denotes this continued fraction, then theta n is a rational number and as one would expect, this proof follows by induction on n.

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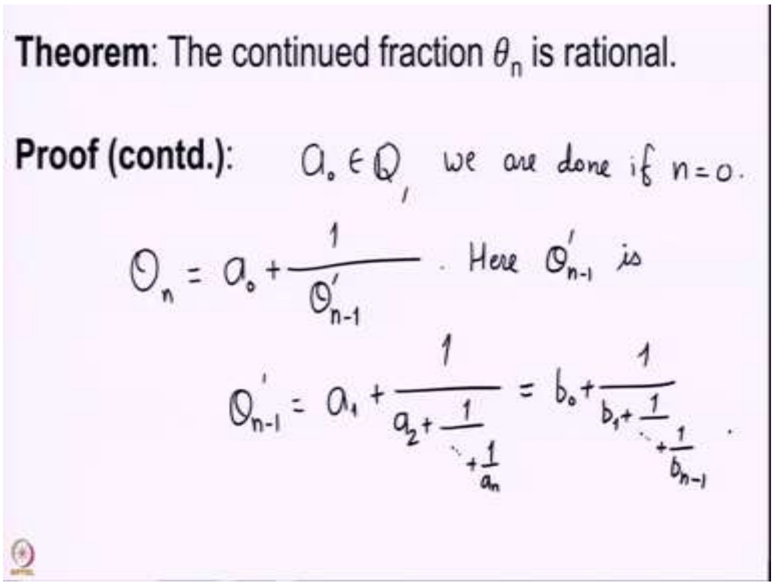

So, induction will start with n equal to 0 or n equal to 1 wherever you want. So, we observe that a0 is ofcourse a rational number, so if your n was 0, so we are done if n is 0, moreover any theta n can be written as 1 a0 plus 1 upon here I will write it as theta prime n minus 1, where theta prime n minus 1 has a continued fraction expansion or here theta prime n minus 1 is a1 plus 1 upon a2 plus 1 upon dot-dot-dot plus 1 upon an, which we write as b0 plus 1 upon b1 plus 1 upon dot-dot-dot 1 upon b n minus 1 and if we assume the induction hypothesis, because we are done with n equal to 0.

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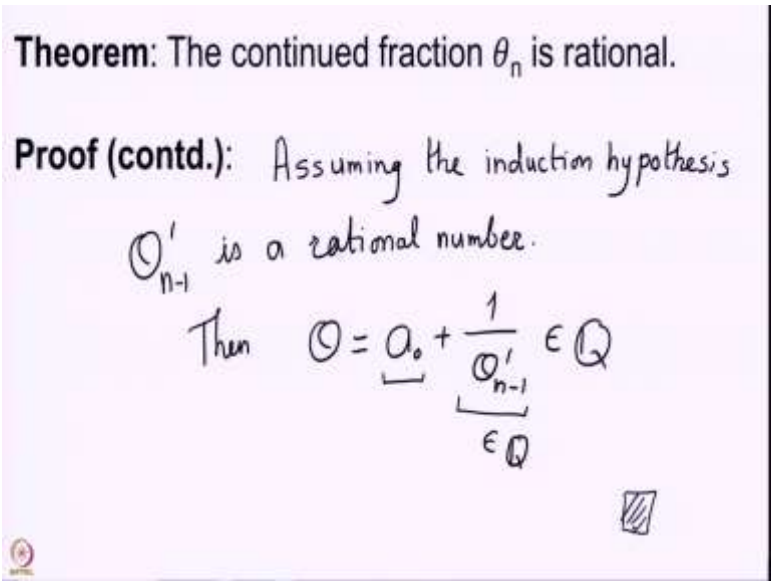

So, assuming the induction hypothesis, assuming the induction hypothesis theta n minus 1 prime is a rational number and then theta n a0 plus 1 upon theta n minus 1 prime also has to be a rational. Because once theta n minus 1 prime is rational this quantity is a rational number being reciprocal of this rational and then you are simply adding an integer. So, ultimately the number that you get theta n also has to be a rational number.

