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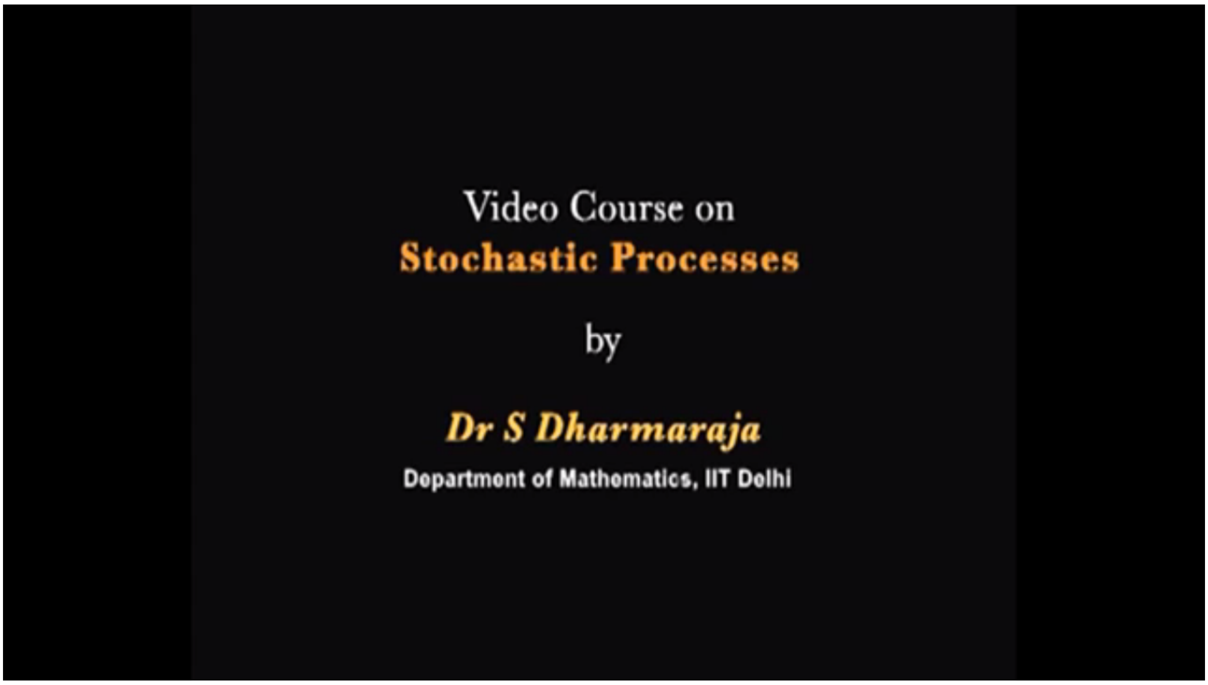

