

Alternative Renewal Process

- ▶ Let X_1, X_2, \dots be i.i.d. random variables which constitutes *on times*.
- ▶ Let Y_1, Y_2, \dots be i.i.d. random variables which constitutes *off times*.
- ▶ Assume that $E(X + Y) < \infty$ and $X + Y$ has distribution F .
- ▶ Suppose that, renewal occurs at the end of every X_i where as no renewals at the end of every Y_i .
- ▶ Assume that, X_i and Y_i are independent random variables.
- ▶ Then $\{X_i + Y_i, i = 1, 2, \dots\}$ is called an alternative renewal process.



Now we are moving into the one type of renewal process that is called alternating renewal process. Let X_i 's are iid random variable which constitute on times and Y_i 's are iid random variable which constitutes off times. Assume that mean is the existent it is finite and the X plus Y has the distribution F . Suppose that the renewal occurs at the end of every X_i 's whereas no arrivals at the end of every Y_i 's. Assume that X_i 's and Y_i 's are independent random variables also. Then the X_i plus Y_i are called the alternative renewal process.

Example

- ▶ For example, consider the following situation.
- ▶ A machine works for time X_1 , then breaks down and has to be repaired (which takes time Y_1), then works for a time X_2 , then is down for a time Y_2 , and so on.
- ▶ If we suppose that the machine is as good as new after each repair, then this constitutes an alternative renewal process.



See the example for this situation. A machine works for the time X_1 and then breaks down and has to be repaired which takes a time Y_1 . Then works for the time X_2 then it is down for a time Y_2 . That's a second repair time and so on. So X_i 's are nothing but the machine works and the Y_i 's are nothing but the repair time. If you suppose that the machine is good as new after each repair then this constitute alternative renewal process. So this is the example of alternative renewal process with the proper assumptions.

Delayed Renewal process

- ▶ It is not always reasonable to insist that the first renewal occurs at time $S_0 = 0$.
- ▶ For instance, in applications where the occurrences are at times when a component of a system must be replaced, one might well be interested in situations where there is already a working component in place at time 0.
- ▶ For this reason, we define a delayed renewal process to be a sequence S_0, S_1, S_2, \dots where

$$S_n = S_0 + \sum_{i=1}^n X_i \text{ and}$$

- ▶ (i) the inter-arrival times X_1, X_2, \dots are positive and i.i.d., as in an ordinary renewal process, and
- ▶ (ii) the initial delay $S_0 \geq 0$ is independent of the inter-arrival times X_i .

Now we are moving into the next renewal processes that is delayed renewal process. It is not always reasonable to insist that the first renewal occurs at time S_0 that is equal to 0 in the origin itself, at the time 0 itself. For instance in applications where the occurrence are not or at times when a component of the system must be replaced. One might well be interested in situations where there is already a working component in place at time 0. For this reason we defined a delayed renewal process to be a sequence S_0, S_1, S_2 and so on where S_n is nothing but S_0 plus the summation of first n X_i 's and the inter arrival times X_i 's are positive and iid random variables as in the ordinary renewal process. And the initial delay S_0 which is great or equal to 0 that is independent of inter arrival times X_i 's.

Delayed Renewal process . . .

- ▶ Notice that the distribution of the initial delay random variable S_0 is not required to be the same as that of the inter-arrival time random variables X_j .
- ▶ Hence, a delayed renewal process is a renewal process in which the first arrival time, $X_1 = t_1$, independently, is allowed to have a different distribution $P(X_1 \leq x) = F_1(x); x \geq 0$, than F , the distribution of all the remaining i.i.d. inter-arrival times $\{X_n, n \geq 2\}$.
- ▶ t_1 is then called the delay.
- ▶ When there is no such delay, that is, when $X_1 \sim F$ as usual, the renewal process is said to be a non-delayed version.



Notice that the distribution of initial delay random variable X_1 is not required to be the same as that of the inter arrival time random variables X_i 's. Hence a delayed renewal process is a renewal process in which the first arrival time X_1 is independently and is allowed to have a different distribution that is F_1 . The distribution of all remaining iid random variables that distribution is F . So the F_1 is different from F . The X_1 is equal to t_1 that is nothing but that delay and there is no such delay then X_1 is also distributed in the same way so the distribution of X_1 is F as usual. Then the renewal process is said to be a non delayed version.

