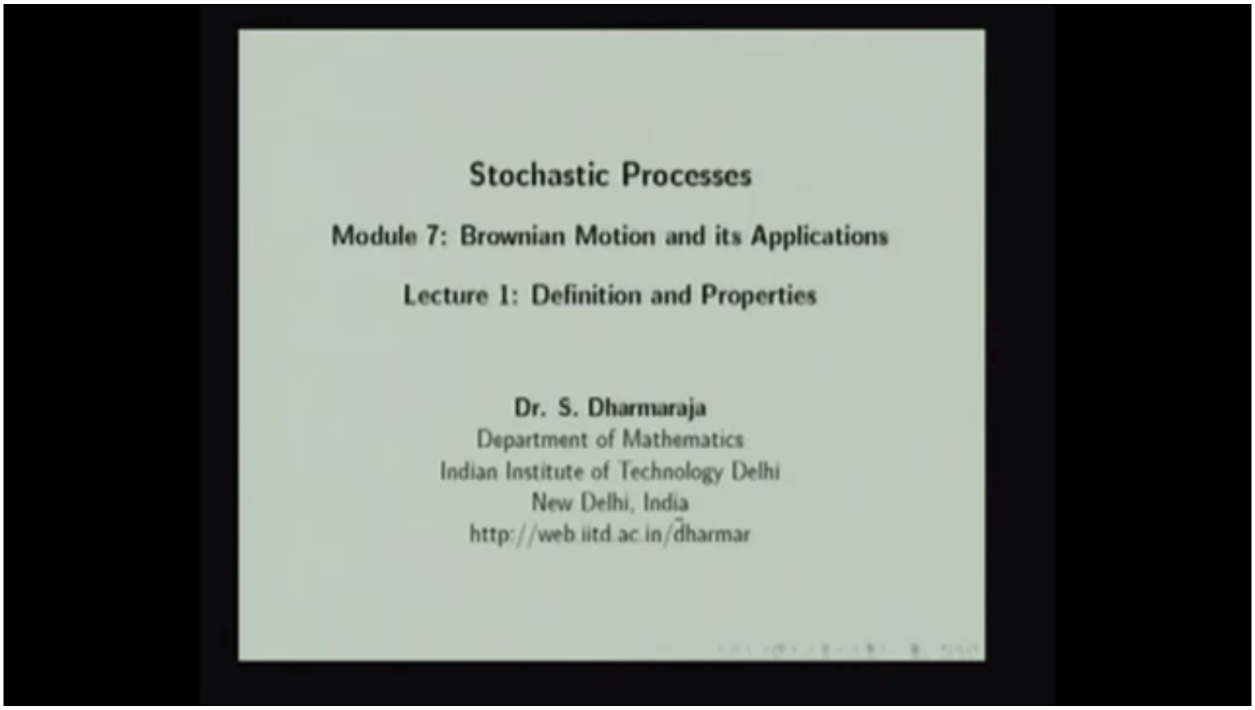


Video Course on

Stochastic Processes -1

By

Dr. S Dharmaraja

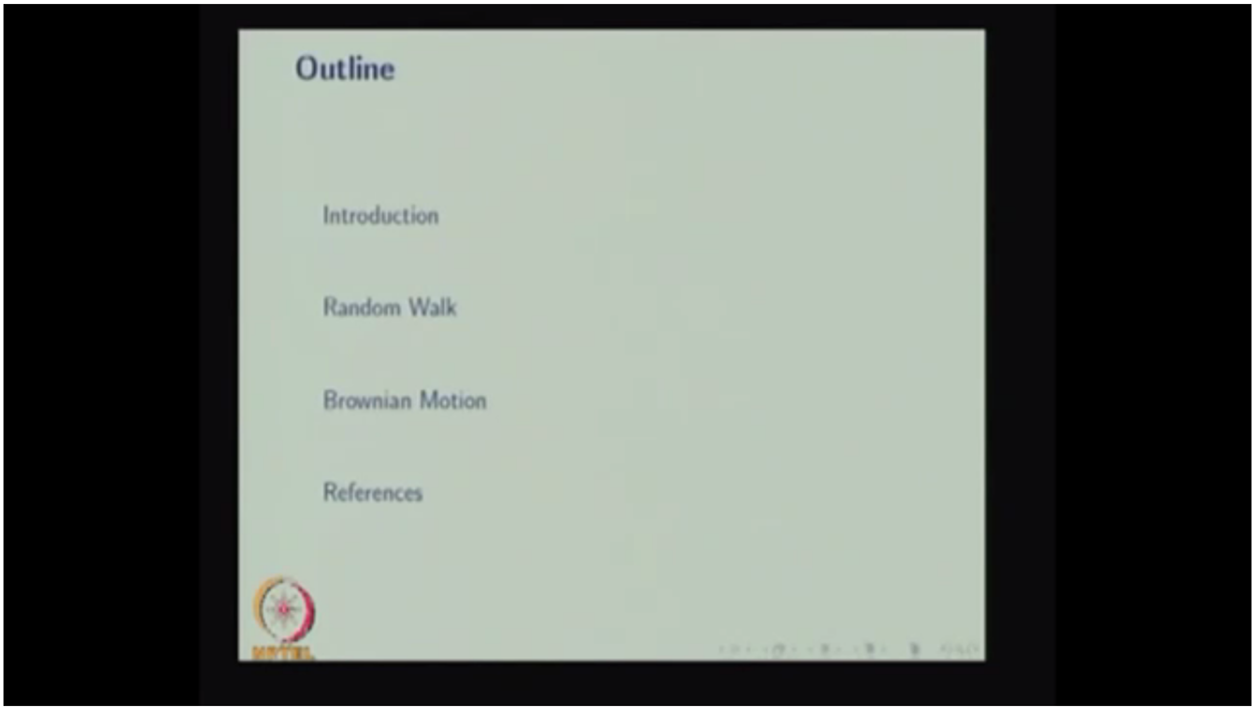
Department of Mathematics, IIT Delhi

Module #7: Brownian Motion and its Application

Lecture#1

Definition and Properties

This is the stochastic processes module 7; Brownian motion and its properties. Lecture 1; definition and properties.

