Mathematical Portfolio Theory

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Module 01: Basics of Probability Theory
Lecture 01: Probability space and their properties, Random variables

Hello, viewers. Welcome to this NPTEL-MOOC course on Mathematical Portfolio Theory. In this weeks classes what we will do is that we will look at the basics of Probability Theory. So, in the first lecture today, we will do probability theory in general in both discrete and continuous time, and we will talk about random variables. This will be followed by a discussion or expectation variance, covariance and correlation coefficients and then we will talk about two important distributions, namely, the binomial distribution and the normal distribution. And we will conclude this discussion by the end of the week with a discussion on linear regression.

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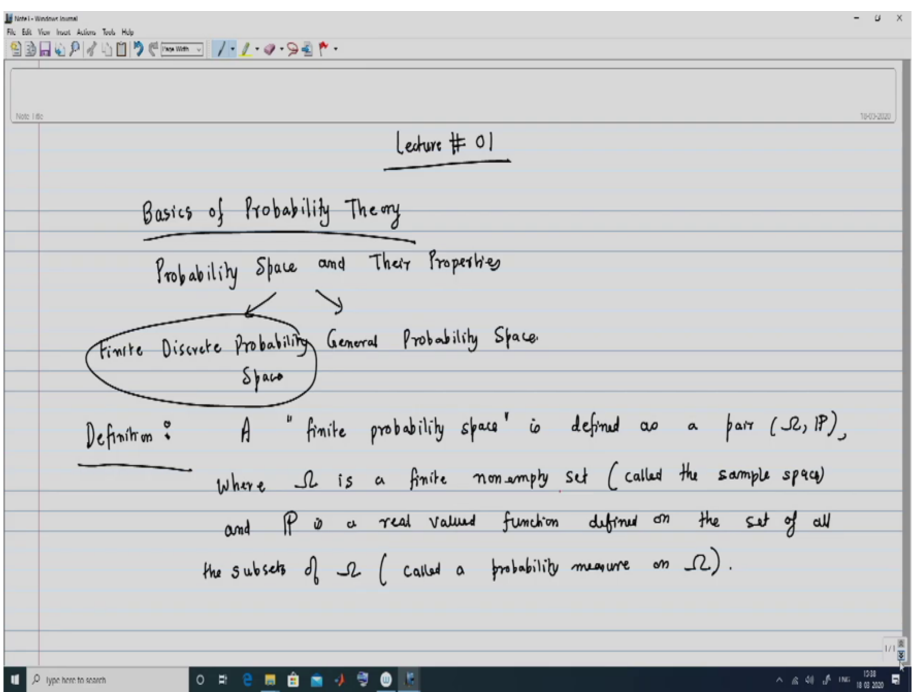
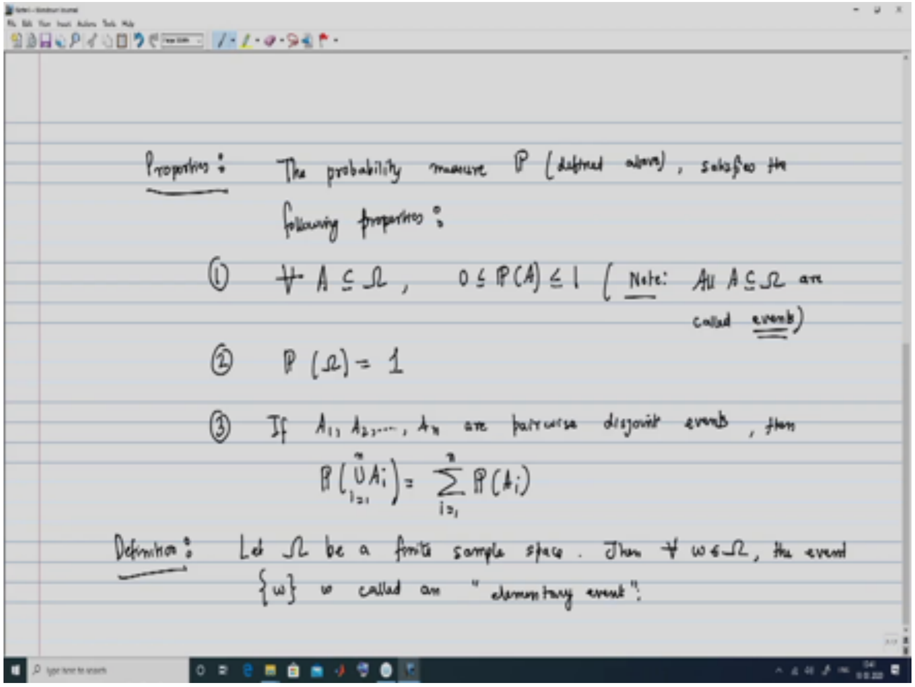


So, we start off our lecture number 1 with basics of probability theory and the first thing we will do is that we will do probability space and their properties. So, what we will do is that we will first essentially look at the finite discrete space and then we will look at a general probability space. So, finite discrete probability space. So, we start off with a definition of what is a finite probability space. So, a finite probability space is defined as a pair $(\Omega, \mathbb{P})$, where the first component omega is a finite non-empty set (called the sample space) and the second component $\mathbb{P}$ is a real valued function, which is defined on the set of all subsets of Ω and this $\mathbb{P}$ is called a probability is called a probability measure on the sample space Ω.



