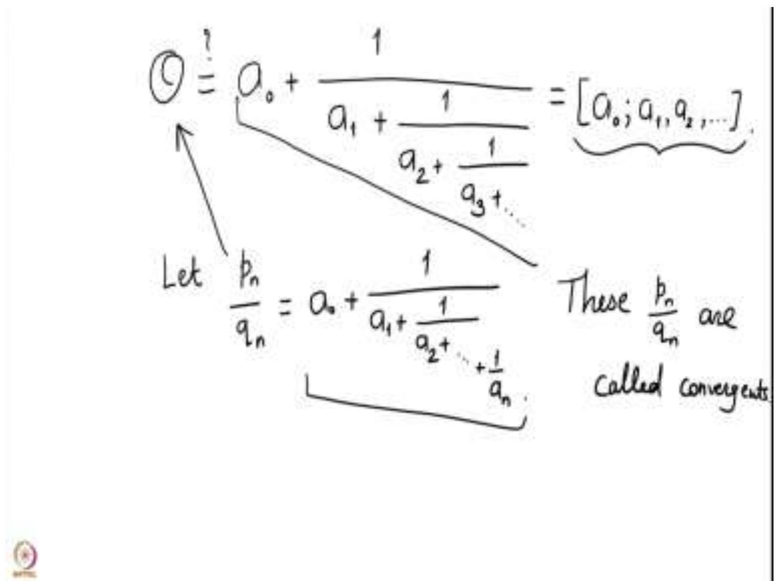
A Basic Course in Number Theory
Professor Shripad Garge
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 53
Continued fraction Expansion for Real Number – l

Welcome back we have constructed a continued fraction expansion in towards the end of our last lecture and we now want to prove that this expansion really converges to the theta we started with. So, coming to our slide let me recall this construction for you.

(Refer Slide Time: 00:38)

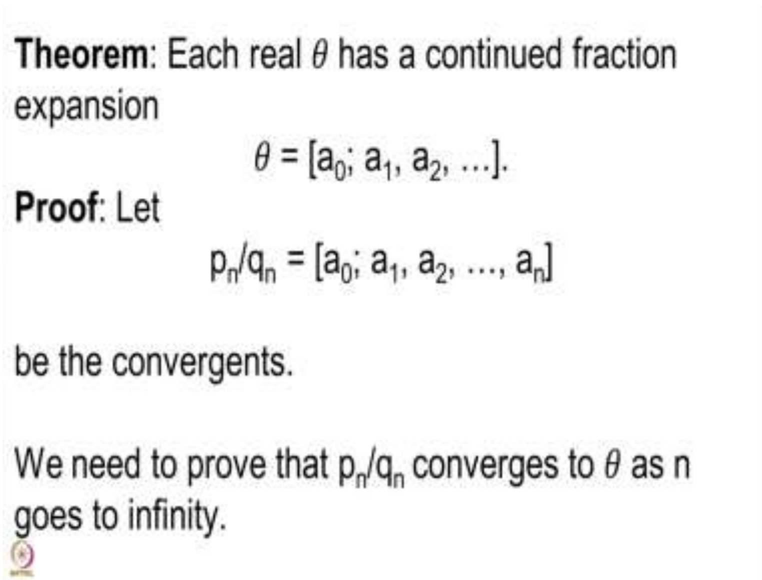


So, we have written every theta as a0 plus 1 by a1 plus 1 by a2 plus 1 by a3 and so on. This is the continued fraction expansion for our real number theta, if you cut it at any finite stage we call it the convergent, so I cut it at the nth stage, I take; do not take the whole continued fraction expansion going all the way to infinity, but we cut it at the nth stage, then we call of course this is going to be a rational number, so we write it as pn upon qn where qn is a natural number pn is an integer.

And we will also have that pn upon pn comma qn is 1 the gcd of pn and qn is 1. We also have a simpler notation to write this continued fraction expansion we write the first integer which is the integral part of theta as a0 we put a semicolon after that and then we write all these integers separated by commas.

So, this is our notation when we write the continued fraction expansion in this way it means that we are looking at this continued fraction expansion and we want to prove this we have not yet proved this, we note that if you cut it at finite stage it is a rational number we call it pn upon qn and we actually want to prove that as n goes to infinity pn upon qn converges to theta, this is what we want to prove. So, the notation is that these are called these rational numbers are called convergents, because they converge to the real number theta.