Video Course on
Stochastic Processes

by

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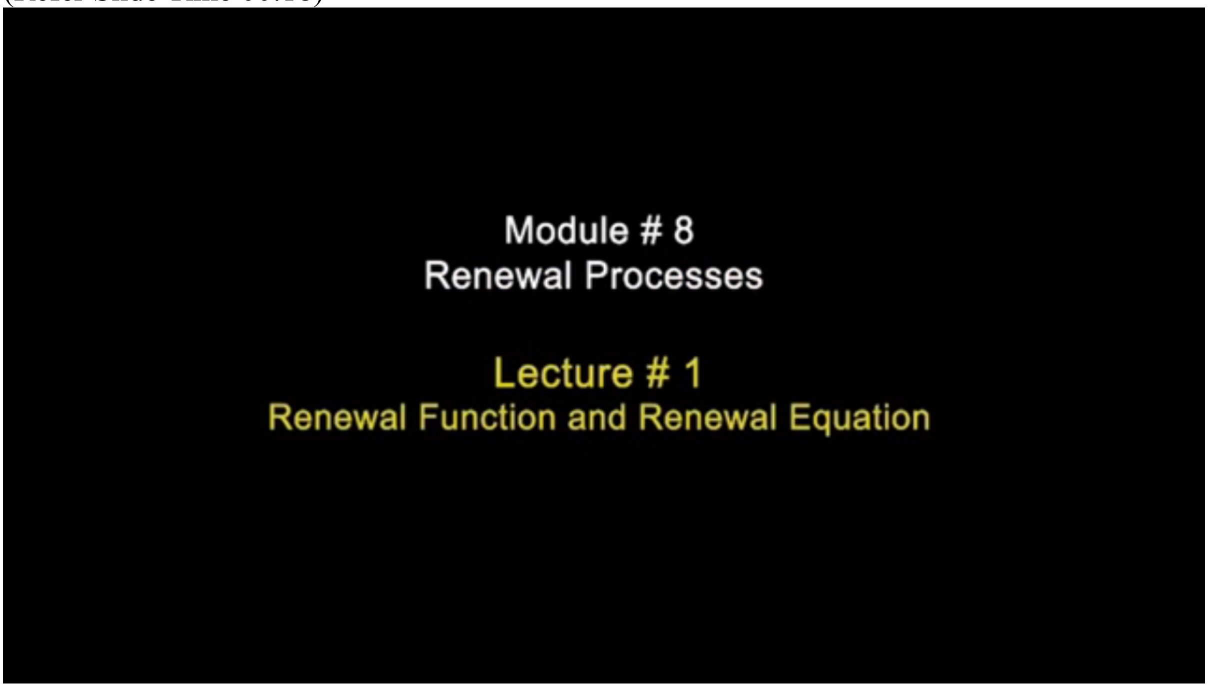

Module # 8
Renewal Processes

Lecture # 1
Renewal Function and Renewal Equation

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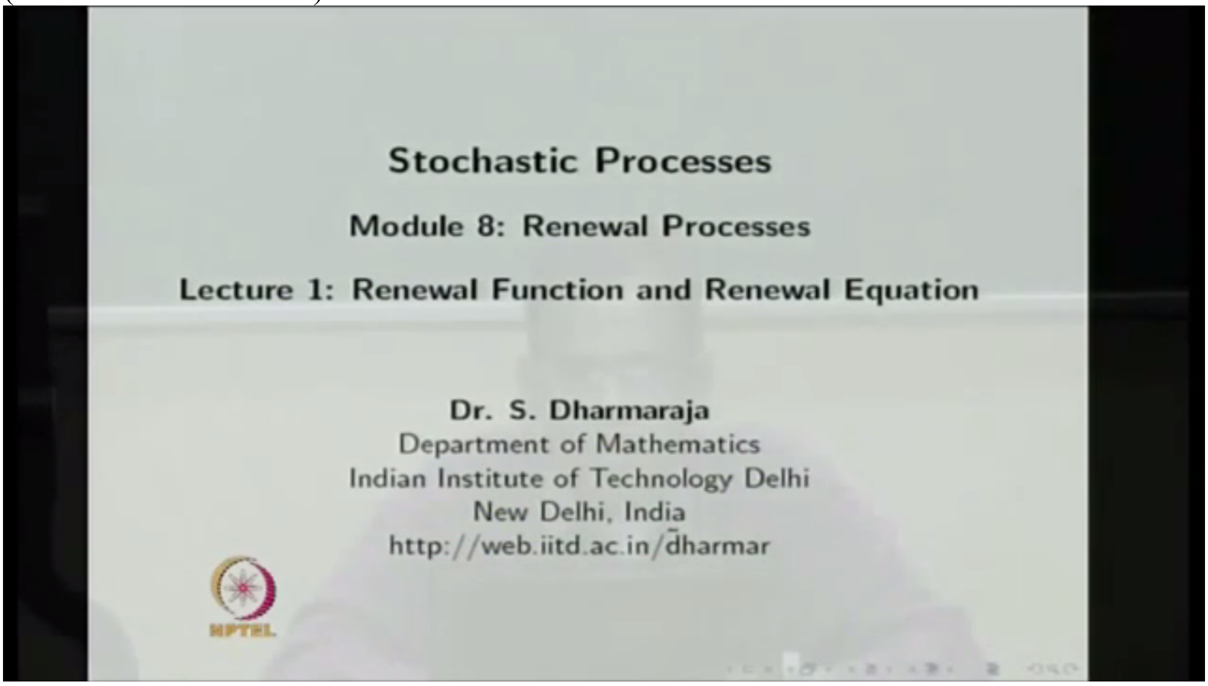

This is Stochastic Processes, Module 8: Renewal Processes.