So, by the induction hypothesis whenever the result is true for n minus 1, the result is true for n once you have a continued fraction expansion, a continued fraction representation having a0, a1 up to an and if the for any a0, a1 up to an minus 1, if you have that the corresponding number is rational, then we prove that the number for a0 up to an is also rational.

Ofcourse, we have proved that four n equal to 0 the result holds. So, by the method of induction, this result is now done. So, every continued fraction is a rational number, is the other way also true if we have a rational number is it equal to a continued fraction that is also true.

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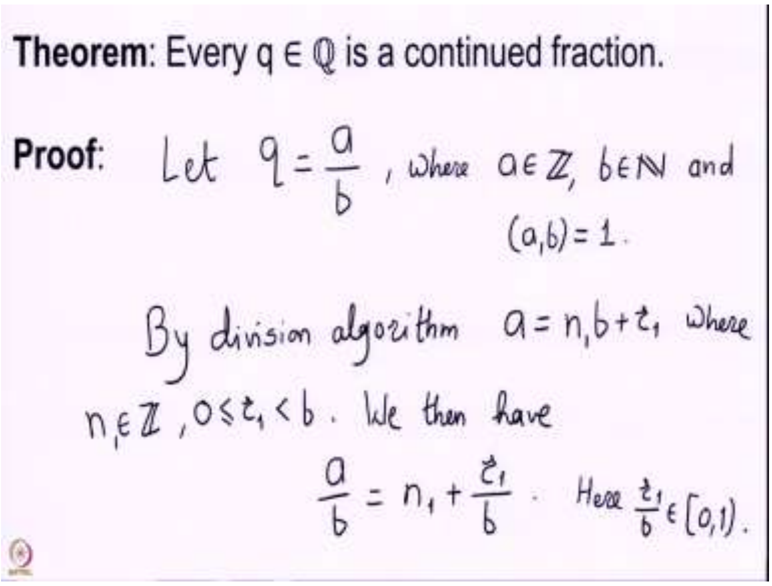

So, every rational number if you start with a q in rationals, then it can be written as a continued fraction. Remember, continued fraction is essentially given by the sequence a0, a1, a2, an, where a0 is an integer and a1, a2, an are natural numbers. So, we want to write q rational number in this form. Suppose our q has the form a by b where a is an integer and b is taken to be a natural number. So our b is taken to be positive and we will also have that a by b is 1, the GCD of a and b is 1, this is our standard requirement.

Now, a might be less than b or a might be bigger than b. So, what we do first of all is that we apply the division algorithm to a and b. So, by division algorithm a is going to be n1 plus r1, a is going to be n1b plus r1 where your r1 is strictly less than b and ofcourse you have that n1 is an integer.

The division algorithm introduced by Euclid that we have studied in our course, requires both a and b to be natural numbers. But we can ofcourse do it whenever b is a natural number and a is any interior we can ofcourse have the corresponding result with the small change that the quotient that you had obtained that quotient can now be an integer.

Earlier when we have worked with a and b to be both natural numbers, we had that the quotient q1 that we had got obtained there was a natural number. But if you have your a to be negative, then it is possible that B into a negative number gives you a plus there is a remainder. So, we will now take the n1 to be an integer and r1 is now a positive integer with the property that it is between 0 and B, but it is not equal to b.

So, with this we then have a upon b equal to n1 plus r1 upon b. So, we have written a by b the given rational number as an integer plus r1 upon d. Now, this r1 can be 0 or it can be non-zero but it is strictly less than b. So, r1 by b belongs to 0, open 1. Since r1 is strictly less than b, r1 upon b is a rational number which is less than 1 and because r1 can be 0, you have that r1 upon B can also be 0, but it is otherwise strictly between 0 and 1. So, if you take the reciprocal of r1 by b, that will be a number which will be bigger than 1.

