A Basic Course In Number Theory

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Lecture - 57

Convergents Are The Best Approximations - 2

Welcome back. We are now going to prove that the convergents for any real number theta are the best rational numbers giving the best approximations. So this is what we want to prove.

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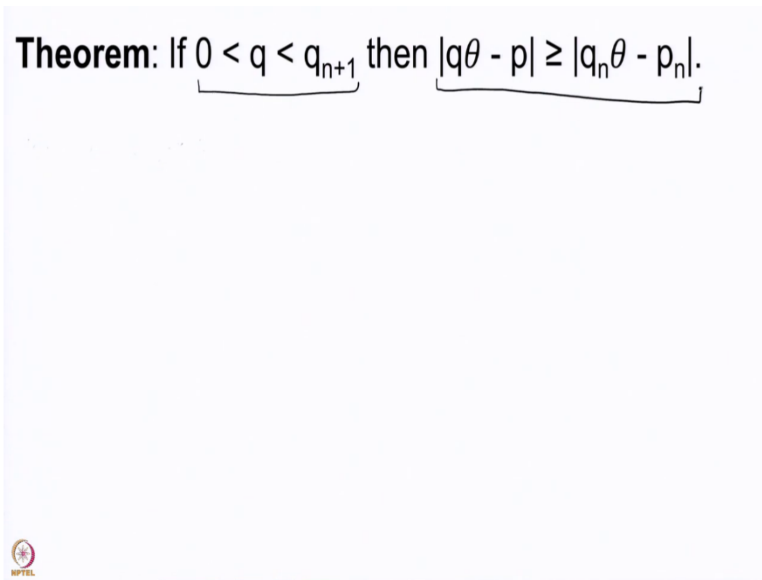

Recall the theorem that we have proved, the last theorem in our last lecture, which says that if you have q to be between 0 and qn plus 1, it is a positive quantity, which is between 0 and qn plus 1 then for any natural number p, any integer p, mod of q theta minus p is always going to be bigger than or equal to the qn theta minus pn. This is the result that we have.

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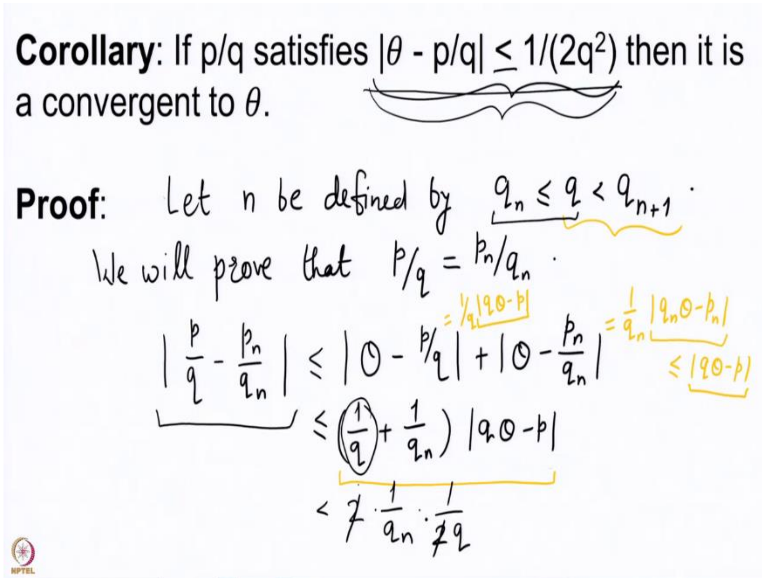

We are going to use this result to prove that if you have any rational number p by q satisfying this particular inequality that theta minus p by q is less than 1 upon 2 times q square then p by q has to be convergent to theta. So it will have to be equal to sum pn by qn.

This says that of course, we have noticed that convergents give you good approximations. We notice that certainly theta minus p by q is less than 1 upon q square and, in fact, one of the consecutive pairs of convergents will give you this inequality, which is referred to in the statement of the corollary and one of the triple will give you even better, and so on.

But if there is any rational number somewhere out there, which is trying to give you the approximation better than the convergents, then it has no alternative but to be itself a convergent. This is a very important theorem, and therefore, we can say that the convergents to a real number theta give the best approximations to the real number theta.

So, of course, we also know that qn go to infinity. So let n be defined by qn less than or equal to q less than qn plus 1. Since we know that the qns go to infinity, we should have some qn which is less than or equal to q and some, after all, q1 is 1, q0 is 1; remember, our p0 upon q0 is an integer a0, therefore, the q0 is in fact 1.