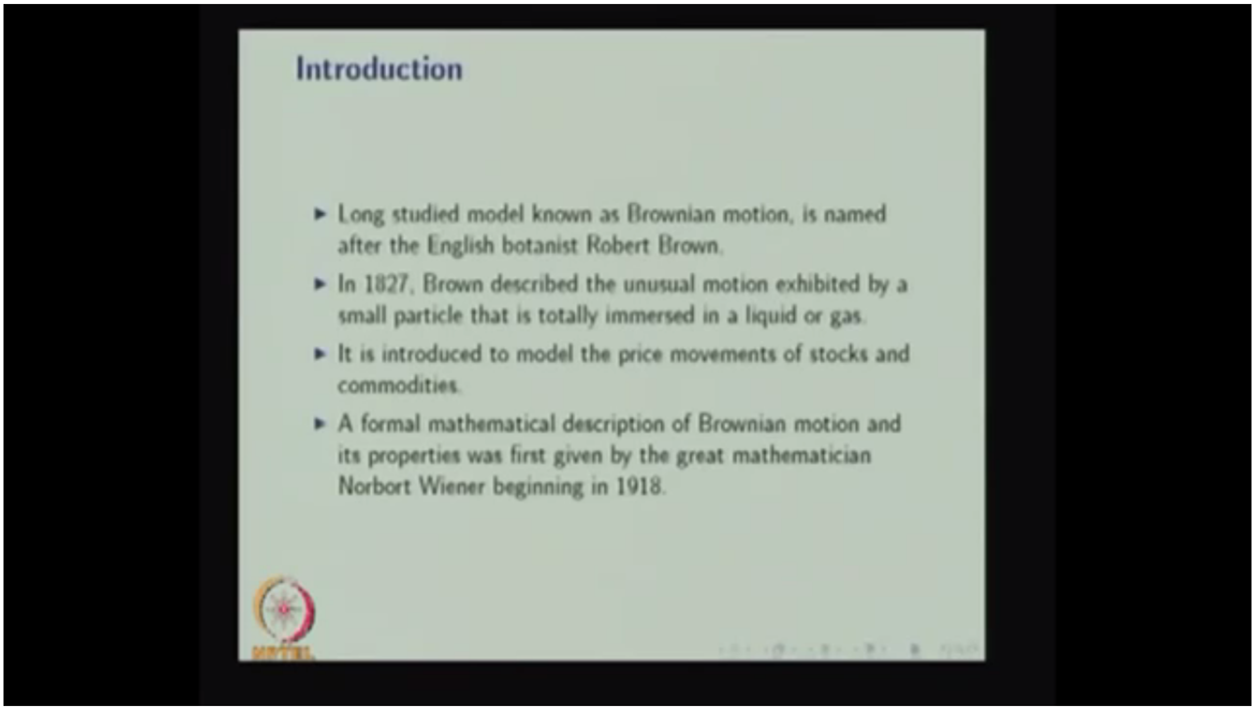


In the last six models we started with the review of probability. It's one model. Then the second model we discussed the definition of the stochastic process and its properties and in the third model we have discussed the stationary process and all the properties. Fourth model we have discussed the the discrete-time Markov chain and in the fifth model we have discussed the continuous-time Markov chain. In the sixth model we have discussed the [Indiscernible] [00:01:22] and this is a seventh model that is Brownian motion and its properties.

