Video Course on
Stochastic Processes

Module # 8
Renewal Processes

Markov Renewal and Markov Regenerative Processes (contd.)

by

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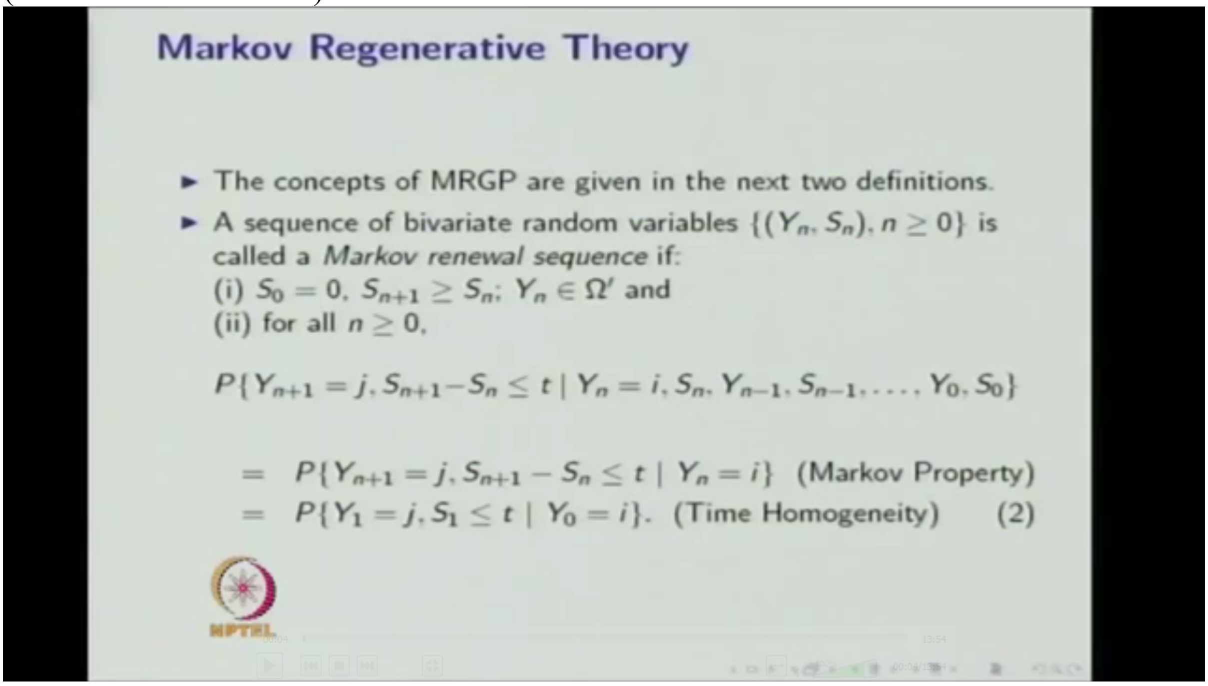

The concepts of MRGP are given in next two definitions.

The first definition, a sequence of bivariate random variables (Y_n, S_n) is called a Markov renewal sequence or Markov renewal process. $S_0 = 0$. In this example also we made it $S_0 = 0$. S_{n+1} is greater than or equal to S_n and Y_n is belonging to Ω' where Ω is the state space, Ω' is the subset of Ω. For all n greater than or equal to 0, the Y_n, the conditional distribution of Y_n has to satisfy this property.

The first line, the probability of Y_{n+1} is equal to j with the difference of time instants is less than or equal to t given that the system was in the state, some state at Y_0 at the time instant S_0 till the system was in the state i at the time instant S_n. This conditional distribution is same as the conditional distribution with the only the latest information the probability of Y_{n+1} is equal to j, the difference of regeneration time points is less than or equal to t given only Y_n is equal to i. That means the conditional distribution depends only the current information or latest information, not the complete history. So that is nothing but the Markov property.

Next, that is same as the conditional distribution of instead of Y_n to Y_{n+1}, you can find out the distribution of Y_0 to Y_1 because of it is a time invariant, because of it is a time homogeneity, this conditional distribution is same as probability of Y_n is equal to j, the first time, the first regeneration time point is less than or equal to t given that Y_0 is equal to i.