So we are starting the sequence qn with 1, therefore, there is always some q which is below our number q, there is some qn which is going to be below the denominator q of the rational p by q and ultimately, qns go to infinity. So there has to be an n such that qn is less than or equal to q, but the next one, qn plus 1 will be beyond q, will be after q, more than q, bigger than q. So we are choosing n with that property.

Remember, we are not putting any condition on p other than the simple condition that theta minus p by q is less than 1 upon q square and now we are going to prove that p has to be the pn, and q has to be the qn. Remember, we are given the real number theta, so we have the continuous-fraction expansion for theta. So, we have pn and qn and that is how we have taken this n.

We will prove that p upon q is pn upon qn. We are actually going to prove that p is pn and q is qn, but that is equivalent to showing that p by q is pn by qn. So how does one prove this? We will consider the difference between these two.

Now, this difference we can insert a plus or minus theta in between and then this becomes less than or equal to theta minus p by q plus theta minus pn by qn, and which is less than or equal to 1 upon q plus 1 upon qn into q theta minus p with a mod.

This is because this term here is 1 upon qn mod qn theta minus pn and this is 1 by q into mod q theta minus p and the inequality that we have here that q is less than qn plus 1 will say that this quantity is less than or equal to mod of q theta minus p. So you have a mod q theta minus p coming from the second term as well as the first term and you have just 1 by q plus 1 by qn mod q theta minus p.

Further, we observe that since qn is less than or equal to q, so 1 upon qn is going to be bigger than or equal to 1 upon q, therefore, we have that this is less than or equal to, therefore, this 1 upon q can be replaced by 1 upon qn. You have it coming from once and twice so, you have 2 times 1 upon qn and we have assumed that q theta minus p, because of this inequality that you have assumed is less than 1 upon 2q. So you have here 1 upon 2q. So these two get canceled. Further, here you have a strict inequality, which will mean that this inequalities also strict.

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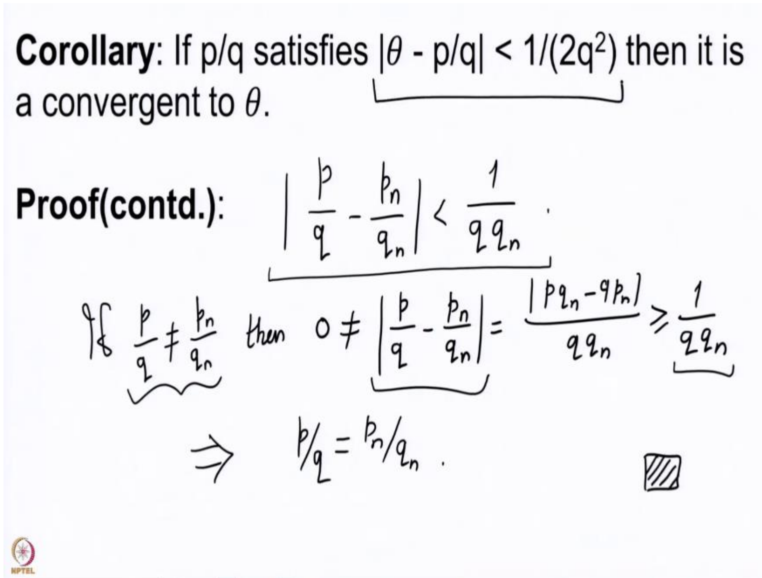

So we get, ultimately, that this p by q minus pn by qn is strictly less than 1 upon q qn. But this is a contradiction because if p by q is not equal to pn by qn then you have that this quantity is not 0. The difference between those 2 is not 0 but the numerator will be p qn minus q pn upon q qn and therefore, this is at least 1 upon q qn.

So if you have that these two are not same then the difference has to be bigger than or equal to 1 upon q qn. But what we prove here is that the difference is strictly less than 1 upon q qn. Hence, there is no other option but p by q has to be pn by qn.

So very remarkable result, which says that if you have a rational satisfying this slightly better inequality than the convergents, convergents will all satisfy where you, instead of 2 you have 1. theta minus p by q less than 1 upon q square, all convergents will satisfy that.

So if you have one rational trying to do something better than the convergents then it has no other option but itself to be a convergents. So, in this way, we can say that the convergents to a real number theta give the best approximations, there is nothing else that give a better approximation to our real number theta.