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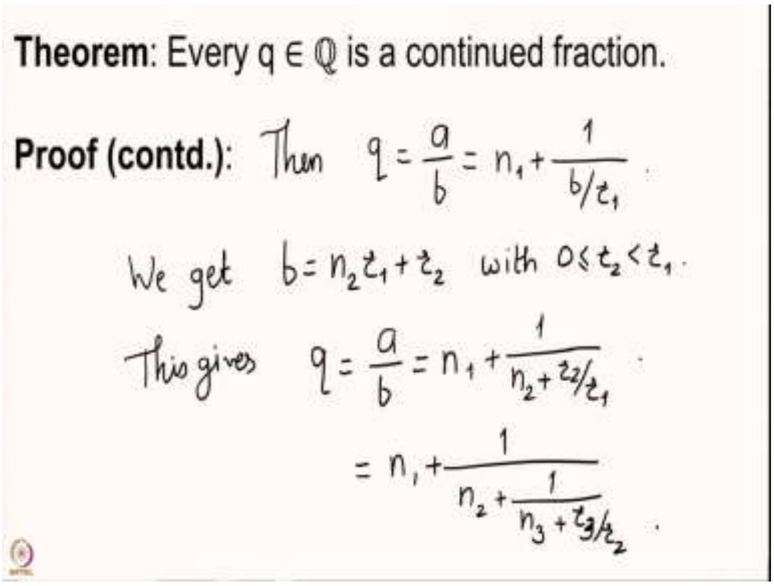

We have q equal to a by b to be 1 n1 plus 1 upon b upon r1. Now r1 is less than b and so we can apply the division algorithm once again to the pair b comma r1. So, we get b equal to n2 r1 plus r2 with 0 less than or equal to r2 strictly less than r1. Now, all these are natural numbers, because our B is a natural number r1 is a natural number. So, n2 is natural r2 is natural and moreover r2 lies between 0 and r1 with possible equality at 0.

So, this gives q equal to a by b which we have written as n1 plus 1 upon b by r1. But we once divided by r1, we get this to be in n2 plus r2 by r1. Once again r2 by r1 is strictly less than 1. So, you invert that and repeat the procedure and you will get r3 by r2 and notice here that the denominators that you are getting at all these levels earlier we had b then we got r1 which is strictly less than b, then we got r2 which is further strictly less than r1 and so on if you continue this way, you are going to get denominators which are strictly decreasing.

And these decreasing denominators will eventually reach the number 1 which is where the Euclidean algorithm concludes to give you the GCD of a and b which is going to be 1 because we have assumed that a and b are co-prime.

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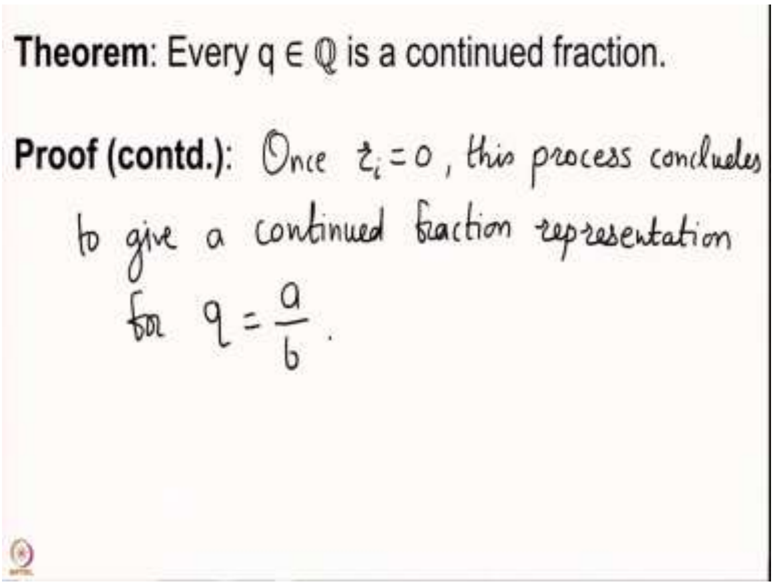

So once this reaches one, then we will conclude this process once ri becomes 0 this process concludes to give a continued fraction representation for q which is a by b. So we proved that while every continued fraction is a rational, on the other hand, every rational number can also be written as a continued fraction. Now, the next thing we would want to do is to look at general real number and see how the continued fractions help us in approximating the real numbers.

