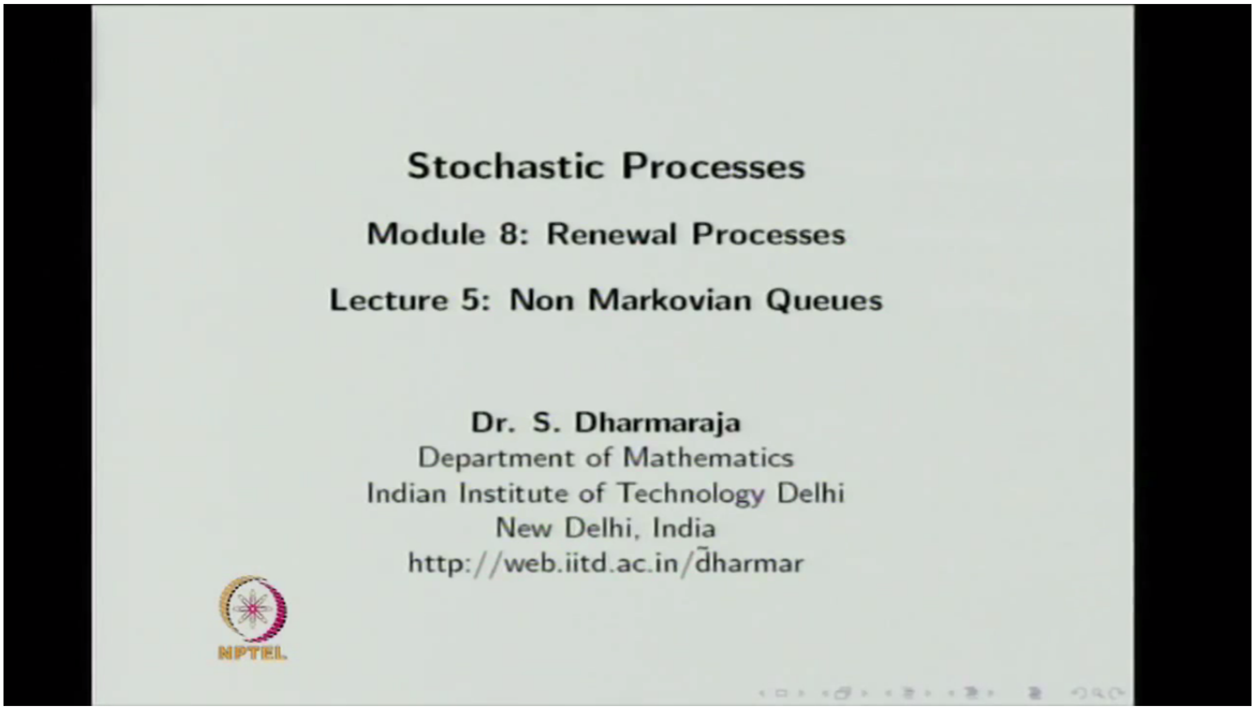

Video Course on

Stochastic Processes -1

By

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