So that means the conditional distribution depends the current state, not the past history including the time homogeneous property. Then the -- that is a way we define the bivariate random variables that is (Y_n, S_n) satisfies this property.

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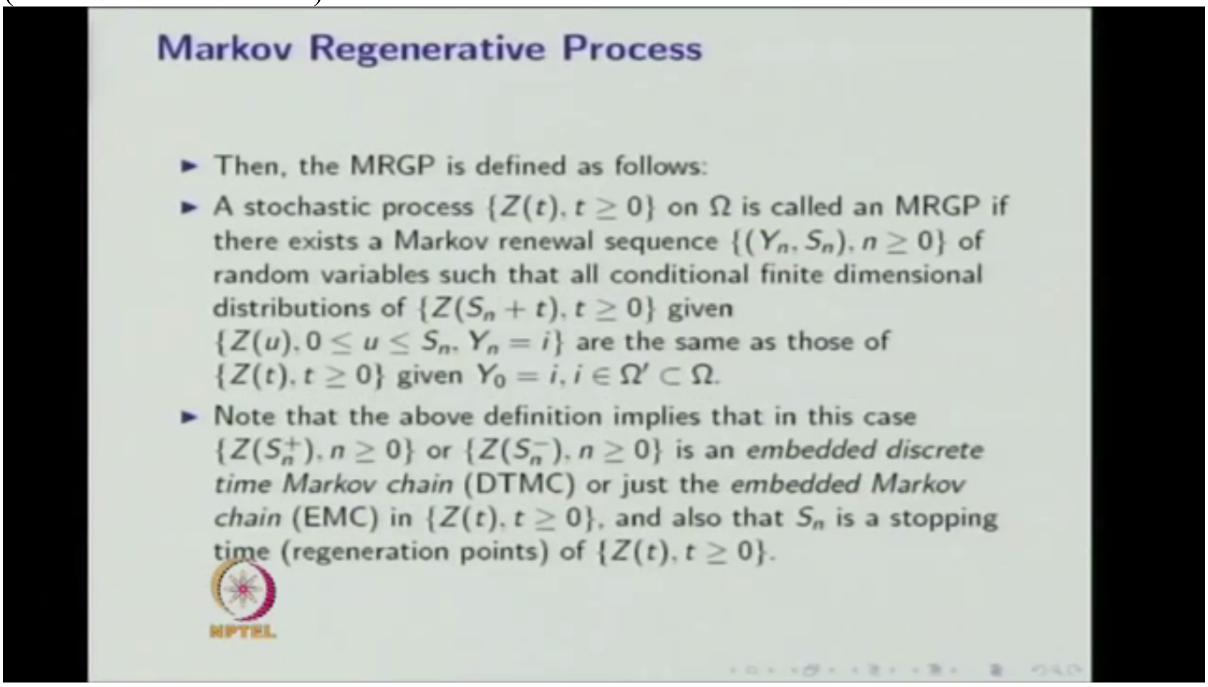

Then the MRGP is defined as follows:

A stochastic process Z(t) with the state space Ω is called a Markov regenerative process if there exists a Markov renewal sequence (Y_n, S_n) such that all conditional finite dimensional distribution of $Z(S_n + t)$ given Z(u) where u is lies between 0 to S_n, Y_n is equal to i are the same as those of Z(t) given Y_0 is equal to i. So this is the probabilistic replica.

The stochastic process Z(t) is said to be a Markov regenerative process if all conditional finite dimensional distribution of $Z(S_n + t)$ given all the past history till S_n including Y_n is equal to i, that is same as the distribution of Z(t) given Y_0 is equal to i. That means it includes a time homogeneity as well as the Markov property.

Note that the abode definition implies that $Z(S_n^+)$ or $Z(S_n^-)$ is an embedded discrete time Markov chain or just embedded Markov chain in Z(t). Also S_n is the stopping time or regeneration points. Stopping time is nothing but the Markov property is satisfied at those time points for the given stochastic process.

So, in this example, before the arrival occurs, $Z(S_n^-)$ is an embedded discrete time Markov chain just before the arrival occurs -- will be an embedded Markov chain in this example.