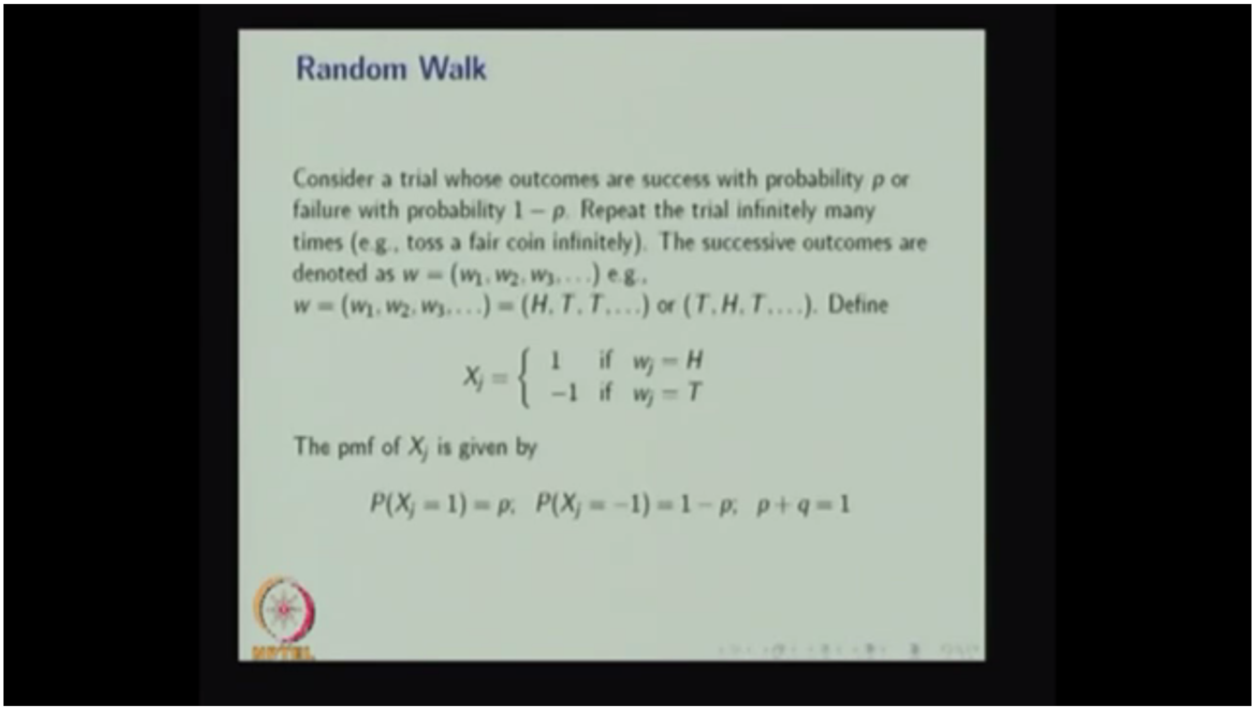
In this lecturer, in this model we are planning to discuss the important stochastic process that is Brownian motion and then later we are going to discuss the process derived from the Brownian motion. Then we are going to discuss the stochastic calculus and followed by that we are going to discuss the stochastic differential equation and Ito integrals and application of the Brownian motion stochastic calculus that is in the financial mathematics.