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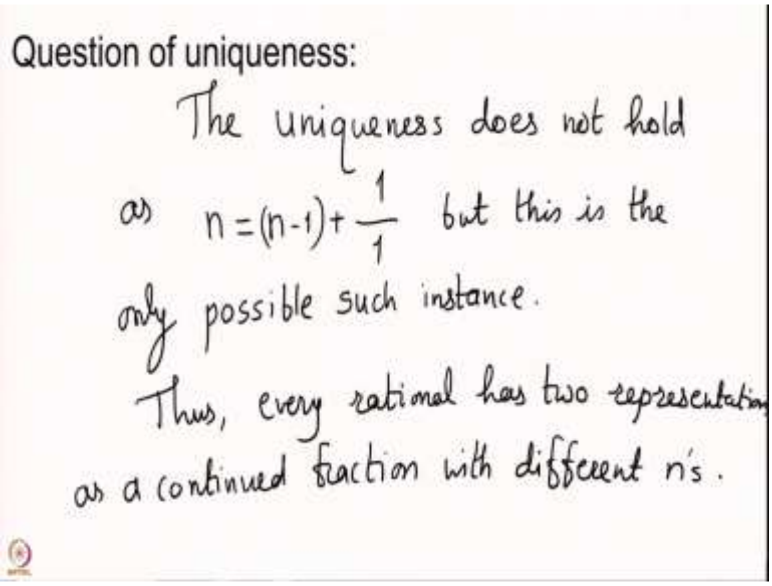

There is of course, the question of uniqueness, but we have already noticed that when you have, the uniqueness does not hold as n is n minus 1 plus 1 upon 1, but this is the only possible such instance. That means, every rational number, thus every rational has two representations as a continued fraction with different ends.

So, if you have a continued fraction representation for n integers involved there, then you will have another with n plus 1 integer or n minus 1 integers. So, this is the only difference that we will have for a continued fraction representation for a rational number. This is the small thing that we have to be careful about, whenever we want to write any rational number as a continued fraction.

But the algorithm that we have expressed is going to give us the continued fraction for the given rational number. Note that even if you have two representation for the same continued fraction, one with n integers involved and say another with n plus 1 integers, the value of the continued fraction does remain the same.

So, unless you have to really work with the numbers, which are there involved in the continued fractions, unless you have to do that, you do not really have to worry about this question of non-uniqueness. So, we would further want to see that every real number theta is a limit of continued fractions in some sense, a unique way.

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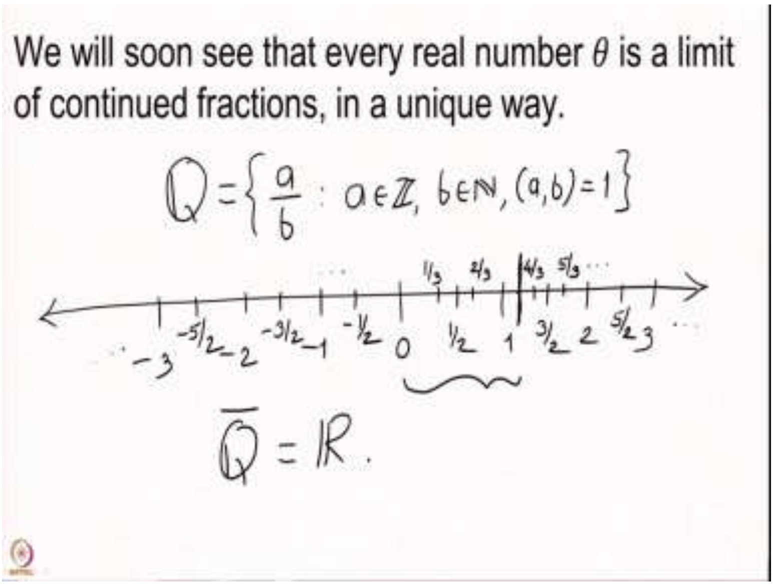

So, I expect that all of you know how the real numbers are constructed, from rational numbers there is some small bit of analysis, which we will have to consider here. So, everybody knows that this is the set of rational numbers you have a by b, where a is an integer b is a natural number and we will also assume that a by the GCD a b is 1, this is the set of rational numbers and if you have the real line, then the rational numbers can be plotted on this real line you have 0, you have 1, 2, 3 minus 1 minus 2 minus 3, then you have 1 by 2, this is 3 by 2, 5 by 2, minus 1 by 2, minus 3 by 2, minus 5 by 2 and so on.