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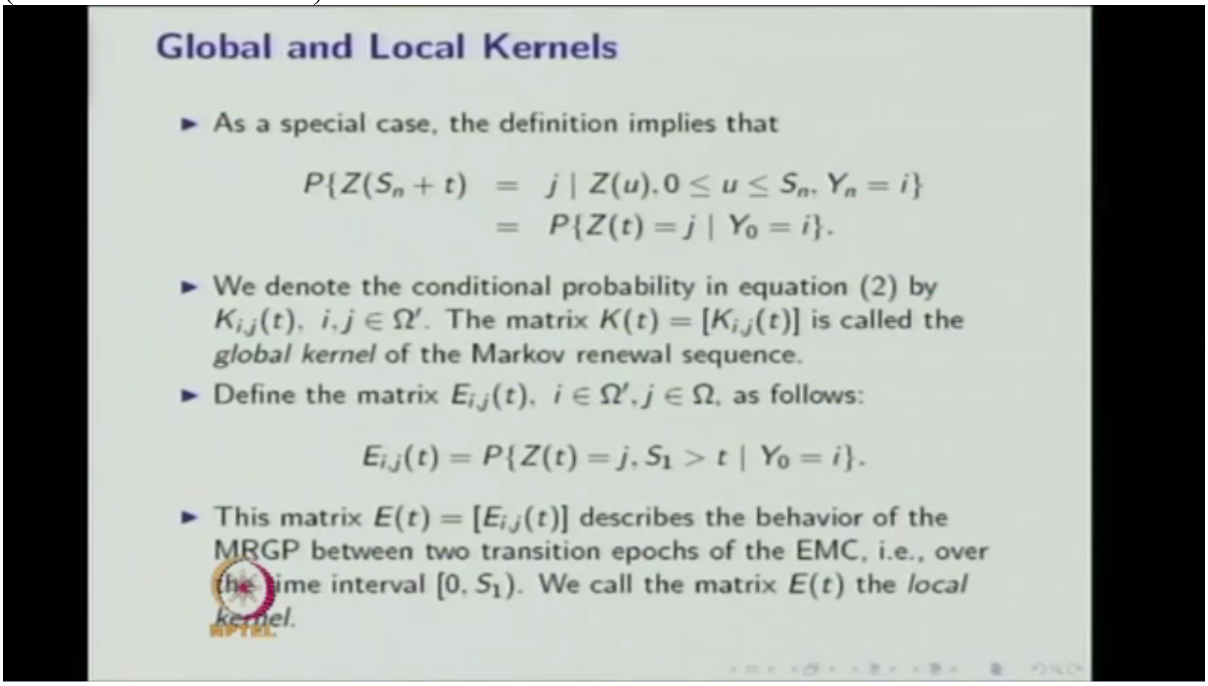

The way we discussed the semi-Markov process with the transition probability matrix and the sojourn time distribution, here we have to explain the global kernel and the local kernel. So that we are going to discuss now.

We denote the conditional probability in the equation number (2) by $K_{i,j}(t)$.