So we are going to discuss the applications of a Brownian motions in the financial mathematics so with that the module seven will be completed.

And this is the lecture one of – this is the lecture 1 of module 7 Brownian motion and its application. In this lecture we are going to discuss the the random walk and the definition of Brownian motion. Then how one can derive the Brownian motion using a random walk and some important properties of Brownian motions also will be discussed.

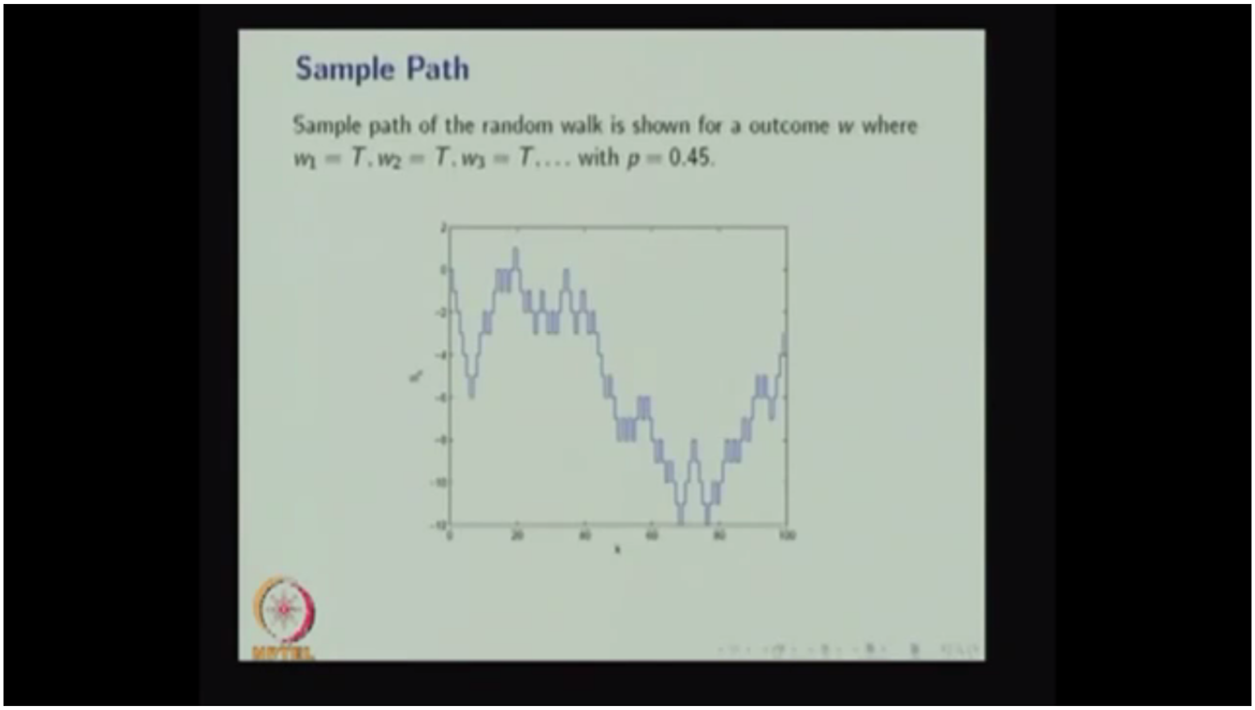


The long studied model known as a Brownian motion is named after the English botanist Robert Brown. In 1827 Brown described the unusual motion exhibited by a small particle that is totally immersed in a liquid or gas. It is introduced to model the price movements of stocks and commodities. A formal medical description of Brownian motion and its properties was first given by the great mathematician Norbert Wiener beginning in 1918. Therefore the Brownian motion is also called as Wiener process.