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So, next we will look at a some of the properties of this measure $\mathbb{P}$. So, the probability measure $\mathbb{P}$ (which is defined above) will satisfy the following three properties. So, satisfies the following properties. The first property it satisfies is that $\forall\ A \in \Omega \subset \Omega$, $0 \leq \mathbb{P}(A) \leq 1$. So, here I make a note that all such A which are subset of omega are henceforth going to be called events. The second property which is $\mathbb{P}(A) = 1$. And the last property of the probability measure is that if $A_1, A_2, \cdots, A_n$ are pairwise disjoint events, then

$$\mathbb{P}\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

So, this brings me to the next definition and this is sort of at a more elementary level than any event A, and so, I start off with that omega be a finite sample space remember we are talking about the finite probability distribution. So, let this be a finite sample space then $\forall \omega \in \Omega$; that means, each element of the sample space Ω. The event of this singleton event given by this $\{\omega\}$ is called an elementary event.

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So, accordingly, once you have defined what is an elementary event. So, accordingly, we can assign the probability (remember as an elementary event is also an event). So, we can assign a probability which we will denote by p_ω for each elementary event $\{\omega\}$ such that and obviously, it is going to satisfy $0 \leq p_\omega \leq 1$ and $\sum_{\omega \in \Omega} p_\omega = 1$. So, once we have both this things set up the probability measure and the probability of relevant event. So, therefore, we can define the probability measure $\mathbb{P}$ which you have already introduced along with the three properties. So, we will define the probability measure $\mathbb{P}$ in terms of elementary events as probability of the elementary event ω, this is p_ω that we have already introduced. Further on, I can make another statement that the probability measure $\mathbb{P}$ in terms of an event. So, let us go back to a generic event A will be the sum of probabilities of all the elementary events in A and is accordingly given by $\mathbb{P}(A) = \sum_{\omega \in A} p_\omega$, where my ω is a member of the event A. So, this brings me naturally to two more definitions. The first one is so, now, that I have defined all these probabilities for each elementary event. So, this particular set $\{p_\omega \mid \omega \in \Omega\}$. So, this set is called a probability distribution and please do not confuse this with distribution function that we will discuss in the later part of this lecture.

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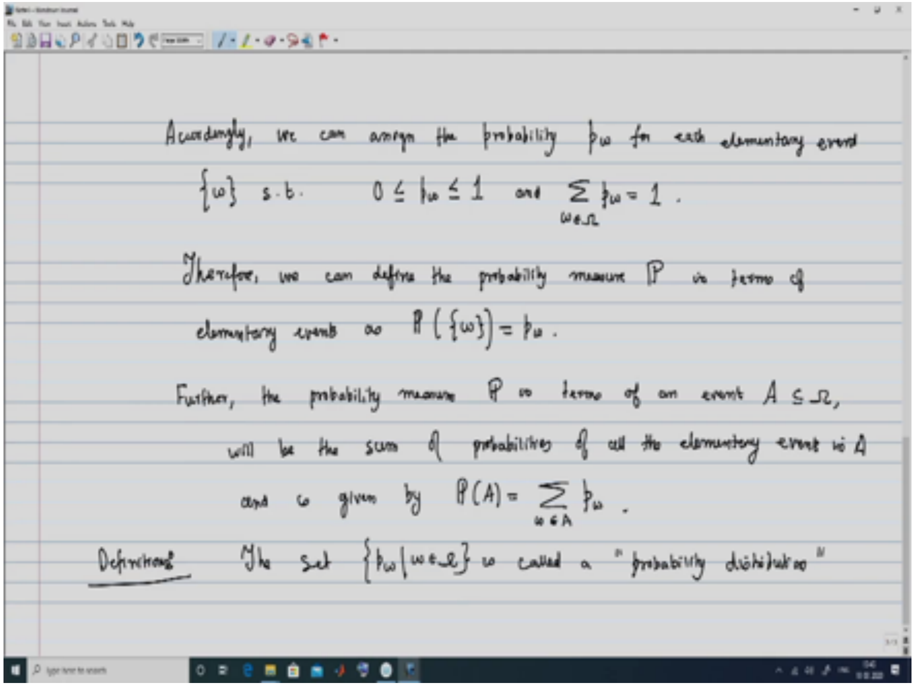
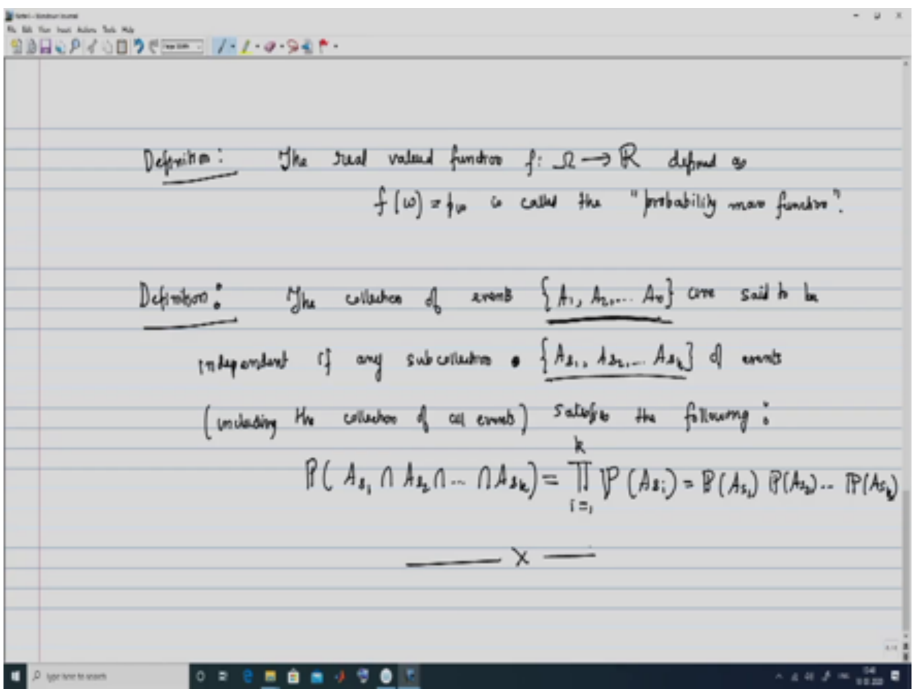
So, the other definition that we have; so, here now once I have all the p_ω, so, I am now in a position to define what is known as the probability mass function. So, the real valued function $f : \Omega \to \mathbb{R}$ defined