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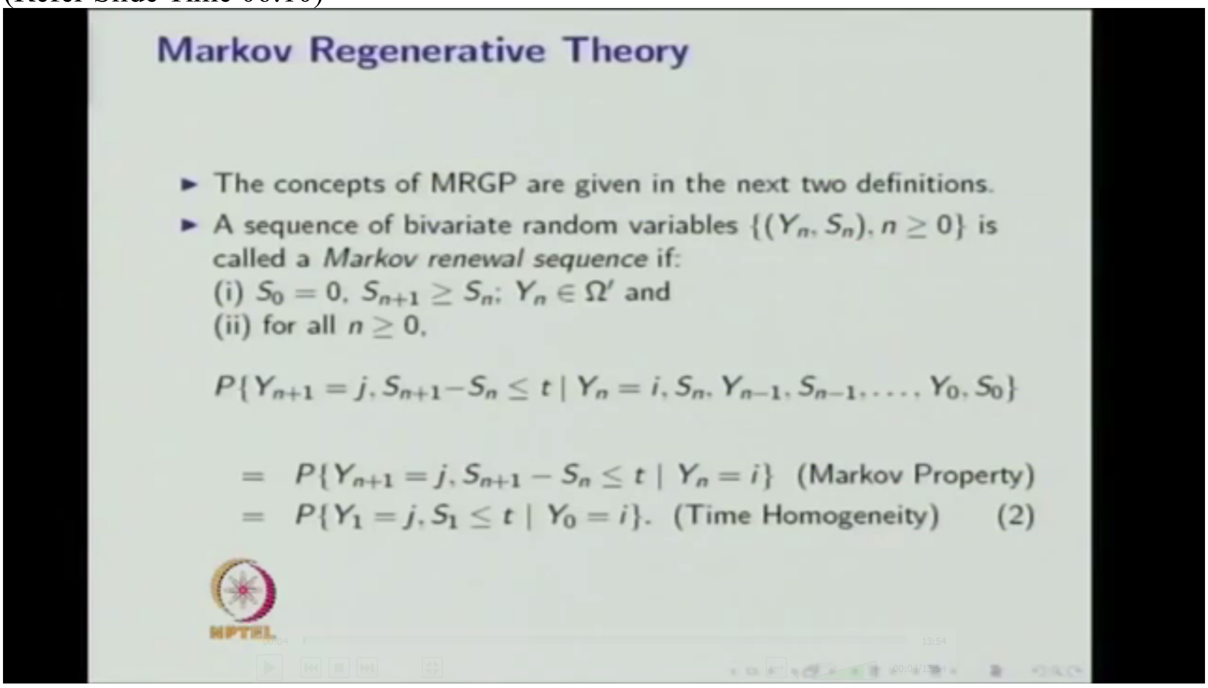

Equation number (2) is nothing but the conditional distribution of Y_{n+1} is equal to j with the difference of time in time -- regeneration times are less than or equal to t. That is same as because of Markov property and the time homogeneous property, this is the probability of Y_n is equal to j, S_1 is less than or equal to t given Y_0 is equal to i.

A Markov renewal sequence is also defined in the bivariate as this and usually this form of definition is frequently used since renewal time, and the state of the time and the state of the system at renewal instant, both are important.

So this conditional probability becomes the transition probability. That is this conditional probability will form a matrix K(t) and that is called a global kernel of the Markov renewal sequence. For the Markov renewal sequence, we can find the global kernel and the global kernel is the matrix K(t) that consists of $K_{i,j}(t)$ where each $K_{i,j}(t)$ is nothing but probability that $P(Y_1) = j$ with S_1 is less than or equal to t given $Y_0 = i$.

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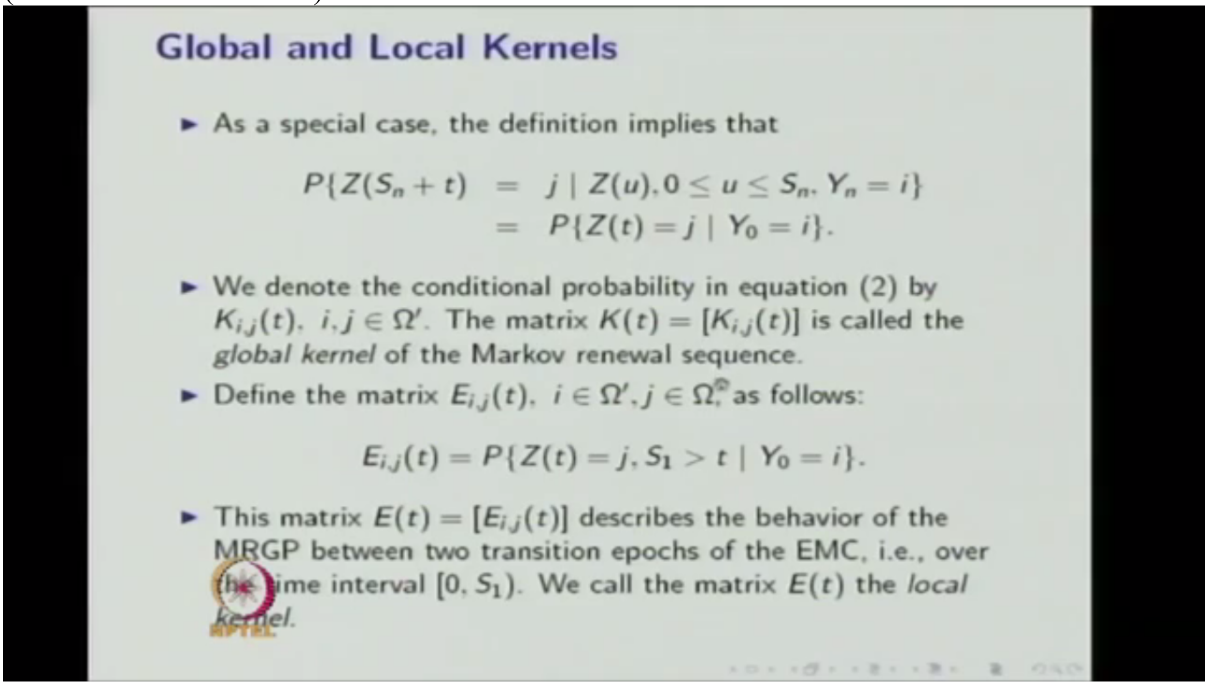