Now, we are going to do some computations. We have not really, except for computing the continued-fraction expansion for the golden ratio, we have not computed the continued-fraction expansions for any other number. So let us start with some of the simplest numbers.

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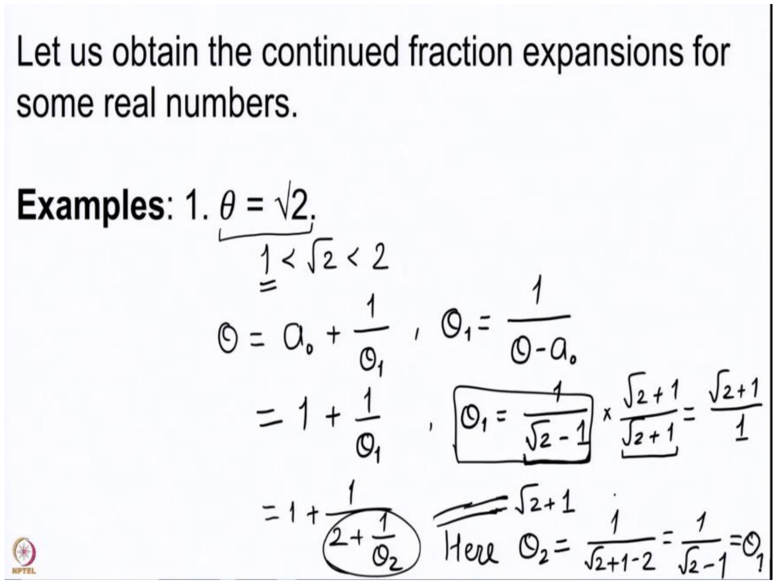

We start with theta equal to root 2 and let us compute its continued fraction expansion. So first of all, we note that 1 is less than root 2 is less than 2. Because 1 is less than 2 which is less than four, so square root of 1 which 1 is less than square root of 2, which is less than square root of four, which is 2.

Therefore, we will write theta which is root 2 as a0 plus 1 upon theta 1. Here theta 1 is 1 upon theta minus a0. So what is a0 for us, a0 is the integral part of theta, which is 1. 1 upon theta, remember, is root 2 and we have the a0 to be 1. So we now want to solve this, we want to write this expression 1 upon root 2 minus 1 in a simpler form. And the standard trick to do that is to multiply the numerator and denominator by the conjugate of this number.

So, we therefore get, the root 2 plus 1 comes in the numerator, here we have something like a minus b into a plus b, which is a square minus b square. So we get root 2 square which is 2 minus 1 square which is 1. So it, this is just 1. We get it to be root 2 plus 1.

So our theta 1 is now root 2 plus 1. The real, integral part of theta which is root 2 was 1, therefore, the integral part of theta 1 which is root 2 plus 1 is going to be 2. You have it to be 1 plus 1 upon 2 plus 1 upon theta 2.

Here theta 2 is we just consider this part. So this is equal to theta 1 which is our number, root 2 plus 1. So this is 1 upon root 2 plus 1 minus the integral part which is minus 2 and therefore, this is root 2 minus 1 but that is what we had here for theta 1 as well. So, theta 2 is nothing but theta 1.

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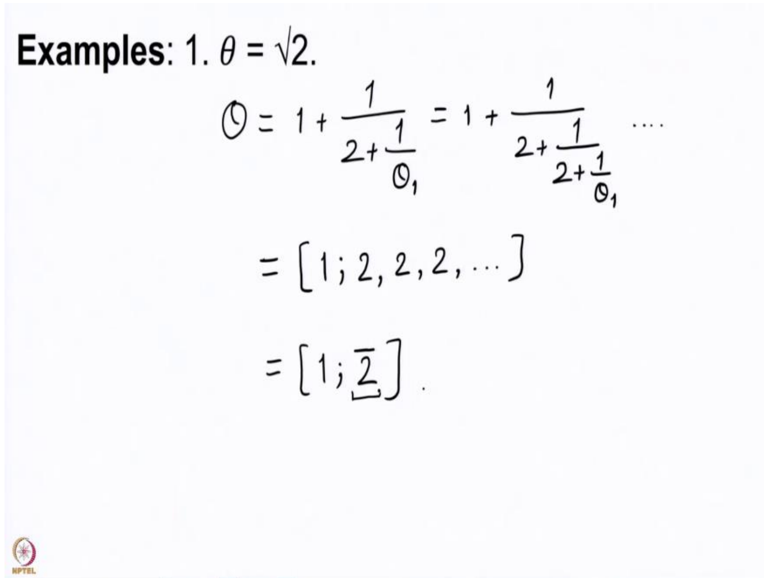

