Mathematical Portfolio Theory
Module - 03: Mean-Variance Portfolio Theory
Lecture 10: Capital Market Line and Derivation of efficient frontier

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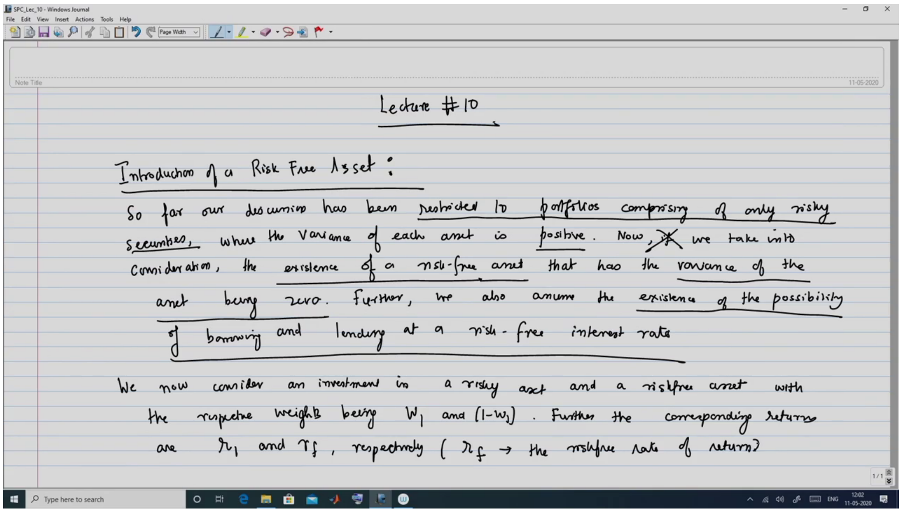
Hello viewers, welcome to this next lecture on this MOOC course on Mathematical Portfolio Theory. We would recall that so far, we have talked about the portfolio optimization problem from the perspective of looking at a portfolio which only involves risky assets. And we started off with an example of two risky asset portfolio, extended to the case of three assets and finally, to n number of assets, and we looked at how we are basically going to determine what is the minimum variance portfolio. And also we talked about opportunity set as well as the efficient frontier. And we looked at this two fund theorem, which basically said that any portfolio in the efficient frontier can be obtained as a combination of two portfolios on the efficient frontier. So, that means, that any portfolio on the efficient frontier is equivalent to a combination of two portfolios in the efficient frontier itself and we determine the weights accordingly. So, now, what you are going to do is that, we are now going to include the risk free assets. So, earlier when we talked about risky assets, we primarily looked at stocks, but now we want to look at what happens when in addition to n number of risky assets, we will also entertain the possibility that we will include in addition to this risky assets we will also include a risk free asset in our portfolio. So, accordingly we begin this lecture with this narrative on inclusion of a risk free asset. (Refer Slide Time: 02:03)



So, we start off with our introduction of a risk free asset. So, far as you will make the following observation, that so far, our discussion has been restricted to portfolios comprising of only risky securities, where the variance of each asset is positive that means, my sigma square is greater than 0. Now, if we take into consideration the existence of a risk free asset that has the variance of the asset that means, the risk free asset being zero ok. So, then we have the following assertion. So, I can say that further, we also assume there is the existence of the possibility of borrowing and lending at a risk free interest rate. So, just to sum up what we have done. We have done so far, that are to a restricted to portfolios