In the Module 1, we have presented probability review and introduction to stochastic processes.

In the Module 2, we have presented the definition of stochastic process and its properties.

Stationary processes is discussed in Module 3.

Discrete-time Markov chain is discussed in Module 4.

Continuous-time Markov chain is discussed in Module 5.

Martingales are presented in Module 6.

Brownian motion and its properties are discussed in Module 7.

Now we are giving the presentation on Module 8: Renewal processes. In this, we are planning to give lecture for renewals starting with renewal function and the renewal equation in the Lecture 1, generalized renewal processes and renewal limiting theorems in Lecture 2 and in the Lecture 3, we are planning to present Semi Markov process or Markov renewal process and Markov regenerative process and in the Lecture 4, we are planning to present non-Markovian queues. In the Lecture 5, we are planning to present non-Markovian queues with respect to the service distribution.

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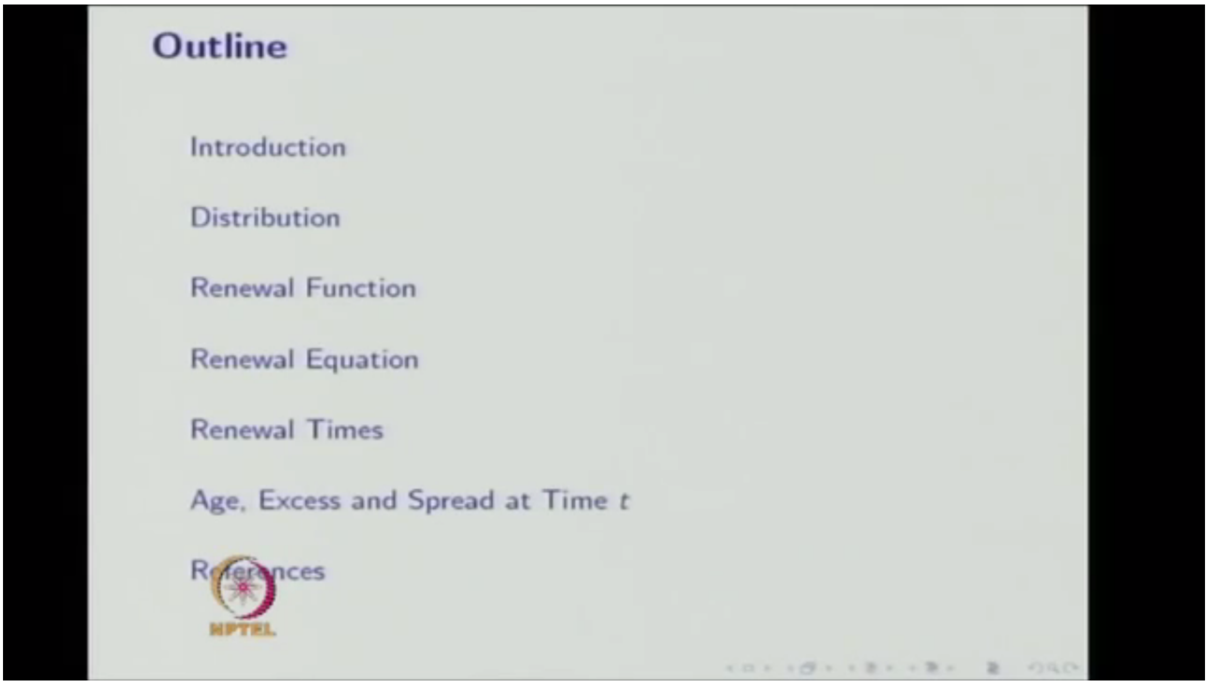

In this lecture, we are planning to give the definition of a renewal process and the related properties. Then we are going to present a renewal equation and then we are going to present renewal times. Then finally we are going to explain what is the meaning of age, excess and spread at time t.