(Refer Slide Time: 03:10)



So, this is the theorem that we want to prove, we want to prove that we actually have equality on the right side of the equation we have a continued fraction expansion a0 a1 a2 dot dot dot and on the left hand side of the equation we have the real theta and the a0 a1 a2 are the ones which are obtained as previous. So, these a0 a1 a2 are related to theta as we have described in the previous lecture and we want to prove that the limit of the convergents is actually our real number theta.

So, the proof, let us fix the notation once again because we are going to prove a major theorem, so pn upon qn these are the rational numbers which are giving the continued fraction a0 a1 a2 up to an, once I call it a continued fraction it has to be a rational number and we are doing it by cutting the continued fraction expansion at the nth stage.

So, these are the convergents that we have obtained, we need to prove that pn upon qn converges to theta as n goes to infinity, this is the result that we want to prove. This proof is very big, there are several steps in the proof and so it is imperative that such a big proof such a long proof be broken into several small proofs.

(Refer Slide Time: 04:46)

Lemma 1: The p_n and q_n satisfy the recursion

$$p_n = a_n p_{n-1} + p_{n-2}, \; q_n = a_n q_{n-1} + q_{n-2}.$$

Lemma 1: The p_n and q_n satisfy the recursion

$$p_n = a_n p_{n-1} + p_{n-2}, \; q_n = a_n q_{n-1} + q_{n-2}.$$

Lemma 2: $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$.

Lemma 3: $p_{2n}/q_{2n} \leq \theta \leq p_{2n+1}/q_{2n+1}$ for every n.

Lemma 4: The sequence q_n goes to infinity.

And we note those statements by calling them lemmas, so this is the first lemma the pn and qn satisfy the recursion pn equal to an pn minus 1 plus pn minus 2 qn equal to an qn minus 1 plus qn minus 2. So, this is the recursion that pn and qn satisfy, we once you know how to construct p0 q0 and once you know what is a1 then you should be able to construct p1 q1, once you know how to construct p1 q1 p0 q0 and you know a2 then you should be able to construct p2 q2.

Ofcourse when you are writing the pn upon qn as the continued fraction expansion, then it is quite natural that you will have a0, a1, a2, an coming in the expansion giving you the formula for pn and qn because you can expand the continued fraction expansion, you can make it simpler and write it as a number upon another number and that will give you the formulae for pn and qn.

But we note that pn qn can be constructed recursively and this recursive construction is very important because recursive construction will tell us how pn plus 1 and qn plus 1 behave compared to pn and qn. So, we will use this method this information to get to our proof, this is the first lemma that we are going to use. There is second Lemma which says that pn qn plus 1 minus pn plus 1 qn this is plus or minus 1, it cannot be anything else than this.

So, this number is always plus or minus 1. The third lemma says that our theta which we had started with is sandwiched between any two successive convergents and the sandwich is such that the even convergent is less than or equal to theta and the odd convergent is bigger than or equal to theta.

So, if these are converging to theta which are going to prove the convergence happens in the following way that you start with the integral part of theta which is on the left hand side then you will obtain p1 by q1 which is going slightly ahead, then you will obtain p2 by q2 which is coming back towards theta and again over shooting theta by a smaller margin and then you again go to p3 by q3 which is again going further than theta.

So, in some sense these convergent are trying to get to theta each time doing a correction of the direction and reducing their distance. So, you had the a0 which is the integral part of theta which is less than theta then the continued fractions machine let us say realized that you have not reached theta, so it starts going to theta but by the time it reaches theta it cannot put brakes.

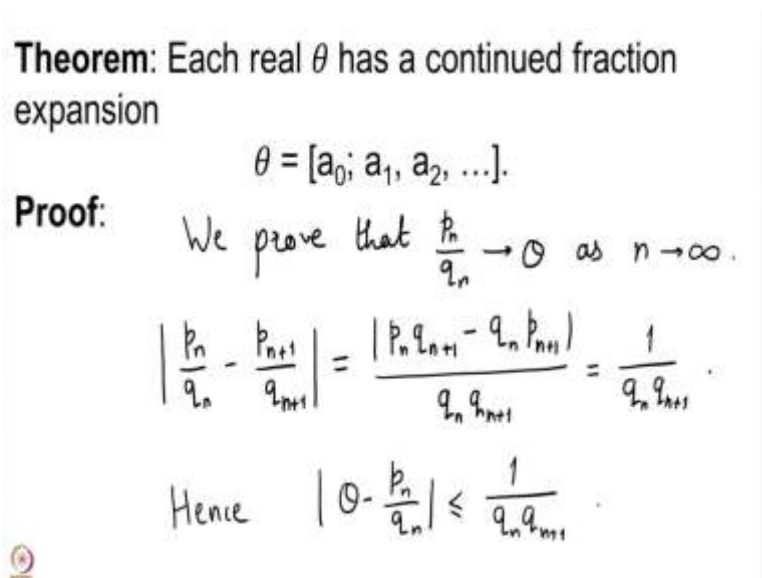
So, once it puts the brakes on its goes slightly beyond theta and then stops it gives you p1 by q1 then it realizes that it has overshot theta, so it turns back again starts traveling towards theta it remembers the previous experience that it has overshot theta by some distance, so this time it goes slowly.