Central Limit Theorem

- ▶ As $n \rightarrow \infty$,

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow Z \sim \mathcal{N}(0, 1)$$

- ▶ As $t \rightarrow \infty$, $N(t)$ becomes normal distributed and both $E(N(t)) \sim \frac{t}{\mu}$ and $\text{Var}(N(t)) \sim \frac{\sigma^2 t}{\mu^3}$.
- ▶ As $t \rightarrow \infty$,

$$Z(t) = \frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} \Rightarrow Z \sim \mathcal{N}(0, 1)$$



Now we are discussing the central limit theorem on the renewal process. As n tends to infinity the random variable S_n that is nothing but the n th time renewal minus n times μ divided by σ times square root of n will be normal distribution with the mean 0 and variance 1. So this convergence takes place in a distribution. So this is a weak distribution, weak convergence.

So as n tends to infinity the random variable S_n and the mean of S_n is n times μ and the variance of S_n is $\sigma^2 n$ and the random variable minus the mean divided by the standard deviation is normal distributed with the mean 0 and variance 1 as n tends to infinity as n tends to infinity the n of t the counting process it's a renewal process becomes a normal distribution in the mean t divided by μ and the variance is $\sigma^2 t$ divided by μ^3 where μ is the mean of inter arrival time as t tends to infinity the random variable n of t minus t by μ divided by σ times square root of t by μ^3 will be normal distribution with the mean 0 and the variance 1. So this is also a convergence in distribution.

Long-run Properties

- ▶ One can study the long-run properties of the renewal process.
- ▶ There are two types of the long-run properties.
- ▶ One is to obtain the long-run (time) average of the quantity of interest and the other is to obtain the pointwise limit.
- ▶ For example, the long-run average of age is given by

$$\lim_{t \rightarrow \infty} \frac{\int_0^t A(s) ds}{t}$$

while the pointwise limit of the expected age is given by

$$\lim_{t \rightarrow \infty} E[A(t)]$$



Now we are going to discuss the long-run properties of renewal process. There are two types of long run properties. One is to obtain the long-run average of the quantity of interest. The other one is to obtain the pointwise limit. For example the long-run average of age that is limit t tends to infinity the integration 0 to t A of s , A of s is the age divided by t . while the pointwise limit of the expected age that is the limit t tends to infinity expectation of A of t .

Long-run Renewal Rate

- ▶ One can study the average number of renewals (per unit time) in the long-run.
- ▶ It is called a long-run renewal rate.
- ▶ For a renewal process $\{N(t), t \geq 0\}$ having distribution function F for inter-arrival times, the long-run renewal rate is given by

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mu} \text{ with probability } 1$$

where

$$\mu = \int_0^{\infty} x dF(x)$$

- ▶ Since $S_{N(t)}$ is the last renewal time prior to t and $S_{N(t)+1}$ is the first renewal time after t .



One can study the average number of renewals per unit time in the long run. It is called the long run renewal rate. For a renewal process having a distribution F for the inter arrival times the long run renewal rate is nothing but limit t tends to infinity n of t divided by t . That will be 1 divided by μ with the probability 1 where μ is nothing but the mean of inter arrival time.

Long-run Renewal Rate ...

► We know that

$$S_{N(t)} \leq t \leq S_{N(t)+1} \quad \text{or} \quad \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)}$$


► But,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{S_{N(t)}}{N(t)} &= \lim_{t \rightarrow \infty} \frac{X_1 + X_2 + \dots + X_{N(t)}}{N(t)} \\ &= E[X] = \mu \quad \text{with probability } 1 \end{aligned}$$

and

$$\frac{S_{N(t)+1}}{N(t)} = \lim_{t \rightarrow \infty} \frac{S_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)} = \mu \quad \text{with probability } 1$$

► Hence,

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{with probability } 1$$


Since S suffix n of t plus last renewal time prior to t and S suffix n of t plus 1 is the first renewal time of t we know the relation of S of n of t with the S of n of t plus 1 and that lies, the t lies between those two renewal times.

You can divide by t you can divide by n of t . Therefore S of n of t divided by n of t less than or equal to t divided by n of t that is less than or equal to S of n plus t plus 1 divided by n of t . Now we can evaluate the first one the S of n of t divided by n of t limit t tends to infinity. That is nothing but the numerator is nothing but the summation of X_i 's t n of t denominator is n of t and that is nothing but expectation of X in a long run – in the long run the summation of X_1, X_2 till X of n t divided by n of T that is nothing but the expectation of X that is same as the μ with the probability 1 . And similarly one can evaluate the last value that is S of n of t plus 1 divided by n of t that is also will be μ with the probability 1 . Since t lies between S of n of t and S of n of t plus 1 and the throughout divided by n of t in all three therefore as limit t tends to infinity n of t divided by t will be 1 divided by μ with the probability 1 .