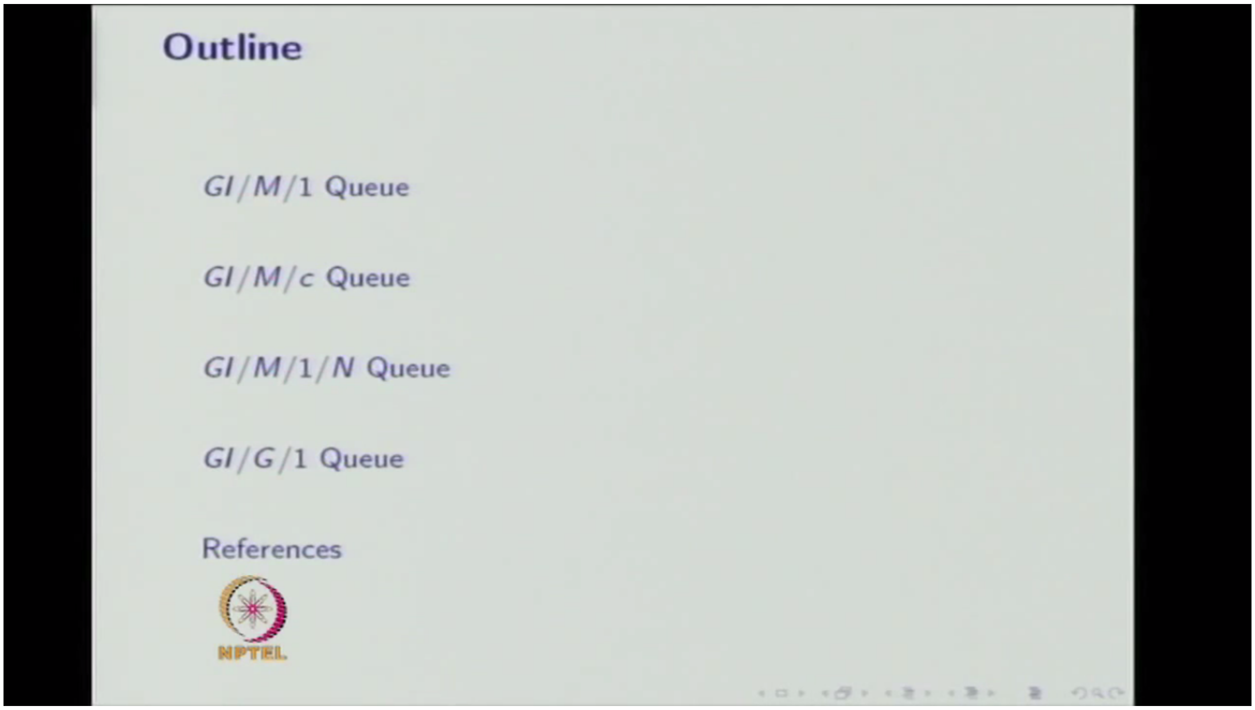
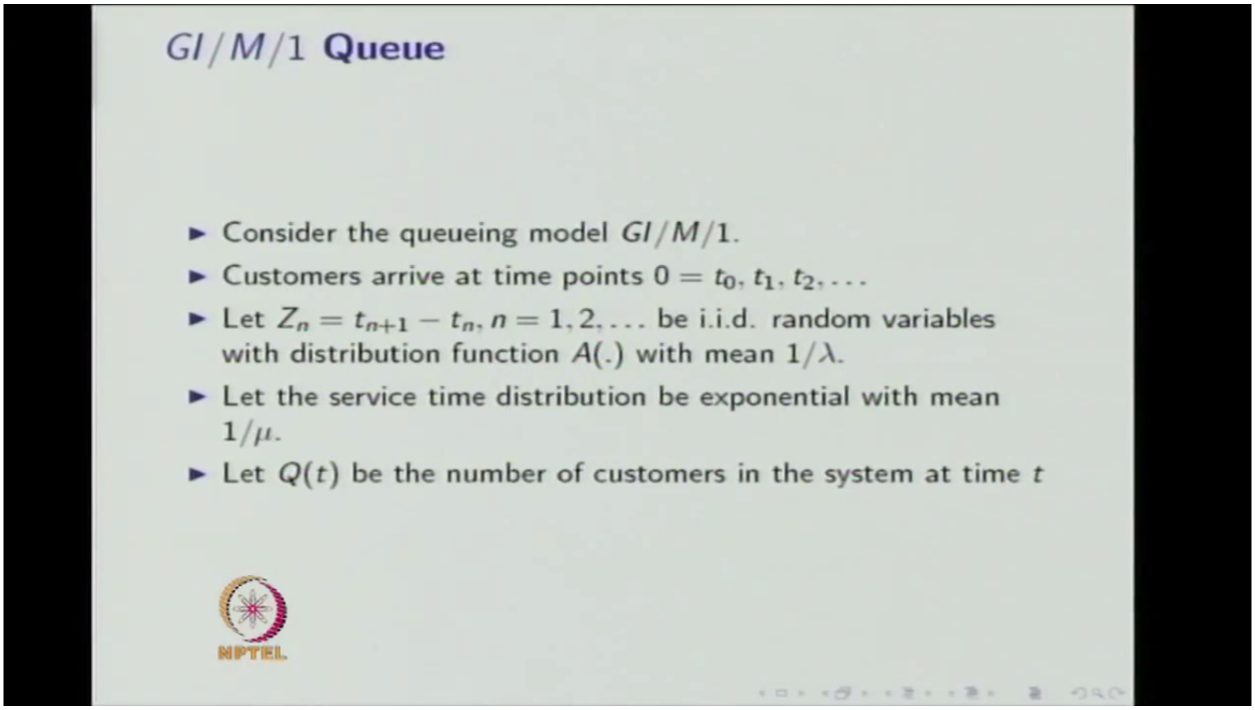
Module 8 : Renewal Process