But again, once it reaches theta it cannot stop there, so it goes further and use you another convergent p2 by q2 and this continues if your theta is irrational then this process will simply continue and you will get this convergents coming closer and closer to theta. So, this is the third lemma which also is a Lemma that we should prove and finally lemma-4 is this very simple statement that the sequence qn goes to infinity.

So, these are the four lemmas which we are going to use. If you remember the proof of the quadratic reciprocity law that we had done that also had a sequence of lemmas, but we proved each of those lemmas first and then we did actually the proof. We are going to do is here the other way we will first assume all these lemmas complete the proof and then we will prove these lemmas one by one.

So, we assume all these proofs, we assume that there is this recurrence relation for pn and qn, we also assume that the successive pn and qn the integers appearing as numerator and denominator for the successive convergents satisfy that their difference the is minus 1 power n plus 1, then we also assume the third lemma which says that theta is sandwiched between the convergents the even one before theta and the odd ones after theta and finally that the qn go to infinity the denominators qn go to infinity. So, with these four lemmas we go towards our proof and we prove this.

(Refer Slide Time: 10:14)



Lemma 1: The p_n and q_n satisfy the recursion

$$p_n = a_n p_{n-1} + p_{n-2}, \; q_n = a_n q_{n-1} + q_{n-2}.$$

Lemma 2: $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}.$

Lemma 3: $p_{2n}/q_{2n} \le \theta \le p_{2n+1}/q_{2n+1}$ for every n.

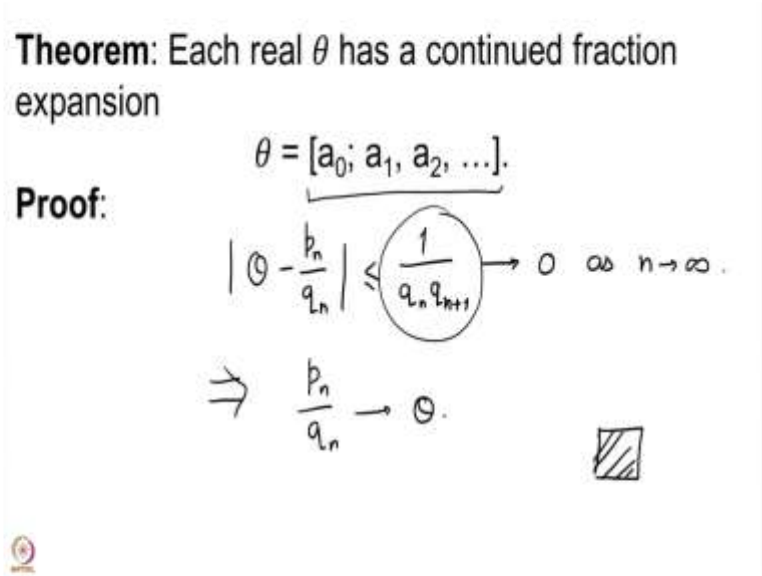
Lemma 4: The sequence q_n goes to infinity.

So, we approve that pn upon qn converse to theta as n goes to infinity this is what we want to prove. So, before we do anything else let us look at this difference. So, pn upon qn minus pn plus 1 upon qn plus 1 this is ofcourse pn qn plus 1 minus qn pn plus 1 upon qn qn plus 1, there is a mod but we note that the denominators are always taken to be natural numbers.