as $f(\omega) = p_\omega$. So, if I define this real valued function this is called the probability mass function. So, continuing with our definition, let us move on to the notion of independence of events. So, accordingly the collection of events $\{A_1, A_2, \cdots, A_n\}$ are said to be independent if any sub collection of this n events. So, if any sub collection which I will denote as $\{A_{s_1}, A_{s_2}, \cdots, A_{s_k}\}$ of events; so, that means, there are s number of a members including the possibility of course, of the collection of all events right. So, this means including the collection of these events this will satisfy the following. So, this sub collection has to satisfy the following property that

$$\mathbb{P}\left(A_{s_1} \cap A_{s_2} \cap \cdots \cap A_{s_k}\right) = \prod_{i=1}^{k} \mathbb{P}(A_{s_1})\mathbb{P}(A_{s_2}) \cdots \mathbb{P}(A_{s_k}).$$

So, once we have done with this definition of independence we move on to the definition of what is known as a random variable. So, this is a very important component from the context of the course.

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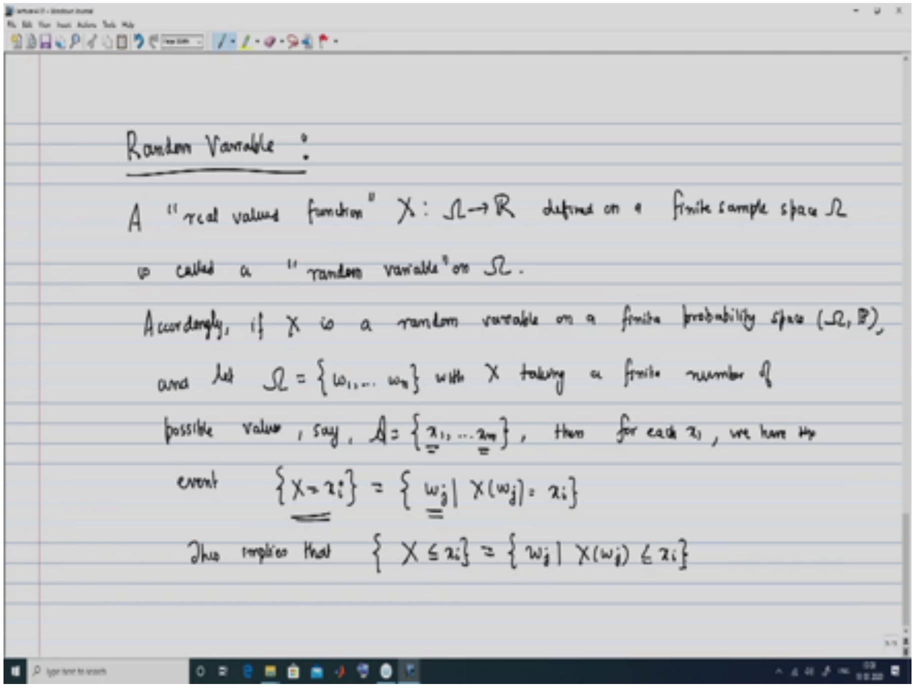

So, eventually what we will do is that we will talk about random variable. And one of the main random variable that we look at during the course is going to be the random variable representing the return of any asset which is in turn going to drive the notion of expected return of an asset. And the risk of an asset and which will then be extended to talk about what is going to be the expected return of a portfolio and what is going to be the risk of a particular asset with the return for each of the asset over you know several time intervals being considered as the random variable. So, accordingly we need to give a great amount of importance to what is going to be the definition of the random variable, both in case of the finite discrete probability space and in case of a general probability space. So, accordingly we will now start off with the notion of random variable. So, random variable is essentially a real valued function and typically we denote the random variable be X. So, a real valued function $X : \Omega \to \mathbb{R}$, that is, it is defined on a finite sample space omega is called a random variable on Ω. So, accordingly, if X is a random variable on a finite probability space which you denote by $(\Omega, \mathbb{P}$, the sample space in the process and the probability measure and let my finite sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ with X taking a finite number of possible values say $x_1, x_2, \cdots, x_m$ which we put them as a set $\mathcal{A} = \{x_1, \cdots, x_m\}$. So, remember that X is a random variable from Ω to $\mathbb{R}$. So, basically for each $\omega_1, \cdots, \omega_n$, the random variable X is going to take some value. So, that means, that the random variable X can take only a finite number of values and here we are looking at a setting where suppose that they take m number of finite values and those I will designate it by $x_1, \cdots, x_m$. So, that means, every each of those $x_1, \cdots, x_m$ is going to be equal to X of one of the ω_j's. So, accordingly so, once we have set this ω, this set A of the random variables it can take. So, then for each of the x_i's; that means, this $x_1, \cdots, x_m$, we have the event and this why I am calling it an event will become clear soon. So, this X can take any one of the values $x_1, \cdots, x_m$. So, the event that $X = x_i$ this is going to be nothing, but all those collection of ω_j such that $X(\omega_j) = x_i$. So; that means, all those ω_j's in the sample space which takes the value x_i will be bundled together and represented as the event of X equal to x_i in the chain bracket. So, this implies that that the event that $X \leq x_i$, this is going to be all those ω_j's such that $X(\omega_j) \leq x_i$, alright.