Lecture# 5

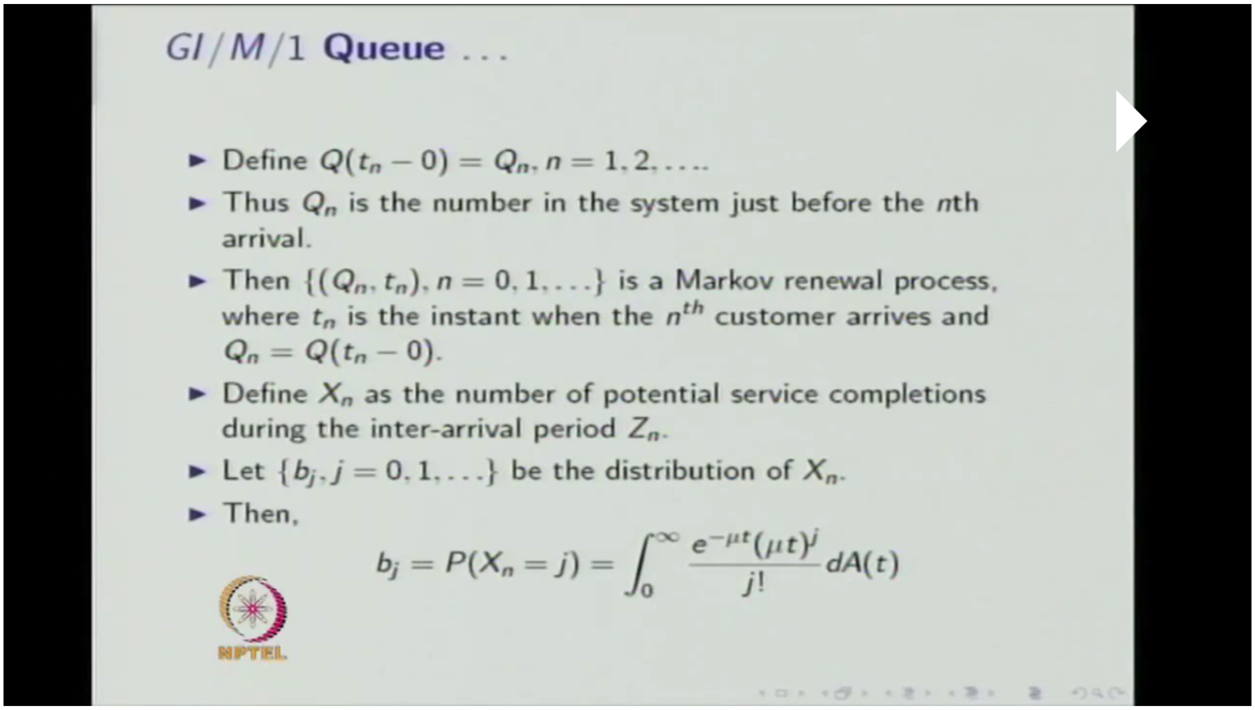
Non Markovian Queues



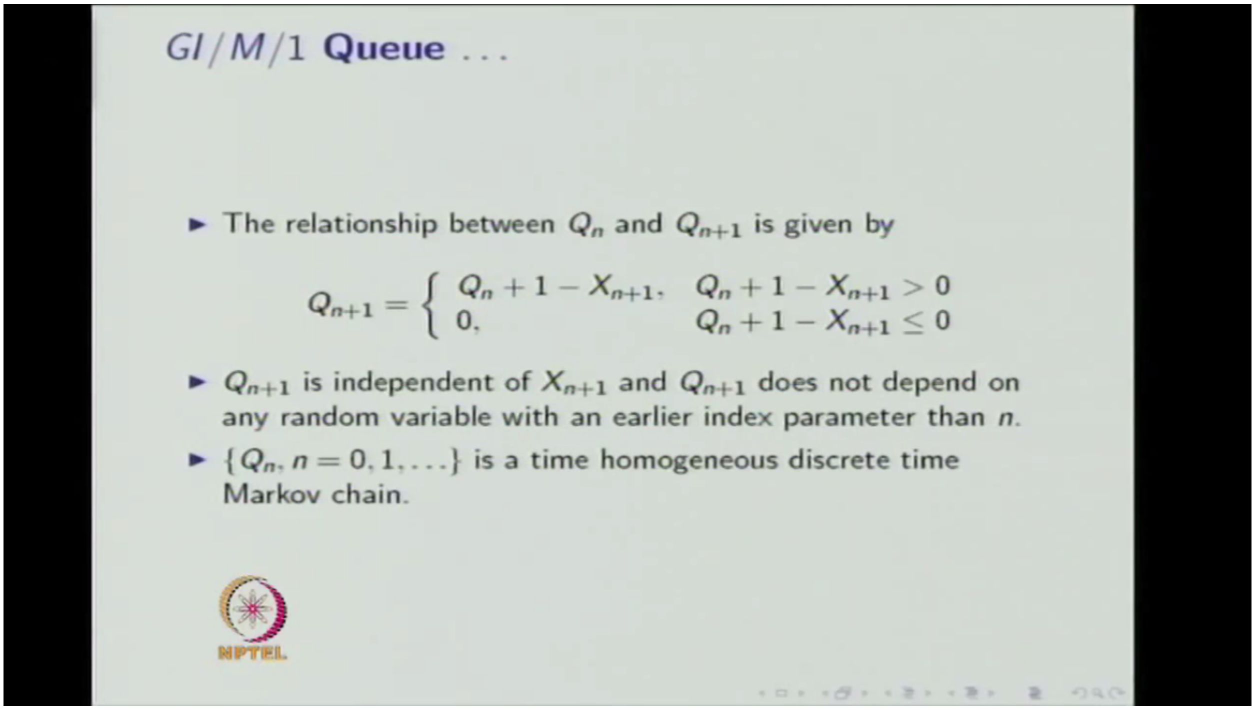
This is stochastic processes module 8 renewal processes. In the lecture 1 we have discussed the renewal function and renewal equation. In the lecture 2 we have discussed generalized renewal processes and renewal limit theorems. In lecture 3 we have covered Markov renewal and regenerative processes. In lecture 4, we have discussed non Markovian queues such as MG1 queue, MG1N queue, MGCC loss systems. This is lecture 5, non-Markovian queues. In this lecture we are going to discuss GI/M/1 Queue, GI/M/c Queue, then GI/M/1/N Queue, and finally GI/G/1 Queue.