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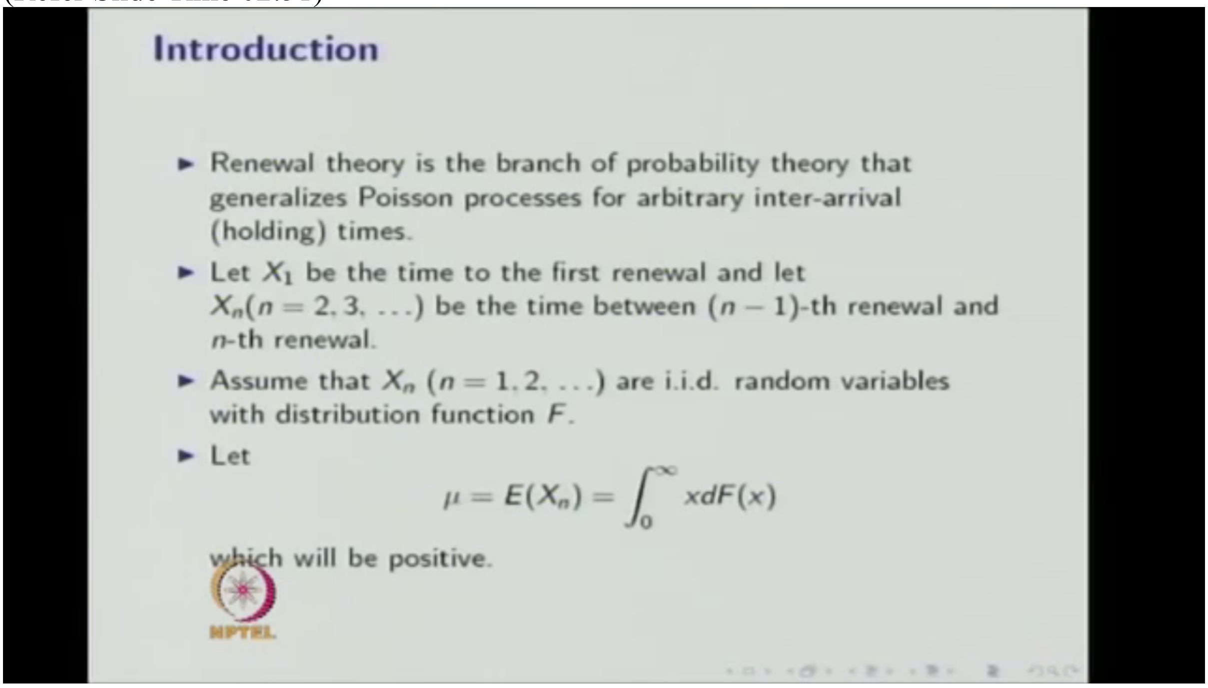

The Renewal theory is the branch of probability theory that generalizes Poisson processes for arbitrary inter-arrival or holding times. Renewal as the word suggests means that an event occur again. In this, when we say that X and X_1 is the time of first renewal, then it refers to the first time an event occurs where event may be defined as required by the user.

Let X_1 be the time to the first renewal and let X_n, n running from 2, 3 and so on be the time between $(n - 1)^{th}$ renewal and n^{th} renewal.

Assume that X_i's are independent, identically distributed random variables with the distribution function capital F. The mean for the random variable X_n that is denoted by the μ that is nothing but 0 to infinity x times integration with respect to dF(x), which will be positive because X_i's are nothing but the time between the $(n - 1)^{th}$ arrival and n^{th} arrival. That is a random variable X_n and the mean is will be a positive.