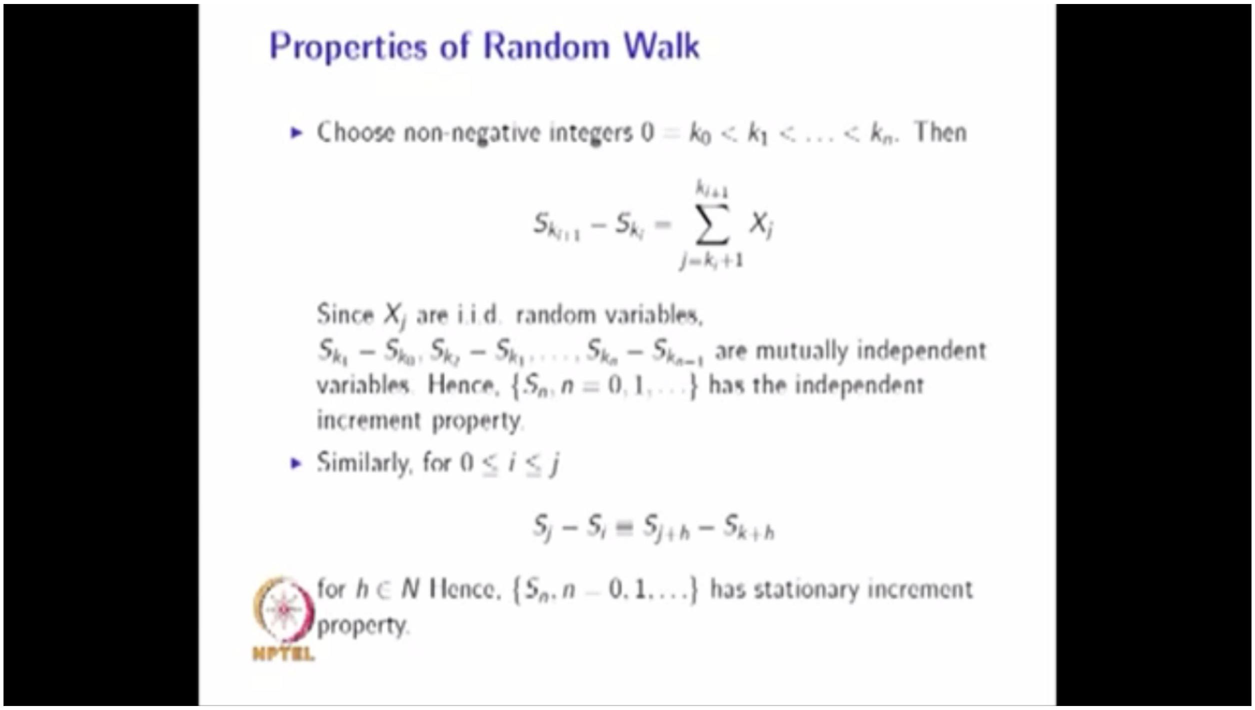
Now we start with the random walk because using this random walk we are going to derive the Brownian motion. Consider a trial whose outcomes are success with the probability p or failure with the probability 1 minus p. Repeat the trial infinitely many times that is equivalent of saying tossing fair coin infinitely many times. The successive outcomes are denoted by the sample w that consist of w1, w2, w3, where each one is the outcome in the nth trial. That means w1 could be head or tail similarly and w2 could be head or tail and so on. For example we have given H, T, or T, H, T and so on. So this collection is the – this all the possible w's that is going to be the sample space.