We can continue this way, but the distance between the two rational numbers will become smaller and smaller once you increase the denominator, so for instance, one third will come somewhere here, two thirds comes somewhere here, then this will be four thirds and this will be five thirds and so on and on this side there are negatives.

So, this is also a fact that any real number, now what is the real number? So, one way you can think about real numbers in an intuitive sort of way is any number which can be plotted on this real line, you take your pen and point it anywhere. Suppose I pointed here, so I am looking at this real number.

Anything that can be plotted, so the real number now is the quantity which is the distance of 0 to this particular point. So, any such point will give us a real number once we have specified where we are putting 0 and once we define what is one that will decide what is the unit, once you have that then any point on the real line gives rise to a unique real number and it is clear that any real number is always a limit of rational numbers.

So, this is expressed by saying that the set of rational numbers is dense in the set of real numbers, there are many ways to obtain the sequence of rational numbers converging to a given real number what you can do is that first of all, you can look at all integers.

Now, distance between any two integers is at most one, any two consecutive integers is at most is equal to 1 and your real number theta has to belong to some such consecutive pair of consecutive integers you are, we have the whole real line, which we are going to cut into several sub intervals of length 1 whose endpoints are integers and then your real number will have to belong to some certain n to n plus 1.

Then you take the smallest one, call that n that will be the first term of the sequence converging to your real number. Because once you have this integer, next you can look at all the numbers whose denominators is equal to 2 and the numerator can be all integers once again. So, you have the number 0 by 2 which is 0, then 1 by 2, then 2 by 2, which is 1, then 3 by 2, 4 by 2, which is 2, then 5 by 2 and so on.

So, once again you have an infinite set. But now the distance between any two consecutive such numbers is reduced by a factor of 2. Earlier we had integers where the distance was 1, now we have these rational numbers whose denominator is 2 or 1 and so the distance between any two such numbers is equal to, any to consecutive such numbers is 1 by 2 and your theta has to belong to one such sub interval.

So, we will again take the one which is just preceding your theta. So, you have gotten another element. Possibly it is the same element that you obtained in the first step, but it could also be some new element. So, this is your second element, earlier element had the property that your theta and your earlier element were bound, the distance between these two was less than or equal to 1.

Now, the second entry of the sequence that we are going to construct has the property that theta and this second entry the difference of the two is less than or equal to 1 by 2, then you will go to the rational numbers whose denominators are 3. So, you will have 0 by 3, 1 by 3, 2 by 3, 3 by 3 which is an integer 4 by 3, 5 by 3, 6 by 3 which is an integer same on the negative side and once again, you get third entry of your sequence with the property that the distance now from theta of the third entry is less than or equal to 1 by 3.

So, we just continue this way and you will get some number a by n for every n you will get an a depending of course on n, such that the distance of theta and a by n is less than or equal to 1 by n and a is an integer and you can also always choose a by n to be less than theta it is on the left hand side.

So, when you look at mod theta minus a by n it is actually theta minus a by n that it is a positive quantity, because theta is bigger than a by n. So, we have this difference to be less than or equal to 1 by n and this can be done for every n as you let n go to infinity, we have that actually a sequence is constructed of rational numbers going to theta.

Now, you could have taken the numbers coming after the theta and that would give you another such sequence or you could take the first term to be before theta next one after theta, third before theta, fourth after theta and so on. So, there are many ways to construct these sequences. But what we want to know is whether there is a nice way to approximate real numbers by rational numbers.

So, what we have constructed so far has the property that theta minus a by n has distance less than 1 upon n. Now we may say that 1 upon n is a very big number, we want the distance to be less than 1 upon n square, we want to have for every theta we would like to have rationals of the form p by q such that mod theta minus p by q is less than 1 upon q square.

We would like these rationals to come very close to our theta while keeping theta as small as possible, this is what we would want to do. So, this and many other things will come in the next lecture. So, I hope to see you then thank you.