comprising of only risky securities and the variance was positive and now we take into consideration, the existence of a risk free asset and this risk free asset obviously, is going to have a variance being zero. And finally, we make the assumption of the existence of the possibility of borrowing and lending at a risk free rate. So, this means that, when you are considering a risk free asset, this basically means that there is no randomness in terms of its return. So, this means that the return as a random variable is just a constant in which case by definition the variance is going to be equal to 0. So, we first take into consideration the existence of such a risk free asset and furthermore we take into consideration the possibility that a borrowing and landing can take place that is, in the context of bonds can take place at this particular risk free rate that we have identified, alright. So, we now consider so, the next thing you do is, we now consider. So, once we have this risk free asset in the picture, we consider an investment in a risky asset and a risk free asset with the respective weights being W_1 and $(1-W_1)$. Further the corresponding returns are r_1 and r_f respectively. So, here we use r_f to denote the risk free rate of return.
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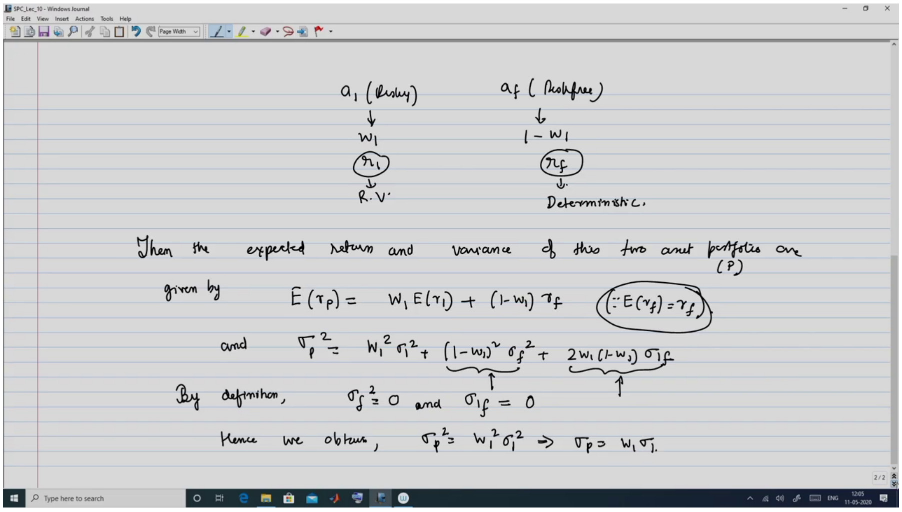

So, essentially this means that I will have two assets so that, there is an asset say a_1 which is risky and say there is an asset a_f which is risk free and the corresponding weights are W_1 and $1-W_1$, and the rate of return is r_1 and $r_f$and note that here r_1, this is going to be a random variable, but this (r_f) is going to be deterministic or constant ok.

So, now you have this portfolio of two assets. So, what you can do now is the following that, we look at the expected return and risk of this portfolio. So, then the expected return and variance of this two asset portfolio are given by:

$$E(R_p) = W_1 E(R_1) + (1-W_1) r_f$$

$$\sigma_p^2 = W_1^2 \sigma_1^2 + (1-W_1)^2 \sigma_f^2 + 2W_2(1-W_1)\sigma_{1f}$$