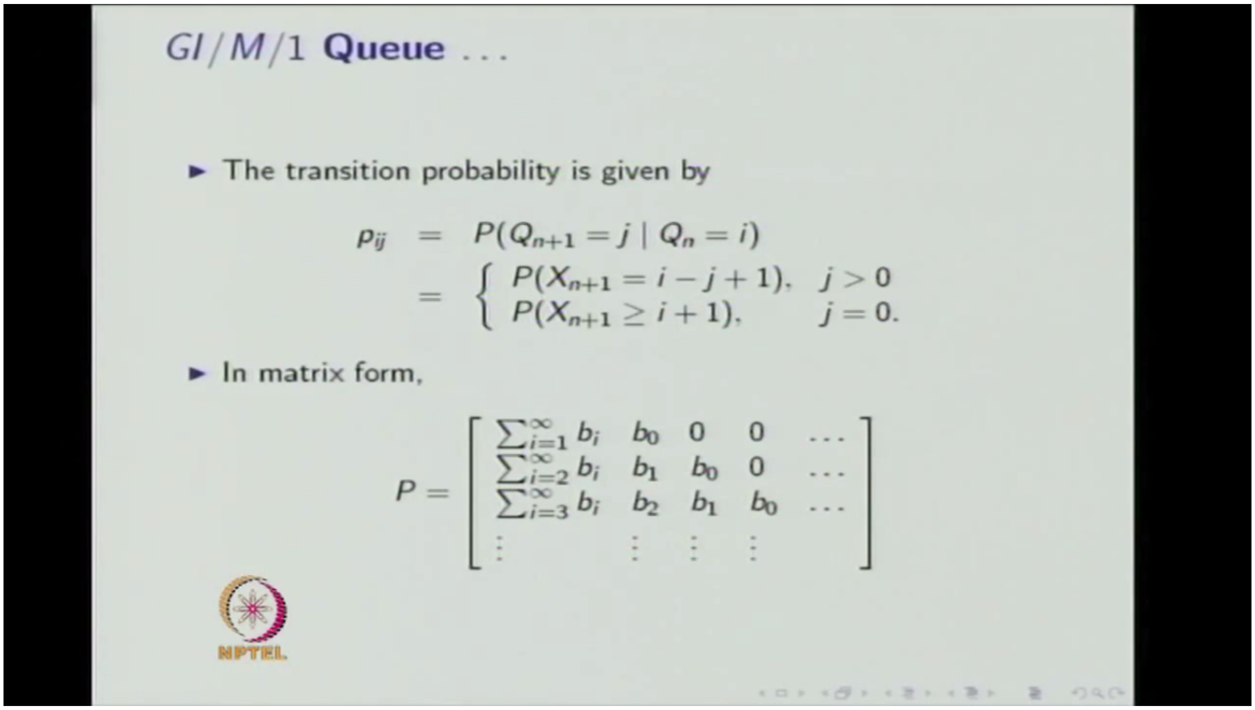
What is GI/M/1 queue? It means that the inter arrival time follows a non exponential distribution which are independent. Therefore the GI some books they use on G as a notation. M stands for the service time is exponential distribution. Only one server in the system with the infinite capacity. So consider the customers arrive at a time point say t0, t1, t2, and so on. Let Zn is equal to Tn plus 1 minus Tn with iid random variables with the distribution function with the CDF of a with the mean 1 by lambda. Therefore, as a special case if you have seen that the inter-arrival time is exponential distribution with the mean 1 by lambda then the arrival follows Poisson process with the parameter lambda but in this GI/1/M model the inter arrival time is a non exponential distribution with the CDF function a with the mean 1 by lambda. Let the service time distribution be exponential with the mean 1 by mean. Let Qt be the number of customers in the system at time t. So Qt for t greater or equal to 0 is a stochastic process. Since the Qt is a number of customers in the system at any time t therefore the corresponding stochastic process is a discrete state continuous time stochastic process. So the underlying stochastic process in the GI/M/1 queue is a Qt, t greater or equal to 0.