Now we are going to discuss the local kernel. That is also a matrix that consists of $E_{i,j}(t)$ where i is belonging to Ω' and j is belonging to Ω. Ω' means the collection of states at which the time transitions of the -- the system satisfies the Markov property at those time instance, those collection of states forms the Ω' and that is a subset of Ω.

So $E_{i,j}(t)$ is nothing but what is the probability that the system will be in the state j with the first regeneration time point is going to be greater than t. That means the system will be in this state j after the time t. The first regeneration going to occur after time t. The system will be in the state j at the time t given it was in the state i at the previous regeneration time point or at S_0 the system was in the state i. So this will form a -- this will form a local kernel.

So using global kernel and the local kernel, one can find the steady-state and the transient behaviour of Markov regenerative process.

Now we are going to discuss the limiting distribution or steady-state measures.

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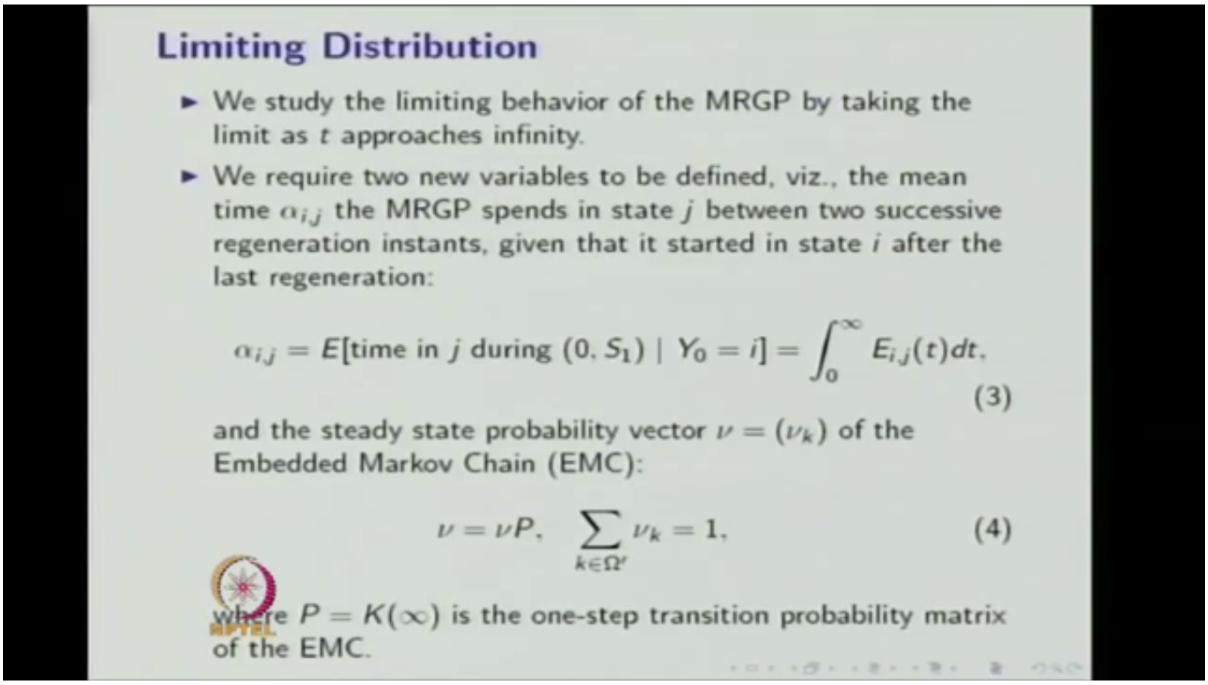