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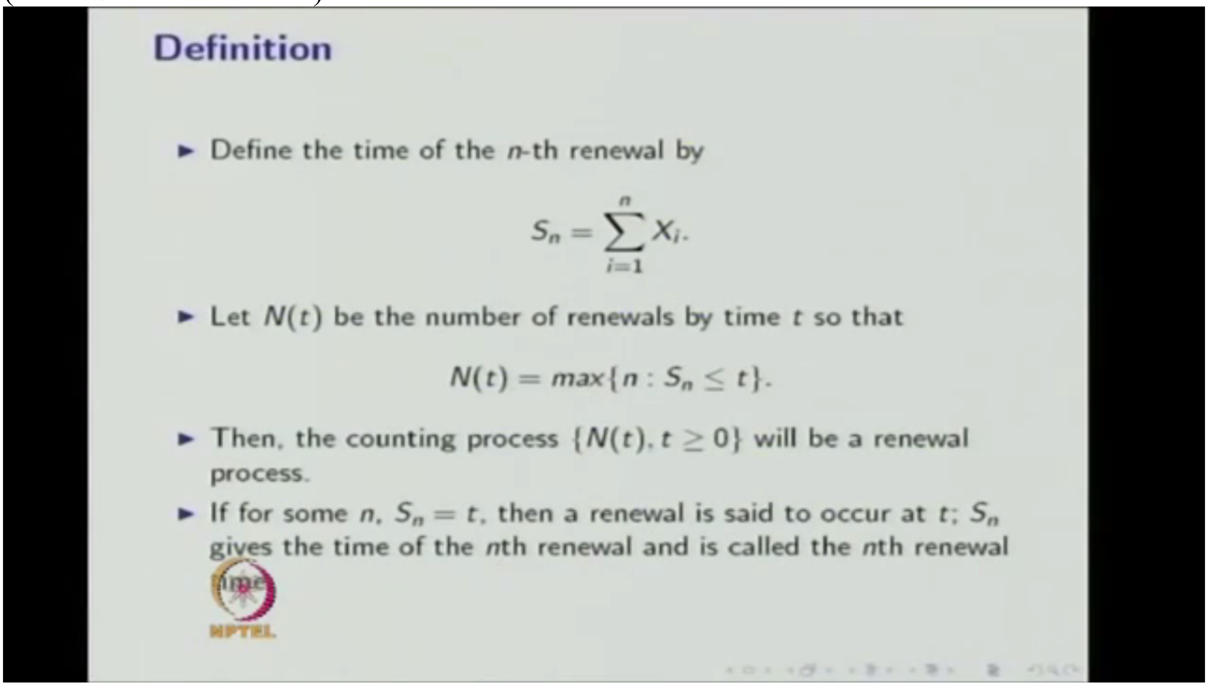

Define the time of n-th renewal by S_N is summation of first n, first X_i random variables. The X_i's are nothing but the time interval between the $(i - 1)^{th}$ renewal and i^{th} renewal and S_n is nothing but the time of n^{th} renewal that is denoted by summation i is equal to 1 to n, X_i. So for every n, you will get a one random variable S_n. So S_1, S_2, S_3 and so on. So this is the time of n^{th} renewal. That's a collection of random variables.

The another random variable let N(t) be the number of renewals by time t so that N(t) is the maximum of n such that S_n is less than or equal to t. That is nothing but how many renewals takes place on or before time t, that will be the random variable N(t).

So one can relate N(t) with S_n with this formula maximum of n such that S_n is less than or equal to t. So the N(t) is nothing but counting how many arrivals, how many renewals takes place till time t. Therefore, this is a counting process and this counting process will be a renewal process.

If for some n, S_n is equal to t, that means a renewal is said to occur at time t or S_n gives the time of n^{th} renewal and is called the n^{th} renewal time. S_n is equal to t, that is nothing but the n^{th} renewal time.