We are in a better situation than most of the other situations because what we obtained is that this is 1 plus 2 plus 1 upon theta 2 has now become theta 1. So we get this to be 2 plus 1 upon 2 plus, next 1 after theta 1 would be theta 2 but that is same as theta 1. And so, we are going to continue in this way.

So the continued fraction expansion for root 2 is where you have 1 in the first place, a0 is 1 but a 1 onwards you just have them to be 2. Since these are repeating, we write it in the way as given here. We put a bar on the head of 2 to denote that this single partial quotient gets repeated until infinity. So this is the expression that we have for root 2. If we have this expression for root 2, solving the continued fraction for 1 plus root 2 is not going to be difficult at all.

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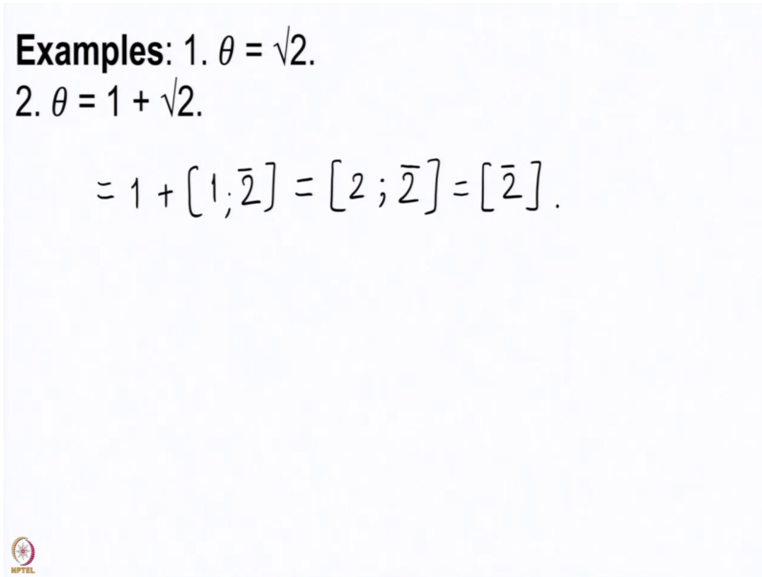

The only change here would be in the integral part. And so, we are going to get 2 semicolon 2 continued or the periodic it, periodic expression for 2 and therefore, this can simply be written as 2 bar, the 2s get repeated periodically. This is the expression for theta equal to 1 plus root 2.

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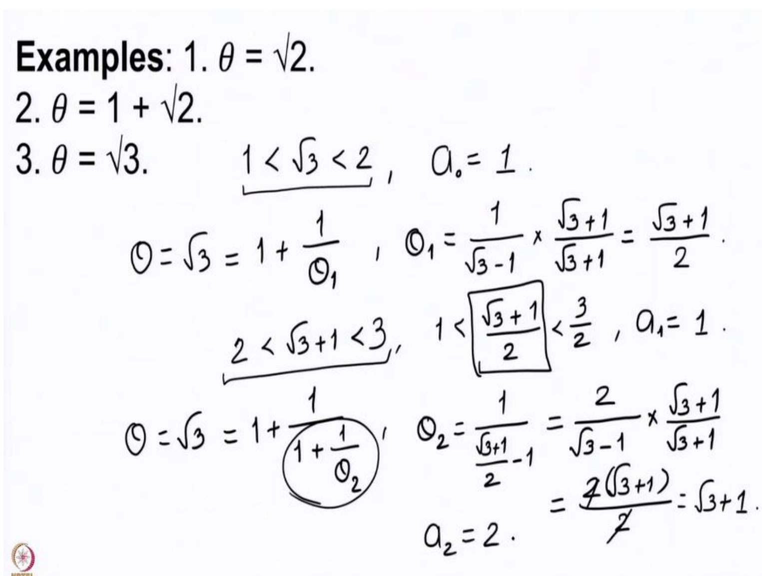

Let us see what happens when we consider root 3. So once again, 1 is less than root 3 is less than 2. Therefore, a naught is going to be 1, so we have theta which is root 3. This is 1 plus 1 upon theta 1, and theta 1 as previous calculation is 1 upon root 3 minus 1, we compute the simplified form of that. By this expression, we have root 3 plus 1 in the numerator but denominator will give you root 3 square minus 1 square 3 minus 1 that is 2.