So, we do not need to put the mod here. Now, the numerator that we have here is equal to plus or minus 1, this is our lemma 2, pn qn plus 1 minus pn plus 1 qn is minus 1 to the power n plus 1. So, the numerator here is 1 or minus 1 and therefore this is qn qn plus 1. So, the difference between any two successive convergents is 1 upon qn qn plus 1.

Now, theta is sandwiched between the successive convergence, therefore the distance of theta from any of these two is going to be less than the distance of these two successive convergence. Therefore mod of theta minus pn by qn is less than or equal to 1 upon qn qn plus 1.

(Refer Slide Time: 12:35)



But then we have lemma four which tells us that the sequence qn goes to infinity mod theta minus pn upon qn which is less than or equal to 1 upon qn qn plus 1, this right hand side goes to 0 as n goes to infinity, because qn go to infinity. So, given any number there are only finitely many qn's less than or equal to that number the sequence qn goes to infinity. So, this is some notion of analysis that I am going to be using here, this is a very basic notion which normally one learns right after one completes ones 12th standard.

So, if you have a sequence of real numbers going to infinity it really means that given any integer let us say any natural number there are only finitely many terms of that sequence less than or equal to that interior. Therefore these qn qn plus 1 these are going to be going to infinity the product will also naturally go to infinity.

And therefore this the 1 upon qn qn plus 1 goes to 0 as n goes to infinity which means that pn upon qn converges to the real number theta. So, the convergents that we get from by cutting this continued fraction expansion at the nth stage do actually converge to our real number theta. So, assuming those four lemmas we have proved our result very quickly, but now we need to prove those lemmas.

(Refer Slide Time: 14:25)

Lemma 1: The p_n and q_n satisfy the recursion

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Lemma 2: $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$.

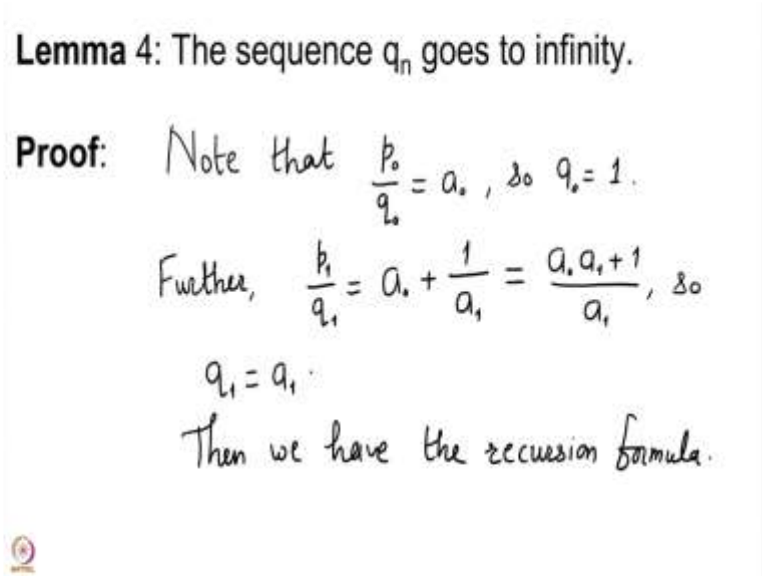
Lemma 3: $p_{2n}/q_{2n} \leq \theta \leq p_{2n+1}/q_{2n+1}$ for every n.

Lemma 4: The sequence q_n goes to infinity.

So, once again, let me recall those lemmas for you that pn qn must satisfy recursion, the distance between the two successive convergents that the theta is sandwiched between the convergent successive convergents and that the sequence qn goes to infinity. So, let me remind you that we have first assumed these lemmas and then prove the theorem, so we went in the reverse direction to what we actually did in the proof of the quadratic reciprocity law, let me employ the reverse direction here also.

So, these are the four lemmas that we need to prove, let me prove lemma four the first, so I will prove lemma four which is that the sequence qn goes to infinity, I will ofcourse need to use the recursive relation, so let me remind you that we have this recursion for qn, qn is an qn minus 1 plus qn minus 2, we have this recursion and using this we will prove that the sequence qn goes to infinity.