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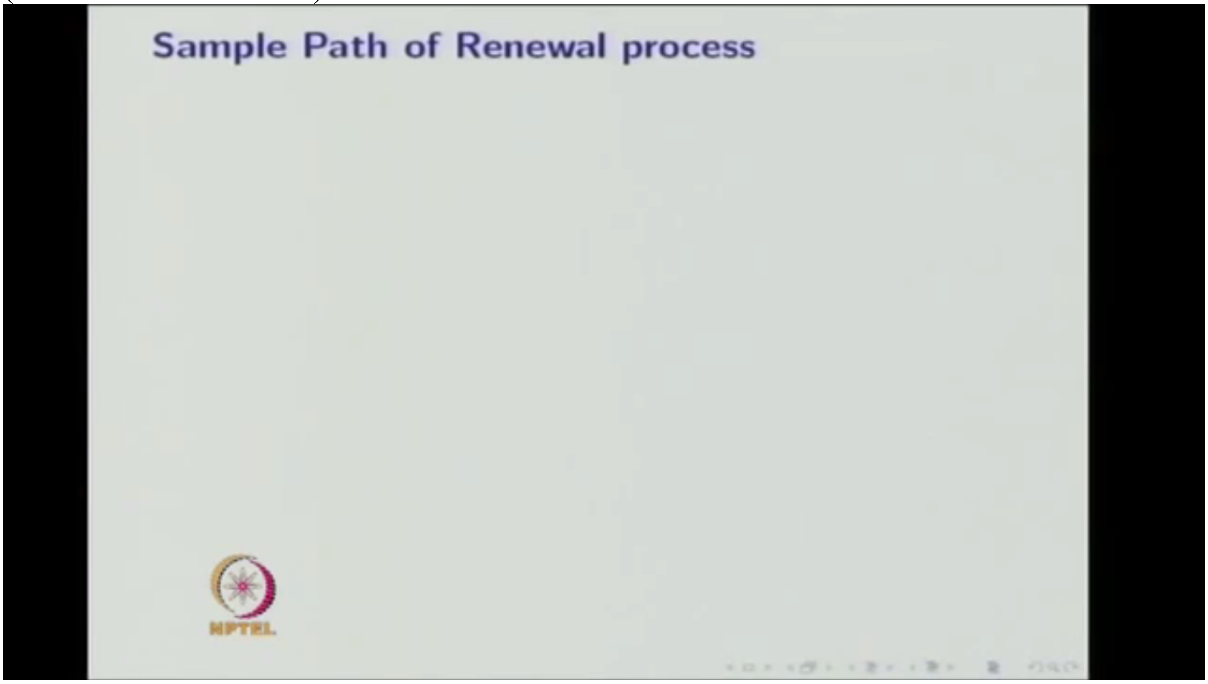

Let's see the sample path of a renewal process.