As, r_f is a non-random quantity, $\sigma_f = 0$ and $\sigma_{1f} = 0$. Hence, we have $\sigma_p = W_1 \sigma_1$
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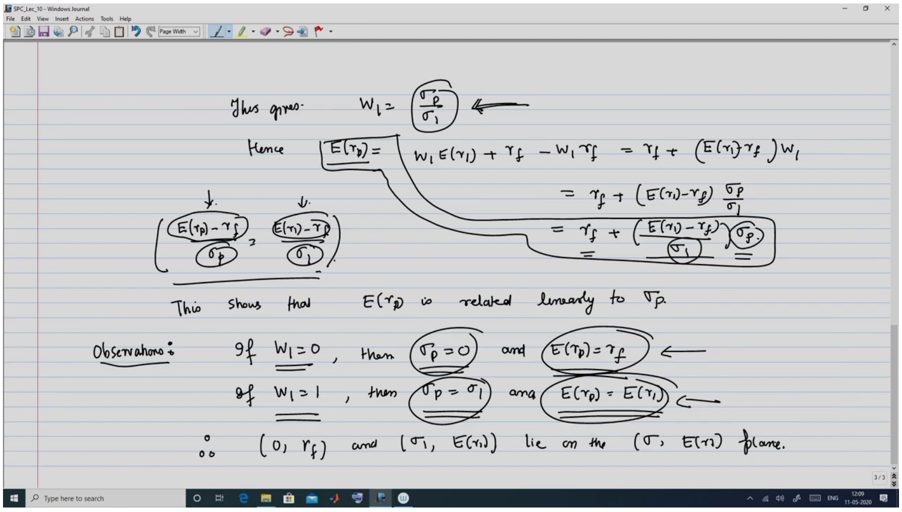
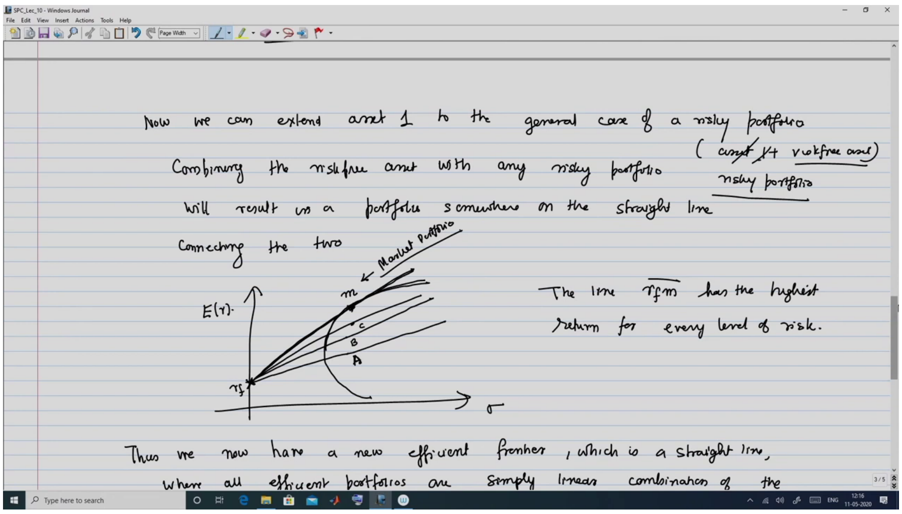


Now we get, $W_1 = \frac{\sigma_P}{\sigma_1}$, which further gives

$$E(r_p) = r_f + (\frac{E(r_1) - r_f}{\sigma_1})\sigma_p$$

So, this basically this means that is the excess return over the risk on the right hand side, we have this excess return over risk for the asset and the on the left hand side we have access return over risk for the portfolio.

So, this is just a way of remembering however, we will essentially use this form of the relation. So, the immediate fall out of this is the following that this shows that $E(r_P)$ is related linearly to σ_P. Now, a couple of observations for the extreme cases are depicted above. (Refer Slide Time: 14:31)



So, next what you need to do is that, we can extend asset 1 to the general case of a risky portfolio. So, combining, so, this basically means that instead of asset 1 plus risk free asset, the portfolio now comprise of instead of an asset it is just going to be the risky portfolio plus the risk free asset. So, combining the risk free asset with any risky portfolio will obviously, in a result in a portfolio somewhere on the straight line connecting the two.

So, this means what? So, this see earlier here what we had? We had this relation it basically the portfolio P will essentially lies somewhere between this on the line connecting this which is the investment is only the risk free asset and this which means that, it is an investment completely in the risky asset so that means, that this new portfolio now instead of just considering the risky asset so, just like this statement I had made that $(0, r_f)$ and $(\sigma_1, E(r_1))$ lie on the (σ, E_r) plane so, instead of $(\sigma_1, E(r_1))$ will consider some generic portfolio.