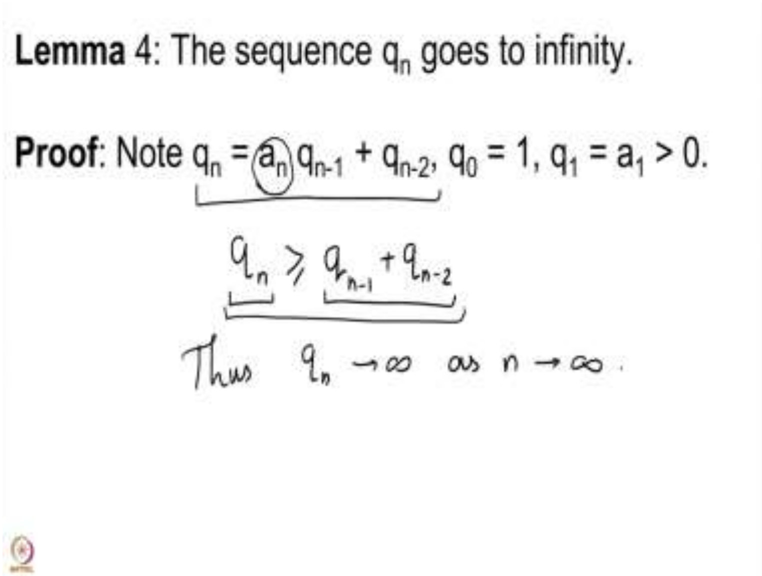
(Refer Slide Time: 15:30)



Note that p naught by q naught this is just a naught, so q naught is 1 our first denominator is 1 because a naught is an integer and you want to write a naught at p naught by q naught where the p naught and q naught are relatively prime and the q naught has to be a natural number then the only way you can do it with q naught equal to 1.

Further p1 by q1 is a0 plus 1 upon a1, we are going to cut the continued fraction expansion at n equal to 1, so we get this and let me simplify this a0 a1 plus 1 upon a1. So, a1 is a1. So, my q naught is 1, q1 is a1 and then we have the recursion formula.

(Refer Slide Time: 17:02)

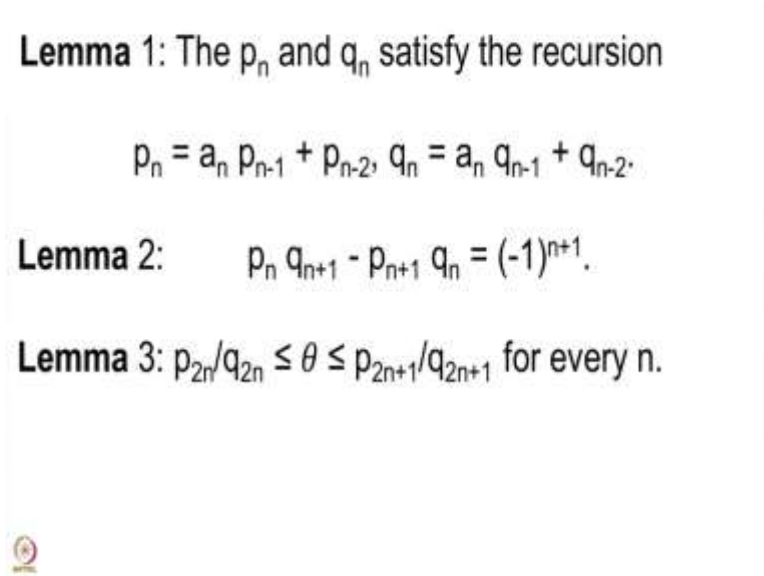


So, this is the recursion formula, qn is an qn minus 1 plus qn minus 2, since an are bigger than or equal to 1 we have that qn is going to be bigger than or equal to qn minus 1 plus qn minus 2 and therefore thus all these are positive quantities and then nht one is at least the some of the previous ones, thus qn goes to infinity as n goes to infinity.

In practice these an's are going to be large integers. So, although we have this innocent-looking inequality, qn is bigger than or equal to qn minus 1 plus qn minus 2, actually we have a much stronger inequality. If you remember the continued fraction expansion of Pi that I had given in the very beginning of this theme, then you may remember that 292 appeared there as one particular an.