We study the limiting behaviour of the MRGP by taking limit as t approaches infinity.

We require two new variables to be defined, namely, the mean time $\alpha_{i,j}$ of the MRGP spends in the state j between two successive regeneration instants, time instants given that it started in the state i after the last regeneration. So this is nothing but the average spending time in the state j given that it was in the state i at the last regeneration.

So $\alpha_{i,j}$ is nothing but the expectation of time in state j during the interval 0 to S_1 where S_1 is the first regeneration time instant given that the system was in the state i at the previous or last regeneration time and the steady-state probability vector v of the embedded Markov chain that means v is equal to vP and the summation of v_k's is equal to 1 where k is belonging to Ω' and P is the one-step transition probability matrix of embedded Markov chain.

So from the global kernel K, that is the K(t), if you make a t tends to infinity, you will get the one-step transition probability matrix P. So from using P, you can get the steady-state probabilities v by solving v = vP and the summation of v_k is equal to 1. Once you solve the -- this using the $\alpha_{i,j}$, you can get the limiting distribution.

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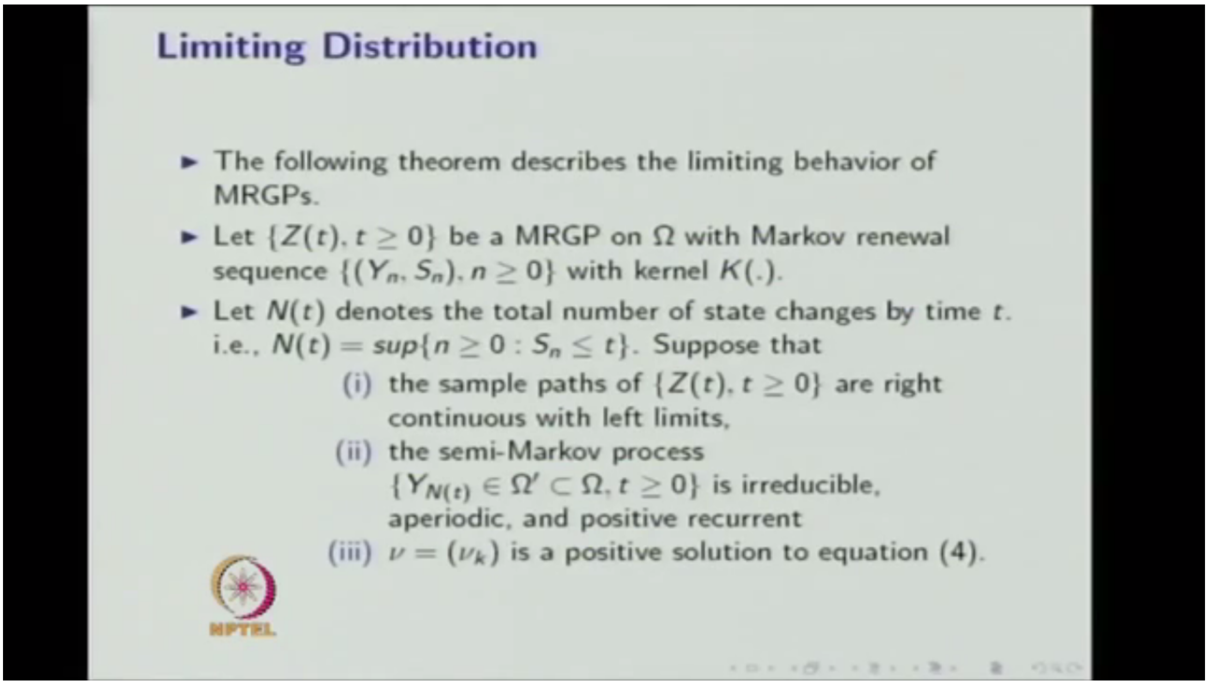

So the limiting distribution is given in the following theorem.