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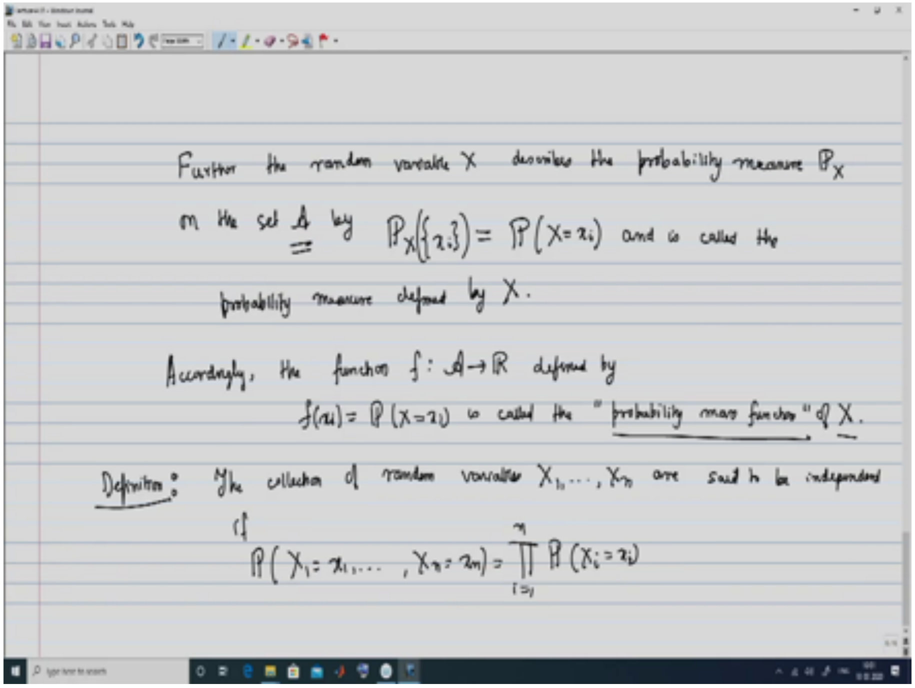
So, just to wind this up I will just say that further the random variable X this describes the probability measure. So, I am bringing the probability measure into the picture again. So, describes the probability measure $\mathcal{P}$, but the subscript X on the set A by probability subscript $X(x_i)$ (remember this is on the set



A. So, A is basically now working like some sort of a sample space). And this is going to be nothing, but probability $X = x_i$ and this is called the probability measure probability measure defined by the random variable X. So, accordingly the function $f : A \to R$, just like we had defined the f earlier to defining the probability of mass function defined by $f(x_i) = P(X = x_i)$ is called just like before this time also we are calling this as the probability mass function of the random variable X. So, here we specify that is a probability mass function for the random variable X. So, accordingly, we now have our definition, now once we have this probability measure and the probability mass function for X, so, naturally we have to talk about independence. So, the collection of random variables $X_1, \cdots, X_n$, some n number of random variables are said to be independent if

$$P(X_1 = x_1, \cdots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i).$$

So, what we have done so far is, we have looked at what is a finite probability space and we looked at what is a probability measure and we talked about random variables. And we talked about independence of events as well as the independence of event under the random variable X. So, now, we need to move on from a finite discrete probability space to a general probability space to have a more general idea with a particular emphasis on a continuous probability space.