So, this means that so, earlier so, if I consider this to be the (σ, E_r) plane and this is my r_f. So, earlier I would have the risky asset somewhere here, now instead of the risky asset now I will consider risky portfolio. So, in this case the portfolio is going to be on the line. So, the portfolio as a result of a combination of a risky portfolio and a risk free asset will lie somewhere on this line. When the risky asset is given by A and of course, here this point is the risk free asset. Likewise, if I choose another portfolio B then, any portfolio combining B and the risk free asset will lie on this line. Now, accordingly; however, this is you will recall is the opportunity set for a portfolio comprising of only risky assets, but once we have been included risky risk free asset, it is basically going to be some combinations like this. And amongst those combination the one which is highest will be tangent to the efficient frontier of the portfolio of only risky assets. So, this means that earlier the efficient frontier is only this part and once you add a risky asset, the efficient frontier becomes this part. So, this line $r_f\bar{m}$ bar that means, this is part connecting this it has the highest return for every level of risk. So, this is synonymous with the definition of the efficient frontier that, for a given level of risk the efficient frontier is the portfolio that ever at that risk level or the same risk class, the portfolio and the efficient frontier is the 1 which has the highest return. And a lot of time this m we use the alphabet m because, this is what is known as the market portfolio. So, under some assumptions it can be shown that, this market portfolio actually mimics the real market. So, it is as if that this is just a replication of the overall market from where you can purchase you know if you have the possibility of choosing any number of assets in the portfolio. So, we can write now. Thus, we now have a new efficient frontier namely this straight line and where all efficient portfolios that means, portfolios which lie on the efficient frontier are simply linear combination of the risk free asset and the tangency portfolio m.

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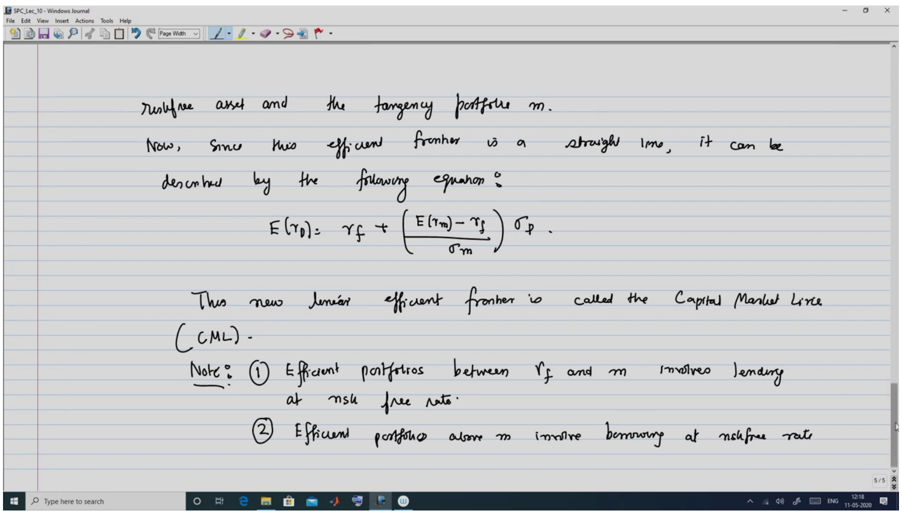

Now, since, this efficient frontier is a straight line it can be described by the following equation. And this equation will come from, this equation that I had here. So, it is going to be so, instead of the asset 1, I will have now this tangency portfolio m. So, accordingly we will get the expression

$$E(r_p) = r_f + (\frac{E(r_m) - r_f}{\sigma_m})\sigma_p$$

So, this new linear efficient frontier is what is called the Capital Market Line or CML, ok. So, now we want to make a couple of observation for this. So, the first observation is that, the efficient portfolios between r_f and m on this straight line. If this involves so, any portfolio this involves landing at risk free rate. So that means, you own a bond. So, this is the situation where $(1 - W_1)$ is positive and efficient portfolios above m involve borrowing at risk free rate.