Let Z(t) be the MRGP with the Markov renewal sequence (Y_n, S_n). Let N(t) denotes the total number of states changes by time t. Then the sample path of Z(t) are right continuous with the left limits and the N(t) is a semi-Markov process, the $Y_{N(t)}$ is a semi-Markov process, which is irreducible, aperiodic and positive recurrent and v is a positive solution to the equation (4) that is this one, summation of v_i is equal to 1 and v = vP. If these properties are satisfied, then the steady-state probability vector π whose elements are π_j's, that is nothing but the limit t tends to infinity probability of Z(t) is equal to j using this formula where v_k's are nothing but the summation of α_k's.

So as long as these three properties are satisfied, that means the sample path has to be right continuous and the semi-Markov process has to be irreducible, aperiodic and a positive recurrent and you need a positive solution, the steady-state probability vector, then you can get the steady-state probability for the Markov regenerative process.

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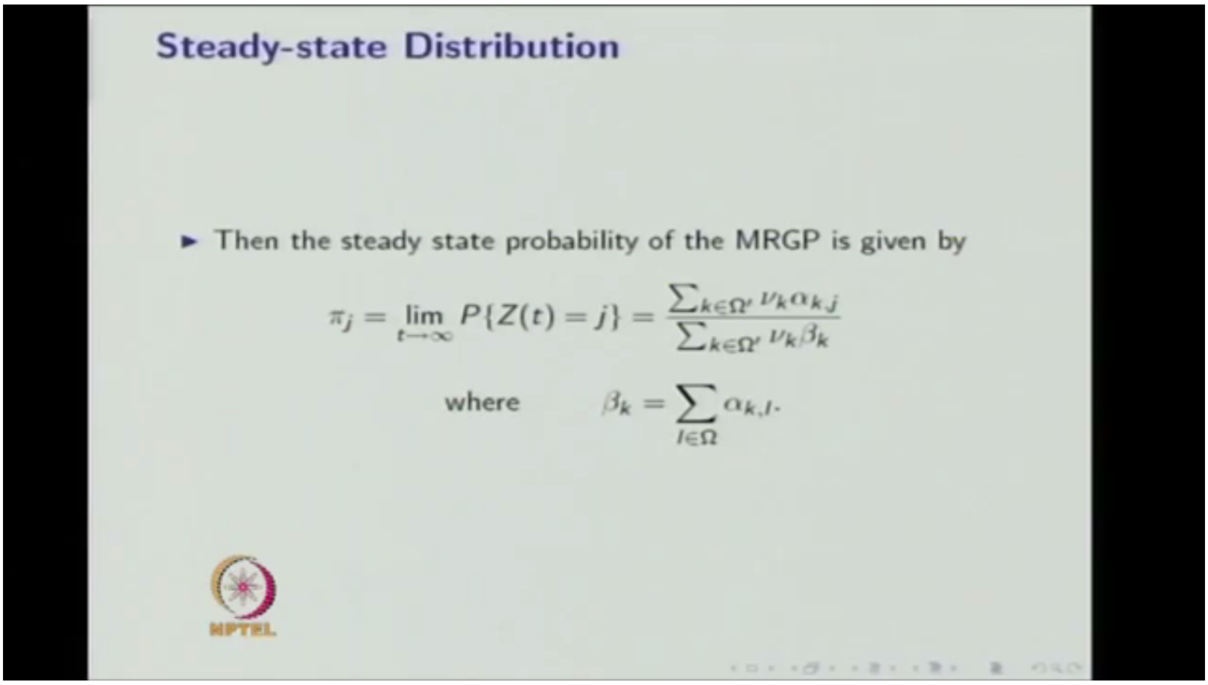

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