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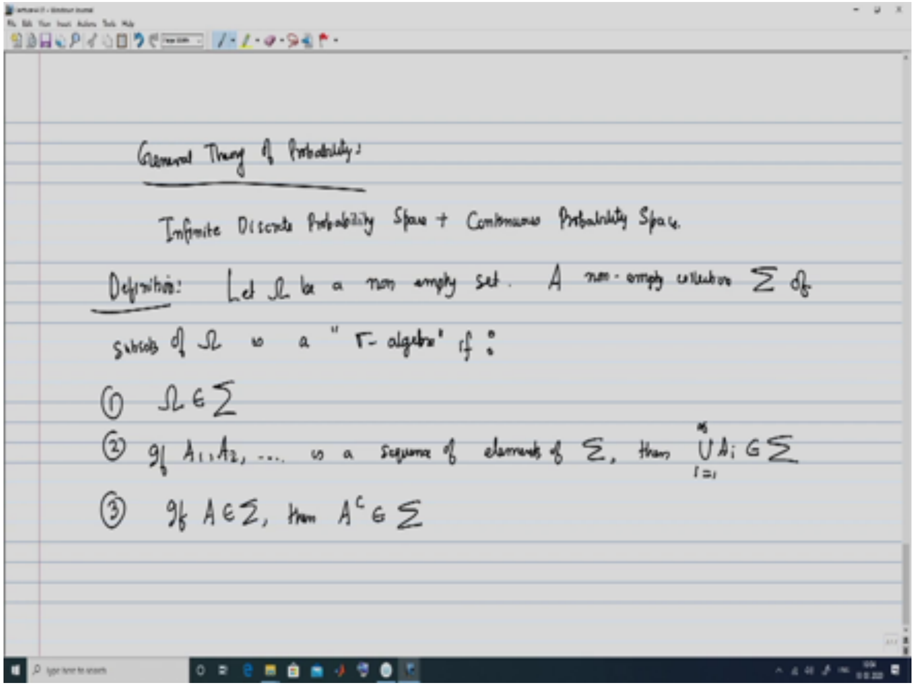
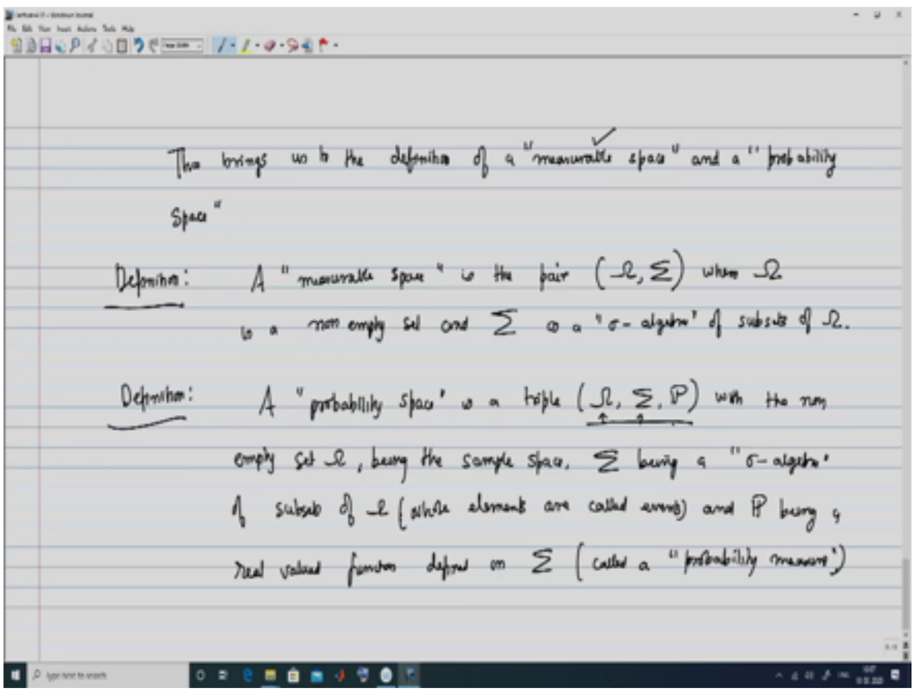
So, accordingly, we now start on the general theory of a probability and we will now move on to two things; one is the infinite discrete probability space and we will talk about continuous probability space. So, we begin with a definition. So, as before let omega be a non empty set. Now, in the previous case we had only talked about a probability space in terms of the sample space ω and the probability measure P. However, we now need to have an additional term here and accordingly we introduce what is known as the sigma algebra. So, accordingly we can write that a non-empty collection which are denote by sigma and this is the collection of subsets of the sample space ω is a σ-algebra if it satisfies the following properties. The first property is that ω belongs to this Σ. second if we have so, this is our second properties of closure under countable union. So, if $A_1, A_2, \cdots$ is a sequence of elements of Σ, then

$$\cup_{i=1}^{\infty} A_i \in \Sigma.$$

And, the 3rd property is closure under complement which says that if $A \in \Sigma$, then $A^c \in \Sigma$, alright.





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So, this brings us to the definition of a measurable space. So, we will first have to define what is the measurable space and then we will define what is the probability space just like we had done in the finite discrete case. So, first definition that is on what is a measurable space. So, a measurable space is the pair of Ω along with this Σ, where Ω is a non-empty set and Σ is a σ-algebra of subsets of Ω that you have just defined. So, this takes care of what is a measurable space and now we are in a position to talk about what is a probability space. So, a probability space is a triple. So, you recollect the earlier probability space only had Ω and P, but now we have Ω, Σ and P with the non-empty set Ω being the sample space, and Σ being a σ-algebra of subsets of Ω whose elements are called events and P being a real valued function defined on Σ and called a probability measure. So, basically the probability space is this triple (Ω, Σ, P), where my Ω is the sample space, Σ is a σ-algebra and P is a probability measure. So, now, that we have introduced what is a probability measure, so, we need to talk about properties similar to the case we had done in case of the finite probability space.

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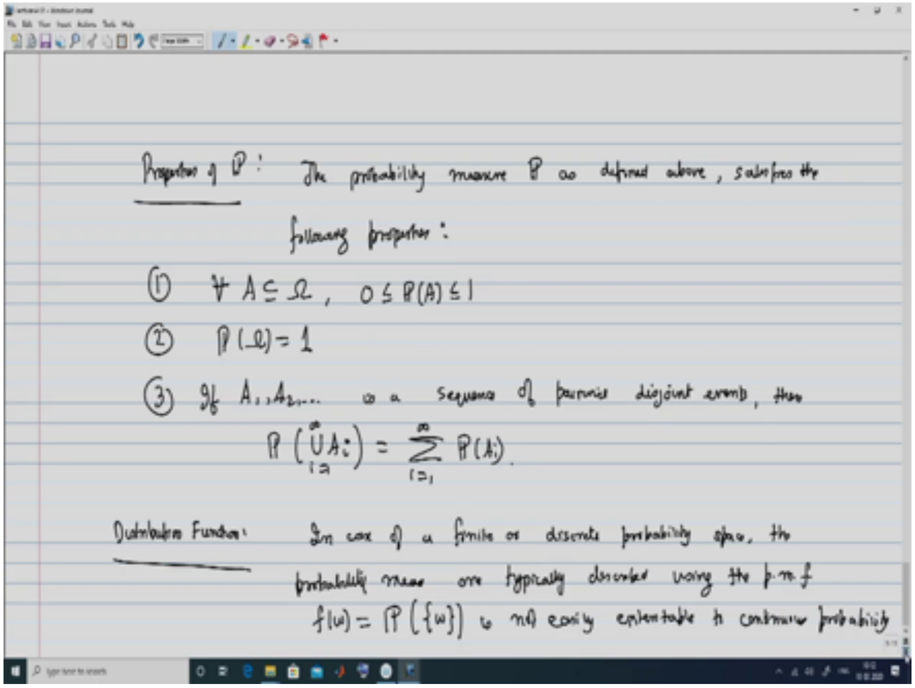