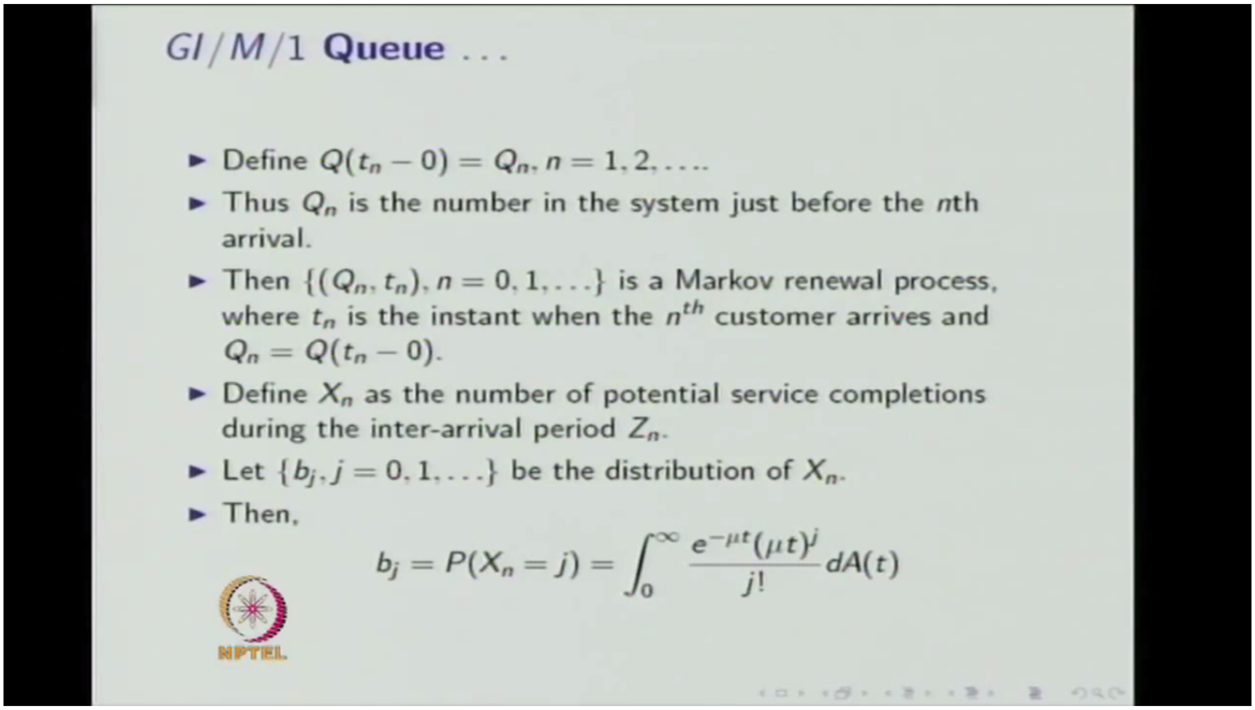
Now define Q of t n minus 0 as Qn. Thus Qn is a number of customers in the system just before the nth arrival. Qn is the number of customers in the system just before the nth arrival therefore the Qn for n is equal to 1, 2 and so on this follows a discrete state discrete time stochastic process. So this is a embedded stochastic process from Q of t, the Q of t is a discrete state continuous time stochastic process whereas Q of n is a discrete state discrete time stochastic process because the Qn is the number of customers in the system just before the nth arrival. The bivariate random variables Qn, tn for different values of n forms a Markov renewal process where tn is the instant when the nth customers arrives and the queuing is defined Q of tn minus 0. Since inter arrival time is a non exponential distribution with the mean 1 by lambda and the service time is exponential distribution with the mean 1 by mu single serve server in the system and infinite capacity. Therefore the Qn, tn from Markov renewal process. And the Tn is a time instant in which the arrival occurs. Now define the random variable Xn as the number of potential service completions during the inter arrival periods at n. Zn is nothing but tn plus 1 minus tn that is a inter arrival time. The Xn is the number of potential service completions during the inter arrival period is added and Pj be the distribution of Xn. Obviously Xn is their discrete type random variable for fixed n and the Pj is a probability mass function for the random variable Xn. Since the number of potential service completion could be 0, 1 and so on so the probability mass function for different values of j, Pj's is nothing but the probability of Xn is equal to j that is nothing but the integration 0 to infinity e power minus mu t, mu t power j divided by j factorial and the integration with respect to A of t where A of t is the distribution function of inter arrival time with the mean 1 by lambda and mu is the parameter for exponential distribution of service time.