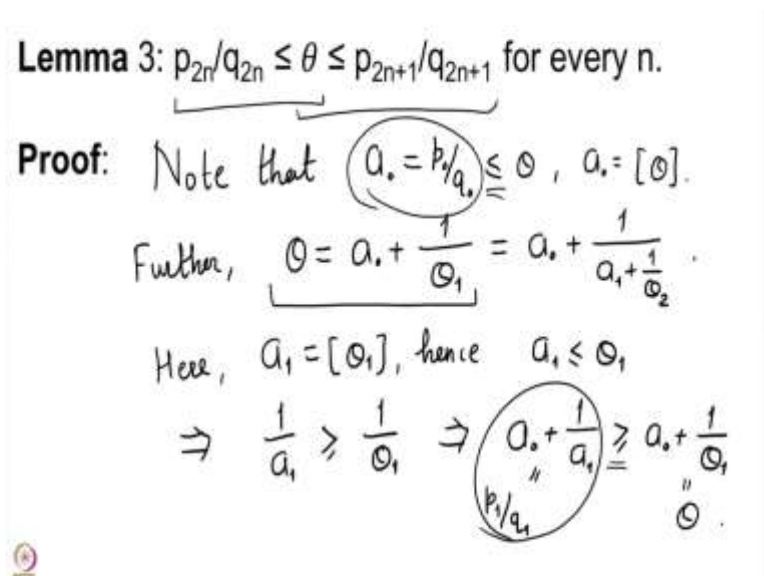
And therefore the qn is going to go strongly towards infinity, qn is going to go quickly towards infinity. But in any case as qn are all positive numbers and the nth one is at least the sum of the previous two we see that qn go to infinity as n goes to infinity. So, thus we have proved our fourth lemma.

(Refer Slide Time: 18:52)



Now, there are these three lemmas that are remaining to be proved. So pn and qn should satisfy the recursion the difference of the two successive convergence and that theta is sandwiched between the even and the odd convergents. So, now I am going to prove the third lemma that p2n upon q2n is less than or equal to theta is less than or equal to p2n plus 1 upon q2n plus 1.

(Refer Slide Time: 19:24)

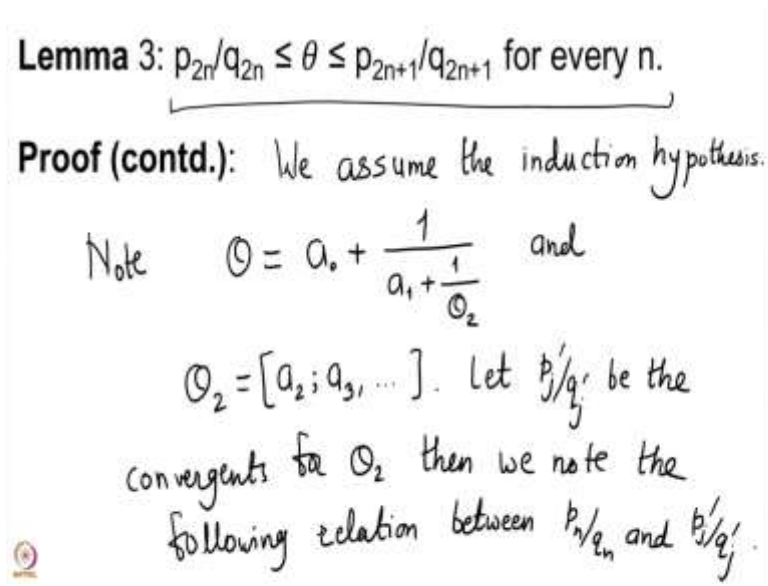


Naturally this proof will follow by the method of induction. So, note that a naught which is our p naught by q naught is less than or equal to theta because a naught is nothing but the integral part of theta. So, when n is equal to 0 this side of the inequality is proved. Further theta is written as a0 plus 1 upon theta 1 where we noted that theta 1 was bigger than 1 and we wrote this as a1 plus 1 upon theta 2.

Now, this a1 is the integral part of theta 1, hence a1 is less than or equal to theta 1, this would give us that if I take the reciprocal I will get 1 upon a1 to be bigger than or equal to 1 upon theta 1 and by adding a naught to both the sides we get a naught plus 1 upon a1 is bigger than or equal to a naught plus 1 upon theta 1.