So this has become somewhat complicated than the computation for root 2 but let us see what we get. So initially, we have 1 plus root 3, 1 is less than root 3 less than 2. Therefore, 1 plus 1 which is 2 is going to be less than root 3 plus 1 is going to be less than 4. You are, less than 3, we are just adding 1 to this pair of inequalities. And once you divide by 2, because 2 is a positive number, the inequalities are respected and therefore, the integral part of this number is 1.

So we get that a1 has some to be 1. So theta which is root 3, now becomes 1 plus 1 upon 1 plus 1 upon theta 2. And here, theta 2 is 1 upon this quantity which we know is root 3 plus 1 by 2, so root 3 plus 1 by 2 minus 1, which is root 3 plus 1 minus 2 upon 2. So that 2 can be put in the numerator and we have root 3 minus 1 in the denominator.

And once again, we solve this, simplify this by multiplying by root 3 plus 1. And we obtain that this is 2 times root 3 plus 1 upon 3 square minus 1 square now, which is simply 2. So we get this to be root 3 plus 1. And root 3 plus 1, because of these inequalities, will tell you that it is integral part has to be 2. So our a2 is 2.

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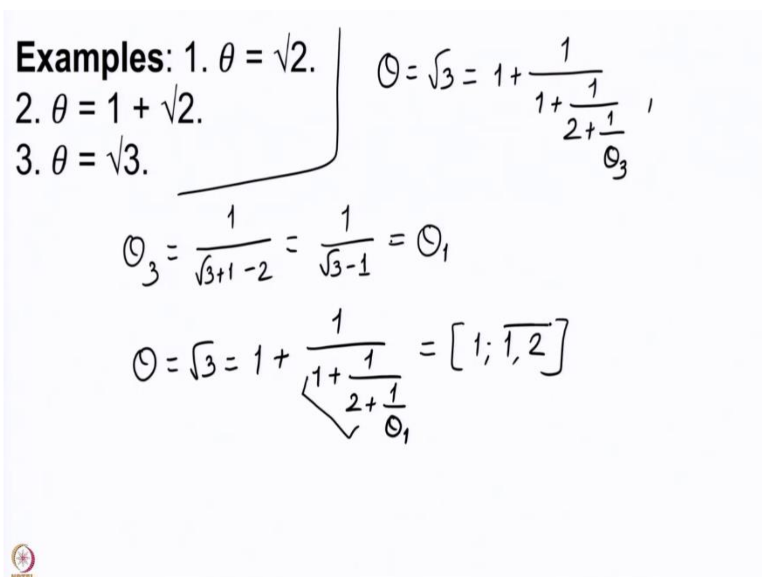

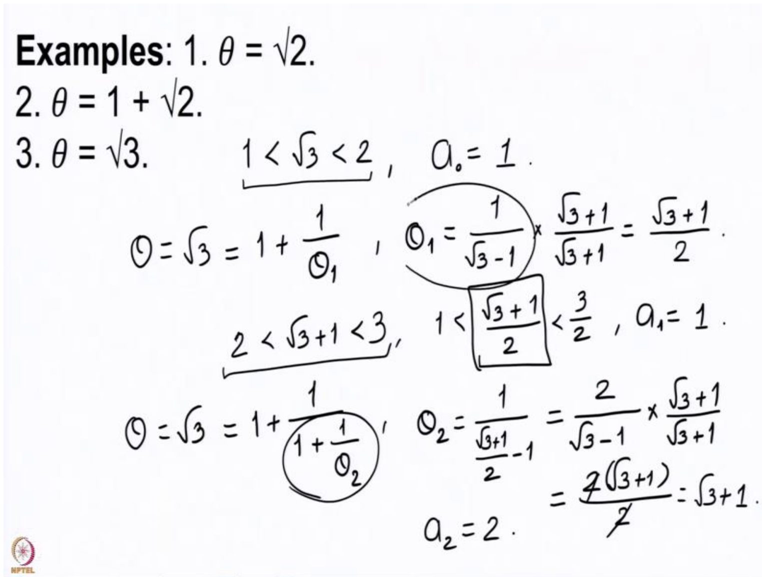

Theta 1, theta 2, and now we have theta 3, where theta 3 satisfies 1 upon root 3 plus 1 minus 2, so this is 1 upon root 3 minus 1 but 1 upon root 3 minus 1 is something that we had already obtained here at theta 1. So this happens to be equal to theta 1.