The way we define the Qn is nothing but the number of customers in the system just before the nth arrival one can relate the Qn plus 1 in terms of Qn. We can find the relationship between Qn and a Qn plus 1 that is Qn plus 1 will be Qn plus 1 minus Xn plus 1 whenever Qn plus 1 minus Xn plus 1 is greater than 0; otherwise it is 0 for Qn plus 1 minus Xn plus 1 is less than 0. The reason is the number of customers in the system just before the nth arrival n plus 1-th arrival the same as the number of customers in the system just before the nth arrival plus the n plus 1-th customer who arrive that is plus 1 minus how many customers are served during the inter arrival period that is Xn plus 1. So if you subtract that Qn plus 1 by Xn plus 1 you will get Qn plus 1. Whenever Qn plus 1 minus Xn plus 1 is greater than 0. If it is less than or equal to 0 then the number of customers will be again 0 just before the n plus 1-th arrival also will be 0. We know that the Xn plus 1 is independent of Qn plus 1. Hence the Qn plus 1 depends only on Qn and Qn plus 1 is independent of Xn plus 1 therefore the Qn forms a time homogeneous discrete time Markov chain. The Qt is a discrete state continuous time stochastic process and the Qn is the discrete time discrete state stochastic process. Since Qn plus 1 is equal to Qn plus 1 minus Xn plus 1 for 0 and Qn plus 1 is independent of Xn plus 1 as well as Qn plus 1 depends only on Qn therefore the Qn for the n is equal to 0, 1, 2, and so on form a time homogeneous means time invariant and also it satisfies the Markov property. Therefore this discrete time discrete state stochastic process is called a discrete time Markov chain satisfying the time homogeneous property therefore it is called the time homogeneous discrete time Markov chain.



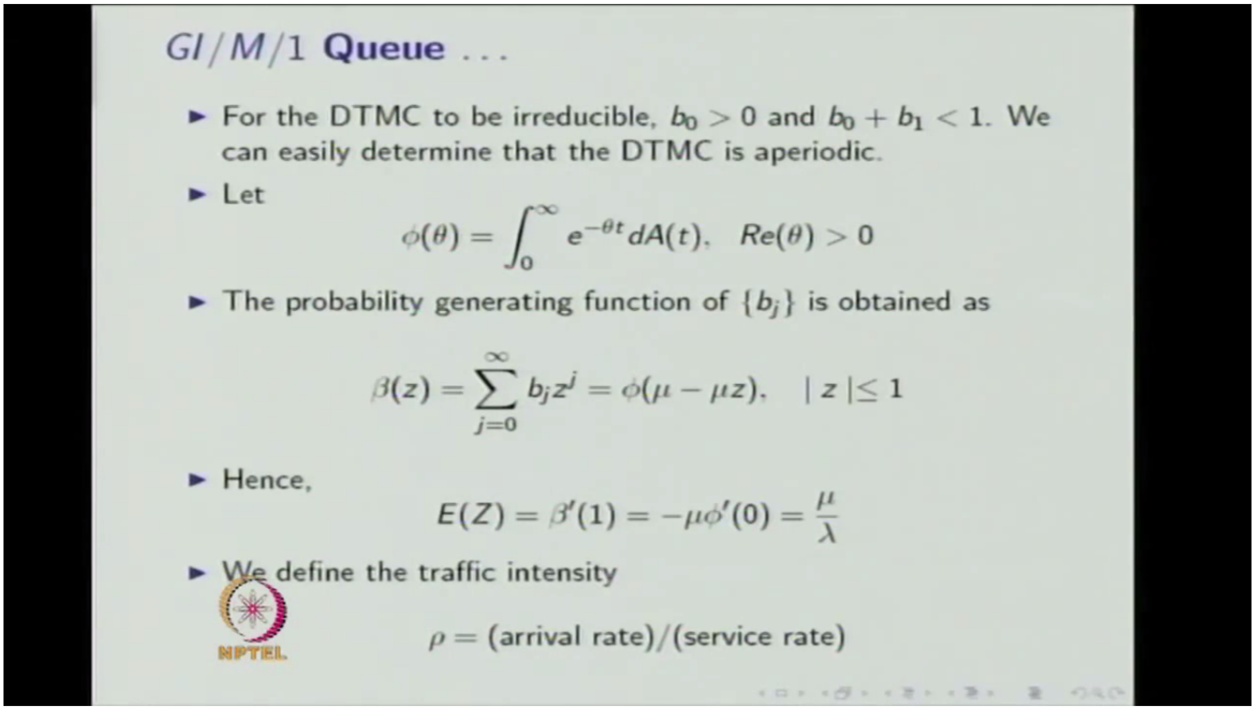
Once we know that Qn is a discrete time Markov chain you can find the one-step transition probability matrix whose elements are Pij's. So the Pij's are nothing but what is the probability that Qn plus 1 will be j given that Qn was i that is same as what is the probability that i plus j minus 1 customers are served during the inter arrival time period. That is a probability of Xn plus 1 is equal to i minus j plus 1 whenever j is greater than 0. If j is equal to 0 then it is nothing but a probability that Xn plus 1 is equal to i plus 1.