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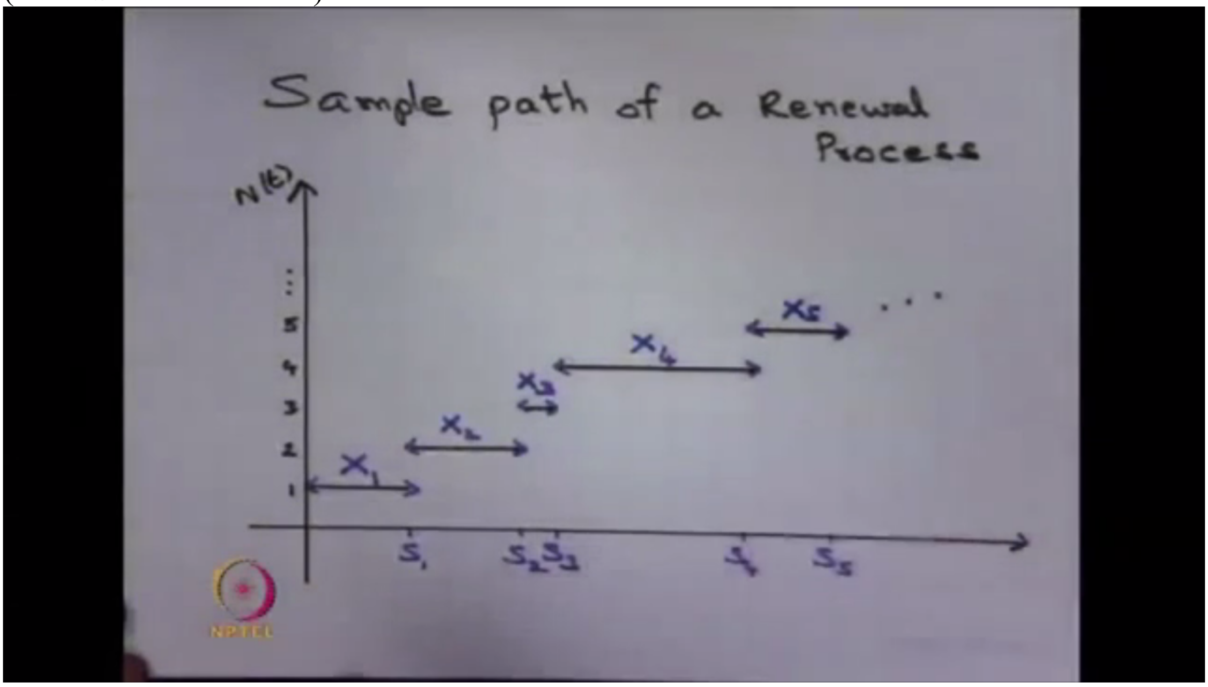

In this sample path of renewal process, the x-axis as the renewal time points, renewal time S_1 is the first renewal, the time in which the first renewal takes place. S_2 is the time in which the second renewal takes place. S_3 is the time in which the second renewal takes place.

The X_1 is the time between the first renewal and the second renewal. The X_2 is the time between second renewal and the first renewal and X_i is nothing but the IID random variables with the distribution function capital F of X. Since it is in between -- the renewal time between the renewal time, therefore, the mean will be a positive.

So in this sample path, we are giving the different values of X_1, X_2, X_3 what is -- what are all the different time points in which the renewal takes place, that is S_1, S_2, S_3, S_4, S_5 and so on and in the y-axis, we are making a count N(t).

So till some time the renewal is only 1. After some time, the renewal is 2. So suppose you stopped it somewhere and asking how many arrivals takes place at this much time, therefore, two arrivals takes place at this much time. After crossing suppose I stopped it here, then there are four arrival takes place at this much time.

Therefore, the N(t) is equal to maximum of n such that S_n is less than or equal to n, S_n is less than or equal to t makes the possible values of N(t). So N(t) for possible values of t, that will be a stochastic process. It is a counting process and this is a renewal process also. And X_i's are the IID random variables and S_n is the time of n^{th} renewal.

A common feature is this example thus is that rather than studying a random value of a sum of fixed number of random variables, one investigates the random number of terms required in order for the sum to attain a certain deterministic value. For non-negative sum means we are lead to the part of probability theory called the renewal theory and the summation, summation process S_n is called the renewal process. So this is the connection between the random walk and the renewal process.

Now we are moving into the remarks of a renewal process.