So, here accordingly, so, what are going to be the properties of P? So, now, the probability measure P as defined above in case of a general probability space. So, I have to you know specify that. So, as it is defined above this satisfies the following properties. The first of property is the range. So, as before for all A, event A in Ω. So, $0 \leq P(A) \leq 1$. So, the probability lies between 0 and 1 are both inclusive. $P(\omega) = 1$. So, these two properties are the same as before. However, in the last case since now we can have an infinite set so, we will have if $A_1, A_2, \cdots$, is a sequence of pair wise disjoint events. Then

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

So, basically now we can say that a probability space here is given as (Ω, Σ, P), where we are specified that the Ω is a non-empty set or the sample space and Σ is a σ-algebra whose properties have been enumerated the three properties and the probability measure also is has been enumerated in terms of its three properties, okay. So, let us now come to the topic of distribution function and we will first of all begin with the motivation of a why one must make use of distribution function and then we will move on to the definition of distribution function in the paradigm of a continuous probability space. So, accordingly we make the statement that distribution function, so, we just give the motivation to begin with. See, what happens is that in case of a finite or discrete a probability space, the probability measure one typically described using the probability mass function which I will denote by pmf and you would recall that this was f of little omega was probability of the elementary event omega is not easily extendable to continuous probability.

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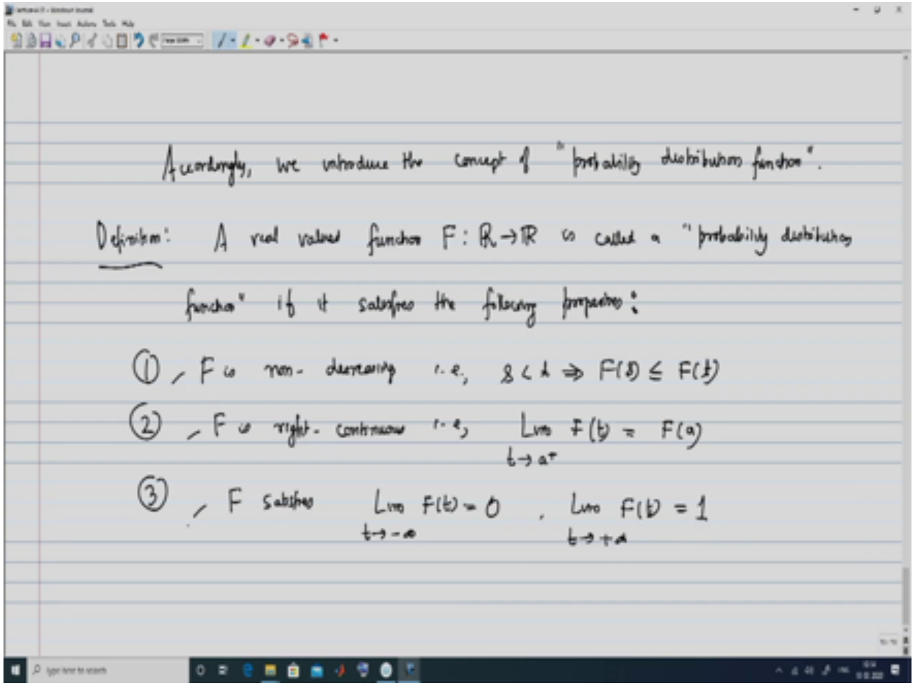
And, so, because of this reason, so, accordingly it is for this reason that we introduce the concept of probability distribution function. So, this natural brings us to the introduction of the definition. So, a real valued function $F : \mathbb{R} \to \mathbb{R}$ is called a probability distribution a function if it satisfies the following three enumerated properties. So, the first property is that F is non-decreasing. So, this means that if $s < t$, this will imply that $F(s) \leq F(t)$. The second property is that F is what is known as right-continuous. So, that is $\lim_{t \to a^+} F(t) = F(a)$, and the last property is that F satisfies $\lim_{t \to -\infty} F(t) = 0$ and $\lim_{t \to +\infty} F(t) = 1$. So, based on this definition with this three property of non-decreasing, right continuous and basically the limit as t tends to $-\infty$ and $+\infty$.