In matrix form you can write it P as the matrix whose elements are Pij so the first element will be P1 plus P2 and so on and the second element in the first row will be b naught, b0. Since Pj's are nothing but the probability mass function for the random variable Xn that is for all n, for all n it is identically distributed. Therefore the probability mass function of Xn is Pj's and the running index for j is 0, 1, and so on therefore if you make the row sum b naught plus b1 plus b2 and so on that will be one.

Whereas in the second row the first element will be b2 plus b3 and so on. The second row second element will be b1. Second row third element will be b0 and so on. substitute i and j in the above equation. You substitute i and j accordingly you will get this values summation of b starting from 2 and b1, b0 and so on. Similarly you can get the third row. You can verify that each element of B capital matrix will be lies between 0 to 1 and the row sum will be 1. It's basically a stochastic matrix. This is a one-step transition probability matrix capital P.

Our interest is to find out the steady state or limiting distribution. For that we need irreducible and positive recurrent and a periodicals also.



So for the irreducible you need a b naught has to be greater than 0 as well as P naught plus P1 has to be less than 1. If this is satisfied then you will get the conclusion the given time homogeneous discrete time Markov chain will be irreducible. That means that each state is communicating with each other states with the condition of b naught is greater than 0 b naught plus b1 is less than 1.

We can easily determine that the DTMC is a periodic. We have discussed the periodic in the discrete-time Markov chain so we can verify that this is a – this discrete time Markov chain is a periodical also that means with the period 1. Now we find out the Laplace transform of the CDF of inter arrival time distribution that is a Pi naught. So Pi naught is equal to integration 0 to infinity e power minus theta times t D of At where At is a CDF of inter arrival time distribution with the real of theta has to be greater than 0. Now we are finding the probability generating function for Pj's that is a distribution of Xn. So that is beta of Z that's the summation j is equal to 0 to infinity Pj's is Z power j. That you can write down in terms of Laplace transform. So this is Laplace stieltjes transform. So Laplace stieltjes transform of A of t so it's a sigh of mu minus mu times Z.

Now we can find out the expectation of Z that is nothing but the beta dash of 1 if you differentiate probability generating function then substitute Z is equal to 1 will be the mean of – mean number of arrivals that is Z.