So theta 3 is theta 1. Therefore, we will have that theta, which is root 3, we will have the expression 1 plus 1 upon 1 plus 1 upon 2 plus 1 upon once again we get to theta 1, and these integers 1 2 1 2 will continue to be repeated. And therefore, we get that the continued fraction expansion for the root 3 has this particular form, it will be 1 semicolon 1 comma 2 comma 1 comma 2 comma 1 comma 2, and so on. So, that is how this is given.

So we have noticed, we have actually computed in a way, we have computed 3 expressions for the 3 continued-fraction expansions for the real numbers. The earlier one was for the golden ratio, 1 plus, root five by 2. Then we have computed for root 2 and for root 3. The number 1 plus root 2 that was just simply obtained from root 2.

So we have computed these three square roots of natural number and we see that the continued-fraction expansions are repeating after some stage. And this is actually one very nice result that these continued-fraction expansions always are going to represent the real numbers, which are quadratic in some sense, and if you are taking these repeating ones that means the continued-fraction expansions are never-ending. Therefore, what you get are not rational numbers.

Because if you have a terminating continued-fraction expansion, if your expansion terminates, you have a finite expansion, that means it is a continued fraction, which will be a rational number. But if you are putting a bar on that, which means that it is simply continues, then you are going to get what is called an irrational number. The numbers are real numbers but they are not rational numbers.

So, they are the irrational numbers, which are quadratic in some sense. So let us make this notion very precise and then we will prove the result about these repeating continued fractions.

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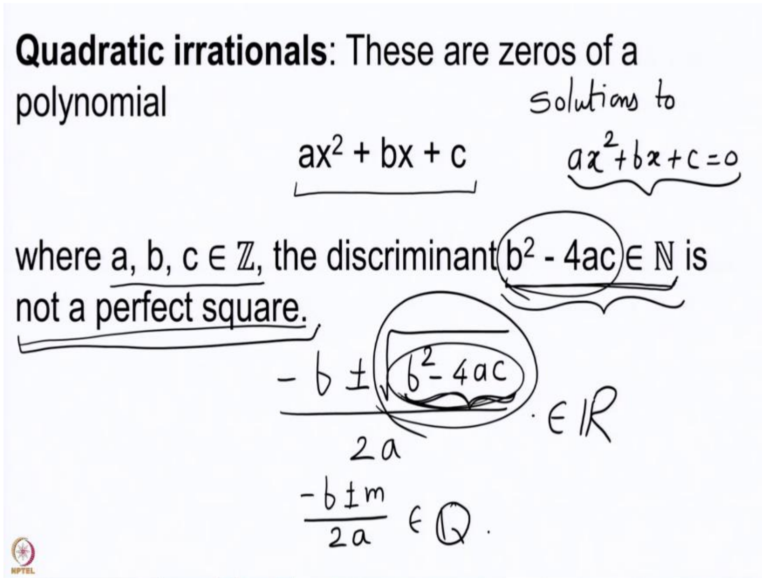

So let us see the definition of what are known as quadratic irrational. So these are the zeros of a polynomial like this. So, or you may say that these are solutions to the equation ax square plus bx plus c equal to 0. These are solutions to this or these are also known as zeros of this particular polynomial that means, when you put the value of this quadratic irrational into this polynomial, you are going to get 0. They are the solutions to these particular equations.

But we have some conditions that the, a, b, c should be integers. They are not allowed to be any random numbers, these are integers. Furthermore, the discriminant, b square minus 4 a c should be positive and should be square free. It should not, should be a non-square; it should not be a square. These are the conditions which are going to give us the quadratic irrationals.

Now, from school level onwards, we know how to find the solutions to this particular equation. They are given by minus b plus or minus under root b square minus 4 a c upon to 2 a. These are the forms of the solutions to this quadratic equation and so, if this guy is 0, then we have that, we get the same root; the 0 minus 2 upon 2 a comes with multiplicity 2. Then, in fact, this becomes a square.

So whenever this quantity is 0, you get the same number. That is why we call that number. Then you get the same root. So whether this number is 0 or not, we will discriminate the number of roots being 1 or 2. So this is called discriminant.