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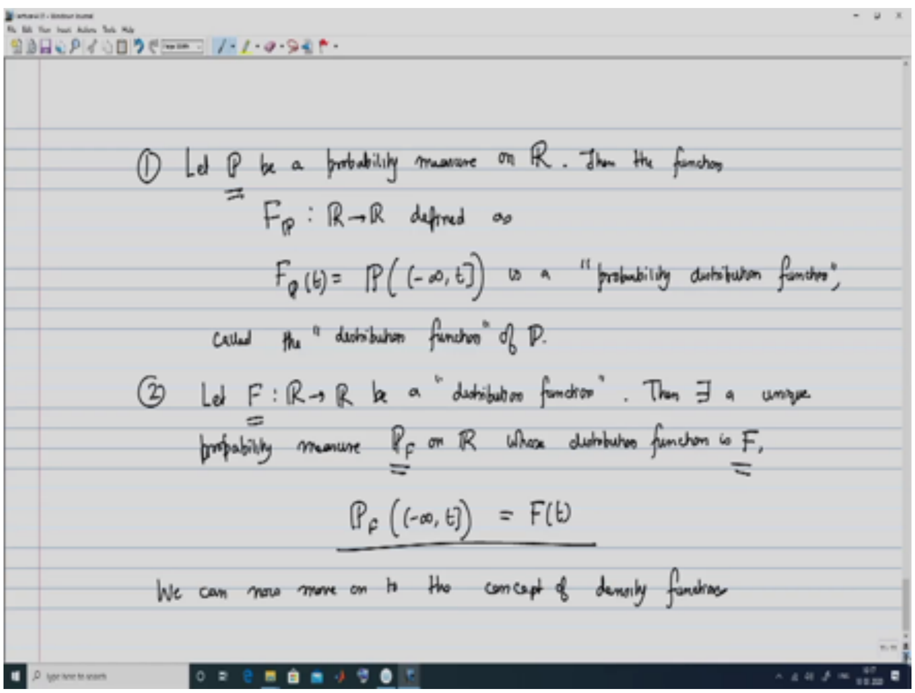

We are now in a position to state two results. So, the first result state the following that let P be a probability measure on $\mathbb{R}$. Then, the function F_P to identify with the in probability measure from $R \to R$ define. So, it is a particular function that I am defining in terms of the probability measure P. So, this function $F_P : R \to R$ defined as $F_P(t)$ is equal to; so, I am defining this as probability of the interval $(-\infty, t]$ is qualifies as a probability distribution function and is called the distribution function of P. And, the second result that I want to state is the following that let $F : R \to R$ be a distribution function as defined in part one; so, this be a distribution function then there exist a unique probability measure P_F on R whose distribution function is F. So, that is that if you are given a distribution function then there will exist a unique probability measure whose distribution function is F. So, that means, when you are given an $F(t)$ you can find a probability measure P_F such that $P_F((-\infty, t]) = F(t)$. So, the next thing is that we can now move on to the concept of a density functions.

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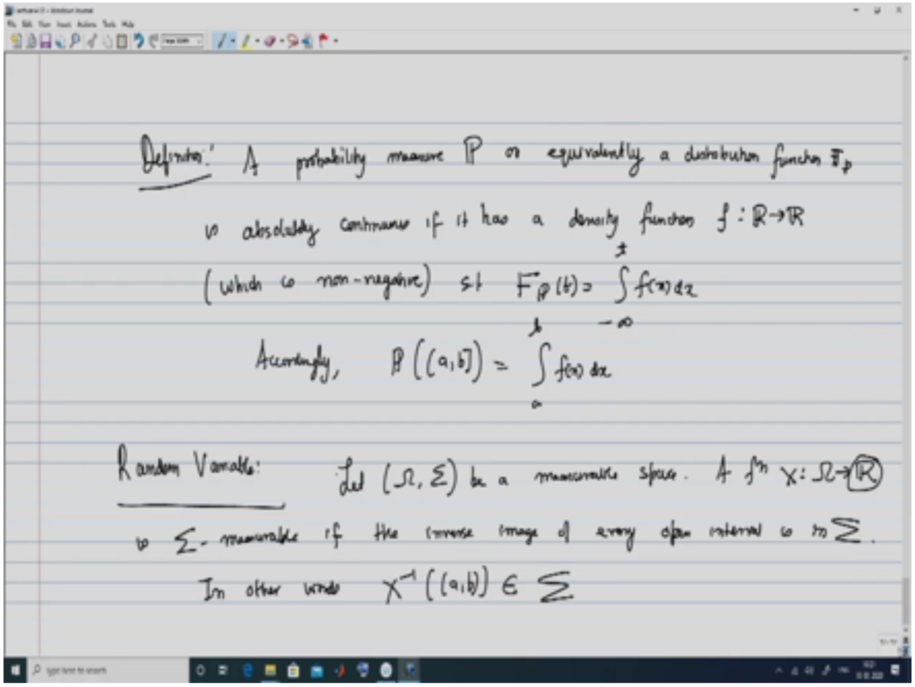

So, we have the definition is the following. A probability measure P or equivalently in light of the results 1 and 2, I can talk about a measure P and equivalently a distribution function F_P is absolutely continuous if it has a density function $f : R \to R$ which is non-negative to confirm with the non negativity of probability. So, which is not negative such that $F_P(t) = \int_{-\infty}^{t} f(x)dx$ and so, accordingly $P([a, b]) = \int_a^b f(x)dx$, alright. So, now, that you have defined what is the distribution function and what is the probability density function. So, we are now in a position to start talking about what is the random variable in the context of a continuous distribution. So, accordingly we revisit random variable in this setup now. So, accordingly so, I will first talk about so, let (Ω, Σ) be a measurable space, alright. So, a function $X : \Omega \to R$ that is a real valued function defined on the sample space R is said to be σ measurable if the inverse image of every open interval is in Σ. Remember that σ was a collection of subsets of Ω. So, what I am saying is that if it turns out that there are a function $X : \Omega \to R$ is said to be σ measurable if the inverse image of every open interval. So, any open interval in R from here if the inverse image of that belongs to Σ, then we say that this function X is σ measurable. So, in other words, $X^{-1}((a, b)) \in \Sigma$, okay.