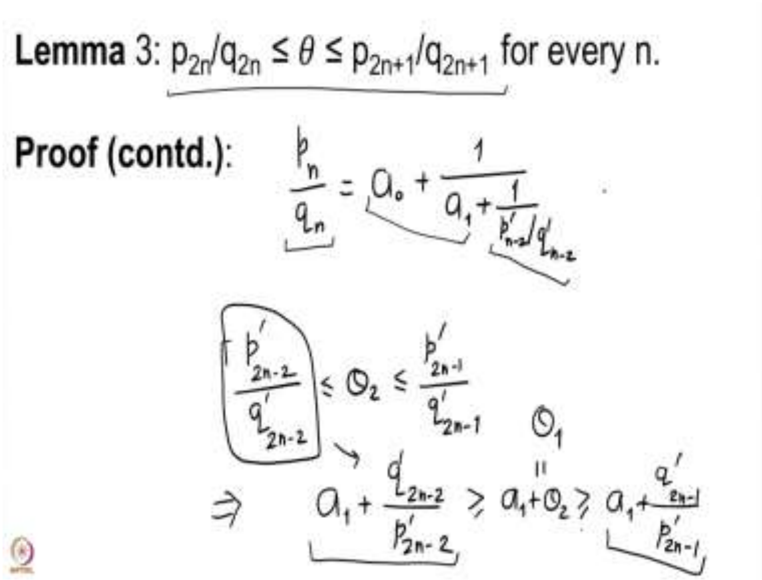
But this side we have p1 upon q1 and this side is nothing but theta. This is theta and this is pn 1 by q and this is our first convergent, this is our 0th convergent a naught which is p naught by q naught that is the 0th convergent, that is less than or equal to theta the first convergent is now bigger than or equal to theta, so we have proved both the sides of this inequality for n equal to 0. And now we will assume that this holds for everything up to 2n and then we will prove it holds for 2n as well as for 2n plus 1.

(Refer Slide Time: 21:58)



So, we now assume the induction hypothesis and prove the general result we are shown the induction hypothesis. Now, note that this theta which we are writing as a0 plus 1 upon a1 plus 1 upon here we will have theta 2 and theta 2 has the continued fraction expansion which starts from a2. So, for theta 2 we have that this result holds whenever we cut at theta 2 by at any stage less than 2n, so let pj prime upon qj prime be the convergents for theta 2, then we note the following relation between pn by qn and pj prime upon qj prime.

(Refer Slide Time: 23:53)



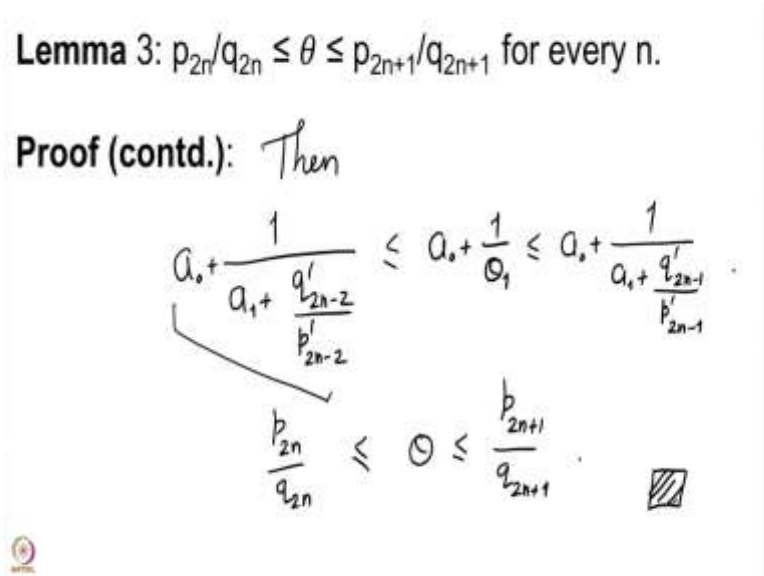
So, there is a natural relation which is that pn upon qn which is a0 plus 1 upon a1 plus 1 upon and then we start for writing the convergents for theta 2 which should have 2n minus 2 because we are taking 2n already here, so this is nothing but p prime 2n minus 2 upon q prime 2n minus 2, it will be p prime n minus 2, p prime n minus 2 upon q prime n minus 2.