We are shown that it is a natural number because, after all, we know that this is going to be an integer; a, b, c are integers, so b square minus 4 a c is also an integer, but if this is a negative integer, then the square root will give you an imaginary number. And we are looking at our continued fractions which concern only real numbers. There may be study for complex numbers using some generalized forms of continued fractions, but we are not considering that yet. We want our numbers to be real numbers.

We want these to be sitting inside R. So these numbers, b square minus 4 a c will have to be positive. This is why we are taking them to be natural numbers. And further, we do not want it to be a perfect square because if this is a perfect square then it says square of m, then you will have that your solution becomes minus b plus or minus m upon 2 a, which turns out to be a rational number in the end. We do not want that to happen. So we want that this b square minus 4 a c should not be a perfect square. So this is the condition that we have on the quadratic irrationals.

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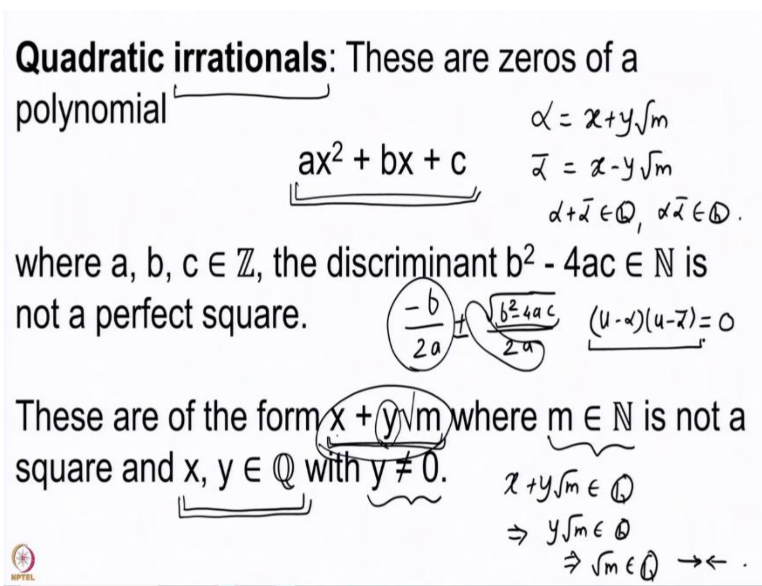

Furthermore, these are always of the form x plus y root m, where x and y are rational numbers, y is non 0, and m is a natural number which is not a square. So these quadratic irrationals, the solutions to these numbers as we have just found are in fact of the form x plus y root m.

So we had minus b upon 2 a plus or minus under root b square minus 4 a c upon 2 a. So we have this minus b upon 2 a, and we have this particular thing. This is not a square. So we have obtained that any solution to this, which is a quadratic irrational has to be of this particular form with x plus y root m, y not being 0 and x and y being rational numbers, and m being a natural number which is not a square.

In other words, every such number, x plus y root m of these particular conditions, satisfying these particular conditions is a quadratic irrational. Because when you have y to be non-zero, then you already get a number which is not a rational number.

Remember, x plus y root m, if this was rational, it would imply that y root m has to be rational because x is after all a rational number. If you have that x plus y root m is a rational then y root m is that rational minus x. So you have y root m to be rational and if you have y root m to be rational, y is non-zero, you can multiply a rational by another rational and still remain in rationales that would tell you that root m is a rational which is a contradiction.

There is a very simple proof using the unit factorization of integers, the fundamental theorem of arithmetic, which will tell you that if your m is not a square, then square root of m cannot be a rational number.

So all these numbers, x plus y root m are irrational and they will clearly satisfy some quadratic equation because you have to simply write, alpha equal to x plus y root m and alpha bar is x minus y root m. Then you notice that alpha plus alpha bar is rational, alpha alpha bar is also rational, and using this you easily get a quadratic equation satisfied by alpha as well as alpha bar.

In fact, u minus alpha into u minus alpha bar will give you equal to 0, will give you a quadratic equation whose coefficients are rational numbers, and then, you clear out the denominators to get the coefficients to be integers.

So these are the quadratic irrationals. In the next lecture, we are going to define what are called ultimately periodic continued fractions and then we will prove that the quadratic irrationals are nothing but the ultimately periodic continued fraction expansions. So see you in the next lecture. Thank you very much.