Now we are defining the random variable Xj it takes the value 1 if the outcome of the jth trial is head. If the outcome of the jth trail is tail then the value is defined for Xj is minus 1. So this is a real-valued function and this will be a random variable since it takes a value 1 or minus 1 this is a discrete random variable and one can find what is a probability mass function for the random variable Xj. So since the trial whose outcomes are success with the probability p success is nothing but the getting head and the failure is nothing but trial land up with tail. Therefore the probability of Xj is equal to 1 that probability is call it the Wj is equal to head that call it as a success therefore this probability is p and the probability of Xj is equal to minus 1 that is 1 minus p and this you can denote it by q therefore p plus q is equal to 1 and denote 1 minus p as quarter ends as p plus quarter equal to 1. Now we are defining the sequence of other random variables that is started with S naught equal to 0 we are defining sum of first k random variables as a Sk where k is running from 1, 2 and so on. Here Xi's are iid random variables. And the sequence of random variables Sk that is the random walk. With the S naught is equal to 0 and the Sk's are nothing but the first k Xi random variables.



You can see the sample path of the random bar whose w1 is a tail therefore takes a value X1 is minus 1. Again if the suppose w2 is t then X2 also takes the value minus 1. Suppose w3 is also T then X3 also takes minus 1. Therefore Sk will be initially it is 0 then S1 will be X1 that is minus 1. S2 will be X1 plus X2 that is minus 1 plus minus 1 that is minus 2. So S2 is minus 2. S3 will be S2 plus X3 that is a again adding minus 1 so S3 will be minus 1. Like that it can take the different values. So here this is a one sample path with the w1 is equal to T and w2 is equal to T and w3 is equal to T and so on with the probability P is equal to 0.45. This is the probability of success or probability of getting head when you toss a coin.

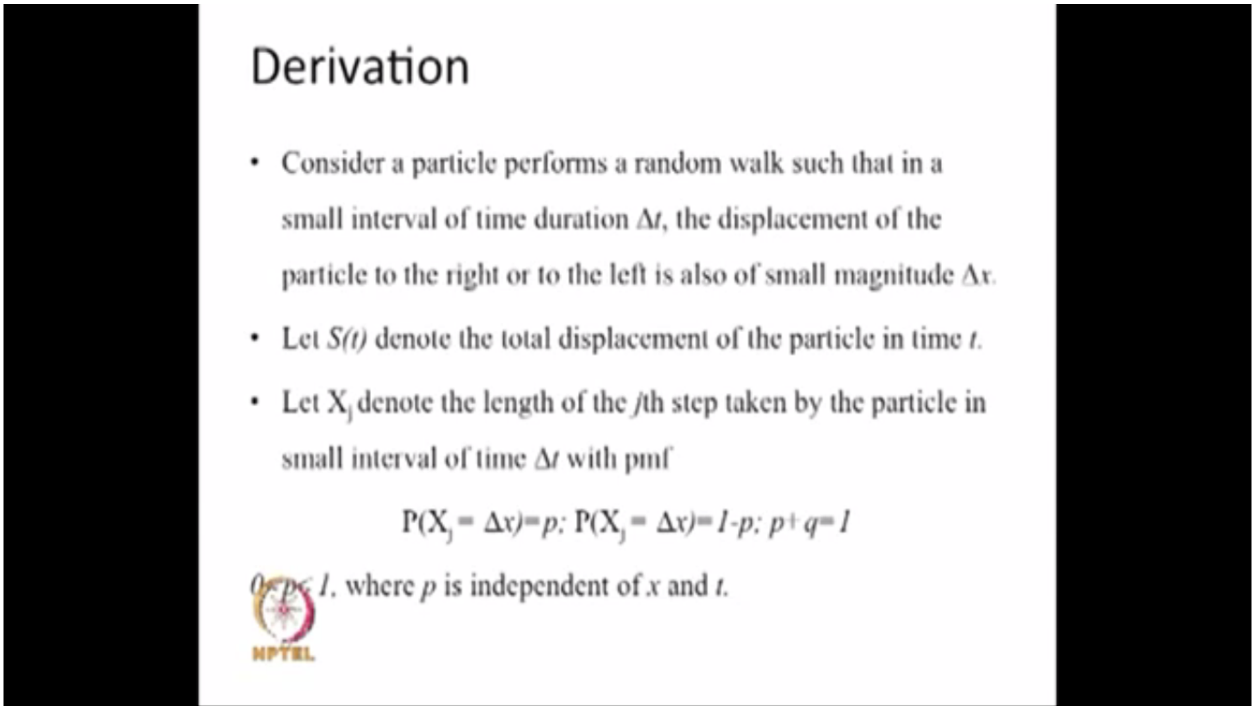
So we are going to conclude later as k tends to infinity using central limit theorem one can conclude this will be a Brownian motion. For that you should understand how the Sk's are are created wher Sk's are the sample path, where Sk's are the random walk.