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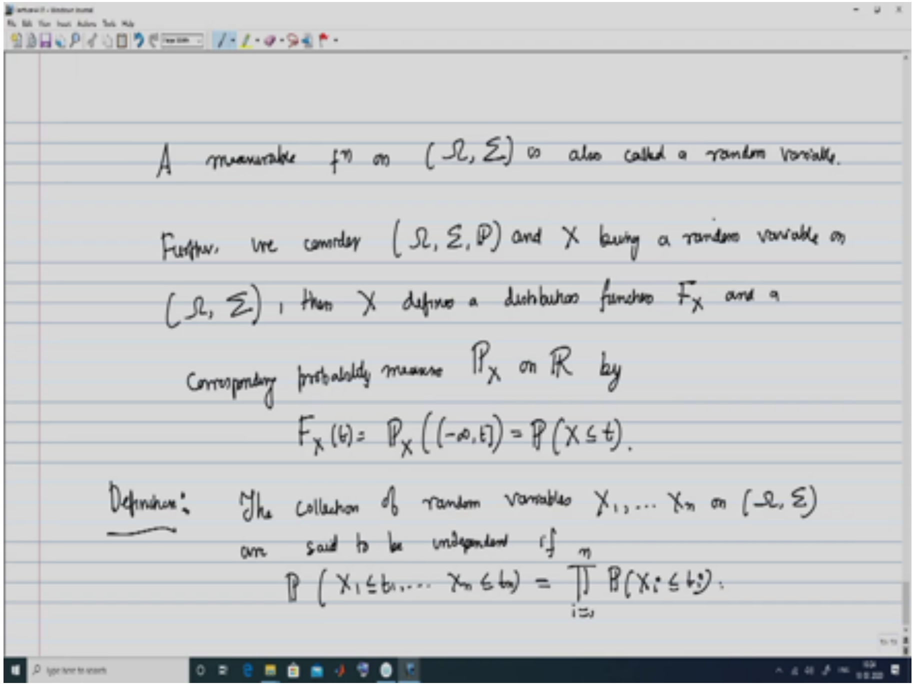
So, a measurable function; so, in our a measurable function on (Ω, Σ) is also called a random variable. So, that is your definition of random variable in the continuous time set up. Further, we consider the probability space (Ω, Σ, P) and X being a random variable on (Ω, Σ), the measurable space (Ω, Σ). Then this random variable X defines a distribution function, which will denote by capital F_X, and a corresponding probability measure P_X on R just like we had done in case of the finite case, and I will denote this by $F_X(t) = P_X((-\infty, t]) = P(X \leq t)$. So, then it brings us to the definition of what is independence of random variables. So, accordingly we can now talk about the collection of random variables, and in this case we talk about random variables in the continuous time. So, the collection of random variables $X_1, \cdots, X_n$ defined on the measurable space (Ω, Σ) are said to be independent if

$$P(X_1 \leq t_1, \cdots, X_n \leq t_n) = \prod_{i=1}^{n} P(X_i \leq t_i).$$

So, just to sum up what we have discuss today we talked about probability space, we talked about a finite discrete probability space, and then we extended this to general probability space, we talked about what is



the probability mass function, probability density function at the distribution function. And, we discussed the definition of the random variables in both the set ups along with that definition of independence in both the discrete and the continuous time set up. In the next class, we will talk about the moments in the probability space framework and namely, we will talk about the first two moments, the expectation and the variance and we will talk about covariance and correlation, coefficients and we will discuss some of the other properties pertaining to them.

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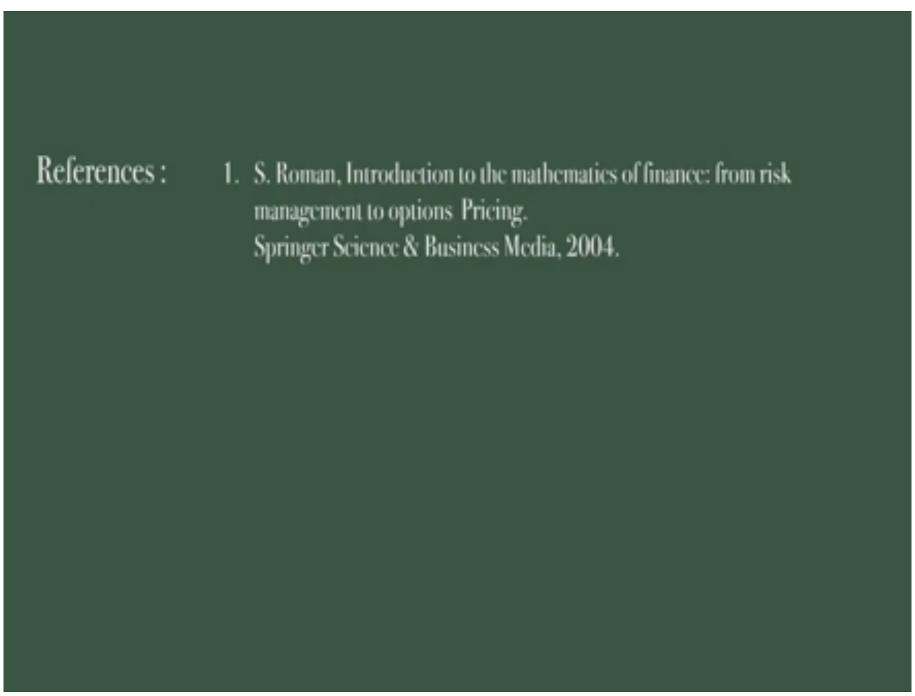

Thank you for watching.