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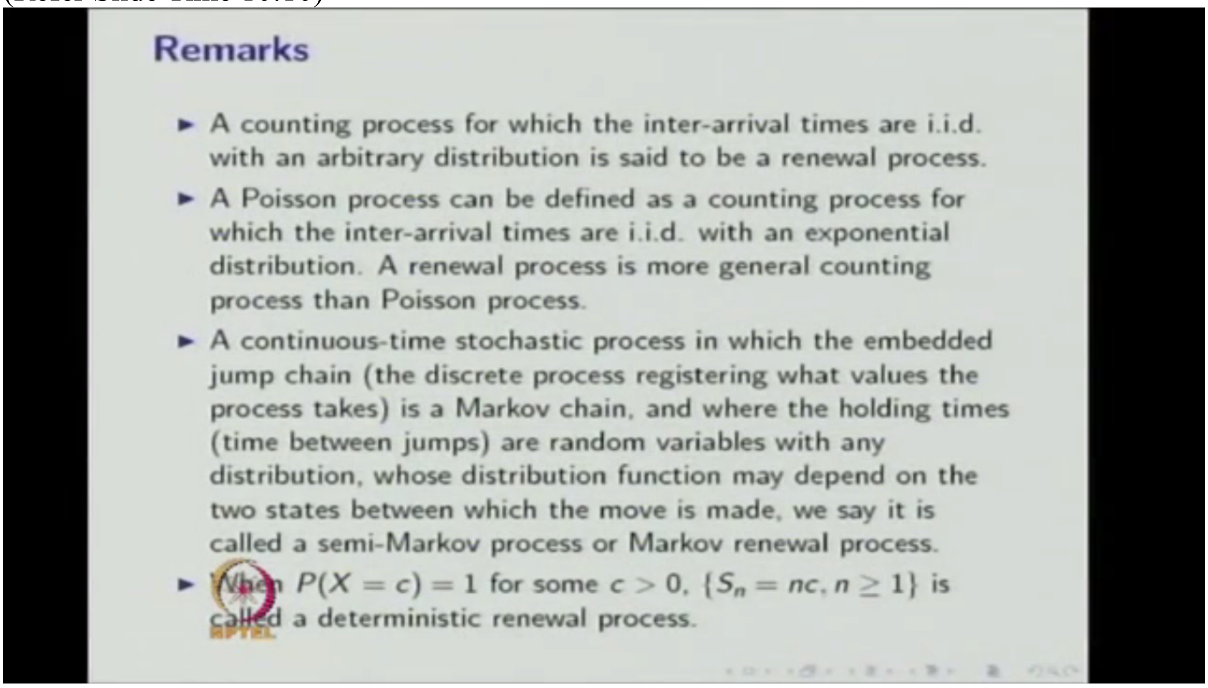

The first remark: a counting process for which the inter-arrival times are IID random variables with arbitrary distribution is said to be a renewal process. Whenever you have a counting process in which inter-arrival times are IID random variables with whatever be the distribution, with the arbitrary distribution is said to be a renewal process.

Second remark: a Poisson process can be defined as a counting process for which the inter-arrival times are IID with the exponential distribution. So the Poisson process is a special case of a renewal process in which the inter-arrival times are exponential distribution. IID as well as each has exponential distribution. Identical, independent and identically distributed random variable.

A renewal process is more general counting process than Poisson process. A renewal process is more general counting process than Poisson process because in the Poisson process, the inter-arrival times are identical, independent and each one is exponential distribution whereas in the renewal process, it could be any arbitrary distribution. Any counting process need not be particularly Poisson process.

A continuous time, third remark, a continuous time stochastic process in which the embedded jump chain that is nothing but the discrete process registering what values the process takes is a Markov chain, and where the holding times that is nothing but the time between jumps are random variables -- are random variables with any distribution, whose distribution function may depend on the two states between which the move is made, we say it is a semi-Markov process or Markov renewal process.

A more -- if this condition is satisfied by a continuous-time stochastic process, it has the embedded Markov chain and the holding times depends, it can be any distribution, but it depends on the two states between the system moves, then the corresponding stochastic process is called a Markov renewal process or semi-Markov process. We are going to discuss the Markov renewal process or semi-Markov process in Lecture 3 in detail and the application of semi-Markov processes in queueing models will be discussed in Lecture 4 and 5.

The remark number four, when the probability of X takes the value some constant c is equal to 1 for some constant c greater than 0, then the corresponding renewal process is called a deterministic renewal process.

This X, X is nothing but the inter-arrival time. For renewal process, the inter-arrival times are IID random variables. Therefore, for one random variable, we are defining it's a constant random variable. If this IID random variables are constant random variable for some constant c greater than 0, then the corresponding renewal process is called a deterministic renewal process.