Now we are going to see the properties of random walk. If you choose non-negative integers k naught, k1 and so on not k1 and so on then if you find that difference the difference is nothing but a sum of Xi's in this range. Since the Xi's are iid random variables if you take a non-overlapping intervals or the increments of Si's then that will be mutual [Indiscernible] [00:10:06] because each these increments will be nothing but the sum of a few Xi's and we know that each Xi's are mutually independent iid random variables therefore non-overlapping increments will be mutually independent random variables. Hence Sn has the property called the independent increment, the the increments of independence.

Similarly for 0 less than or equal to i less than or equal to j Si minus Sj is identically distributed with Sj plus H minus Si plus H for H belonging to natural numbers. Hence the stochastic process Sn has stationary increment property. That means if you find out the n dimensional random variable and shifted by H find out the another n dimensional random variable if though the joint distributions are same for both the n dimension random variable without shifting and with shifting then that stochastic process is called stationary but here the stochastic process is not a stationary the increments are stationary means they have increments and you shifted the increment by some interval H then the distributions are going to be identical. That's what it shows for one less than or equal to i less than j less than or equal to k less than l the difference the distributions are going to be same as long as their length is same. So it is in the increments are time invariants not the actual stochastic process. Therefore this stochastic process has the stationary increment also. Therefore, the random walk has increments are stationary as well as invariants.

Also one can find a mean and variance of increments. The increments are nothing but the difference of those random variables and since each random variable are discrete type random variable with the probability mass function that is discussed in the previous slide so we can find out the mean and variance of those random variables therefore we can find out the mean and variance of increments also.



Now we are going to derive the Brownian motion using random walk. Consider a particle performs a random walk such that in a small interval of time of duration delta t the displacement of the particle to the right or to the left is also a small magnitude Delta x whenever a particle performs a random walk in a very small interval of time delta t the displacement of a particle to the right or to the left that magnitude is Delta x. Now we are defining a random variable S of t denotes the total displacement of the particle in time t. let Xj denote the length the jth taken by the particle in a small interval of time delta t with the probability mass function. So the probability of Xj takes the displacement of the particle to the right side that is Delta x with the probability p the left side that is the Xj takes the value minus Delta x that is 1 minus p that is nothing but a q where p plus q is equal to 1 where p is independent of X as well as time. It is very important. The probability of the displacement to the right or to the left that probability whether p or 1 minus p which is independent of X as well as time. Now the partition of the interval of length t into n equal subintervals of Delta x. Then the n times delta t becomes t and the total displacement St is the sum of n iid random variables Xt. The way we partition the interval 0 to t into n equal parts therefore the S of t the total displacement is nothing but the sum of n iid random variables Xj's where n is nothing but n of t because you are partitioning the interval n sorry you are partitioning the time interval 0 to t the length of t into n parts therefore n is nothing but n of [Indiscernible] [00:15:41] that is nothing but t divided by Delta t. So you know the mean and variance. Therefore you can find out the mean and variance of S of t also. because St is a sum of n iid random variables Xj. Expectation is a linear operator therefore n since it is iid random variable n times expectation of any one random variable whereas variance since the random variables are independent then the variance of S of t is nothing but the variance of sum of random variables. So you can take it out and you can do the simplification.