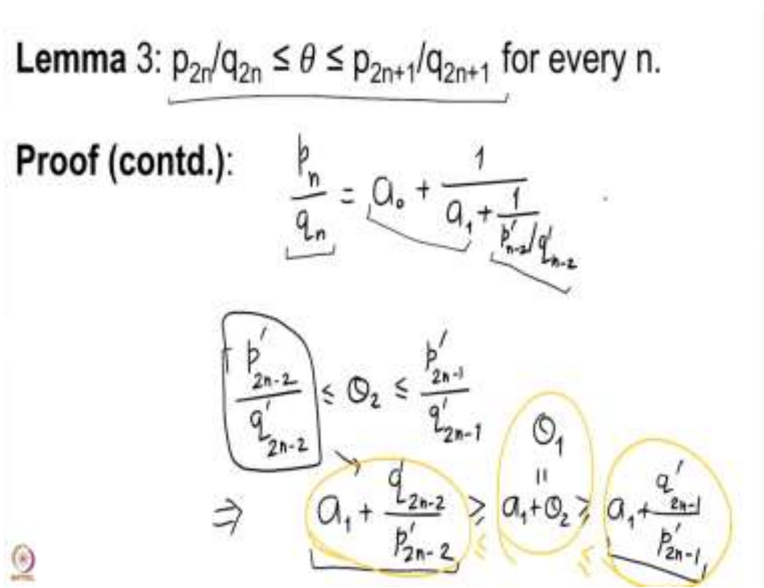
Whenever you have a convergent with the suffix with the subscript K, it should involve K plus 1 integers. So, here you will have n minus 2 plus 1 integers which are n plus 1 and then these two will give you n minus 1 plus 2 n plus 1 integers which is what we should get in the continued fraction expansion for pn by qn.

So, we observe this and now since the theta 2 satisfies this for every m up to n minus 1 we have that p prime 2n minus 2 upon q prime 2n minus 2 is less than or equal to theta 2 which is less than or equal to p prime 2n minus 1 upon q prime to n minus 1, so this inequality holds for theta 2, if we reverse this then we are going to get that p prime we are going to get a1 plus q prime 2n minus 2 upon p prime to n minus 2.

Note that this is nothing but the reciprocal of this one, this is the reciprocal, so we will have it to be bigger than or equal to a1 plus theta 2. And similarly we have the a1 plus p prime 2n minus q prime to n minus 1 upon p prime to n minus 1. And note that this is nothing but theta 1. So, theta 1 has the property that it is sandwiched between these two in this particular way we take the reciprocals of these three quantities involved in this inequality once again and add a naught to that.

(Refer Slide Time: 27:07)

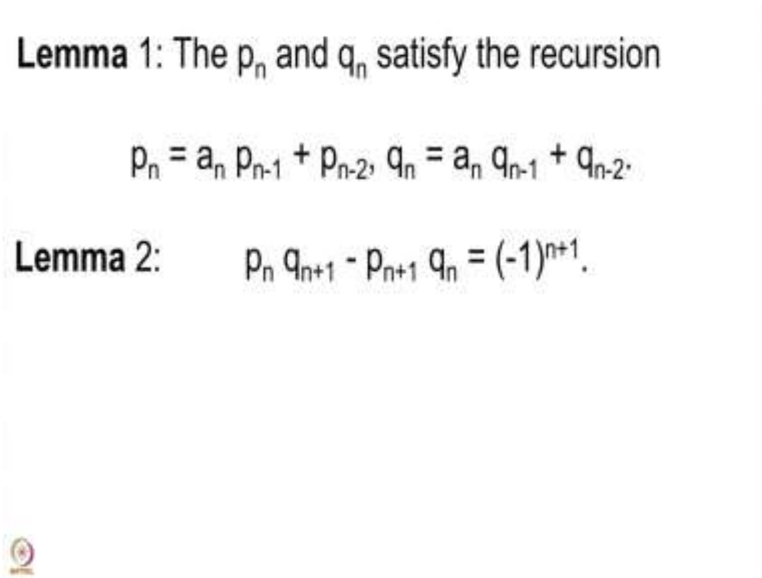




A naught plus 1 upon a1 plus q prime 2n minus 2 upon p prime 2n minus 2 which is what we had here is bigger than or equal to a naught plus 1 upon theta 1 this is less than or equal to because we have taken a reciprocal, so the sign has switched plus 1 upon a1 plus a prime 2n minus 1 upon p prime to n minus 1.

So, what we have done is that we took reciprocals of these three, therefore the sign would change in this way and then we have simply added a0 to each of those terms, adding an a0 will not change the sign the order is preserved, this is something that we have proved in the very first few lectures in our course. Now, this is nothing but p2n upon q2n this is nothing but theta and this is nothing but p2n plus 1 upon q2n plus 1 which proves our result. So, we have proved our lemma and we have also proved the result assuming our lemma.

(Refer Slide Time: 29:01)



Note ofcourse that we have still to prove these two lemmas, we will prove these two lemmas and then we will go on to see how the continued fraction expansion convergents gave a good approximation to our theta. Thank you very much.