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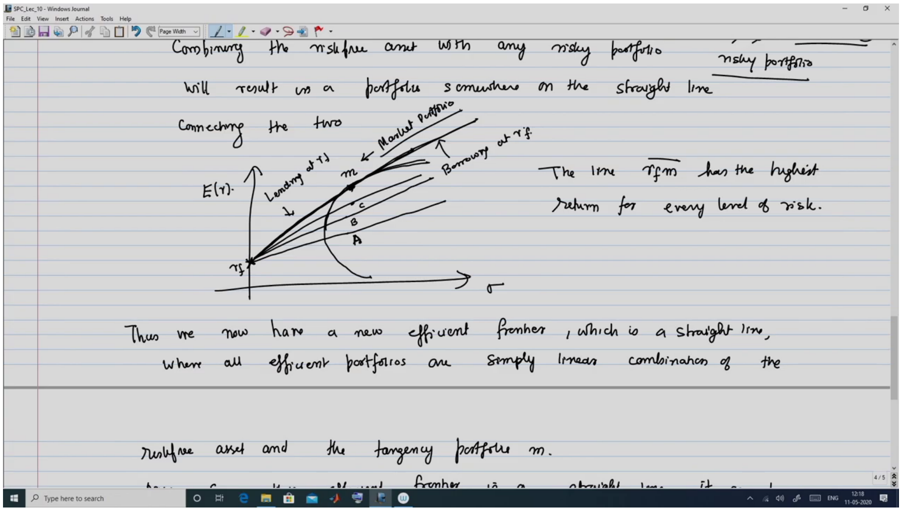

So, here this means that, in this region any portfolio which lies on this part of the line, this means that you have purchased a bond or lending at r_f and beyond this line, all portfolios this will involve borrowing at r_f.

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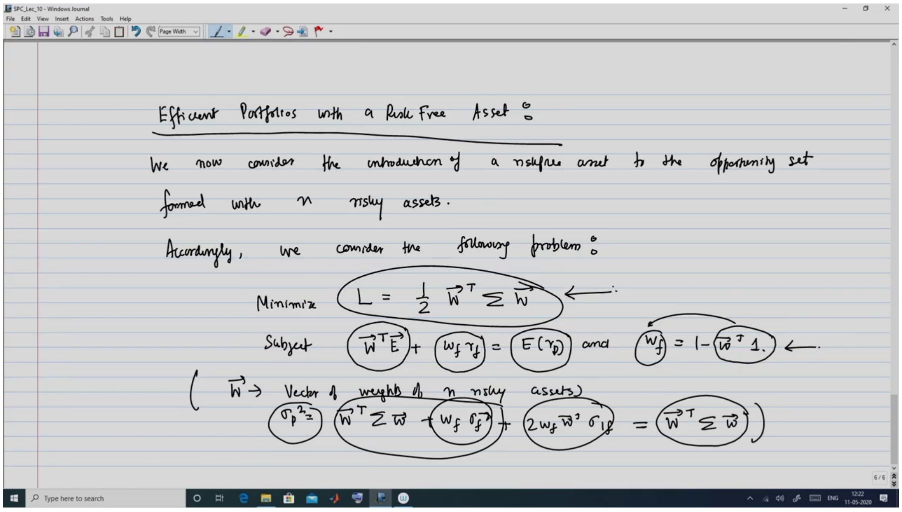

So, now let us move on to a efficient portfolios with a risk free asset. So, let us look a little deeper on the characteristic of the portfolios that lie on this new efficient portfolio which is a straight line. So, accordingly we now consider the introduction of a risk free asset to the opportunity set formed with n risky assets. Accordingly, we consider the following problem and the problem again is going to be since, we are looking at the efficient frontier. So obviously, I will consider the same problem of minimizing the variance that is an remember that we had the factor of half.

So, minimizing

$$L = (1/2)\vec{W}^T \Sigma \vec{W}$$

subject to

$$\vec{W}^T \vec{E} + w_f r_f = E(r_P)$$

and

$$w_f = I - \vec{W}^T 1$$

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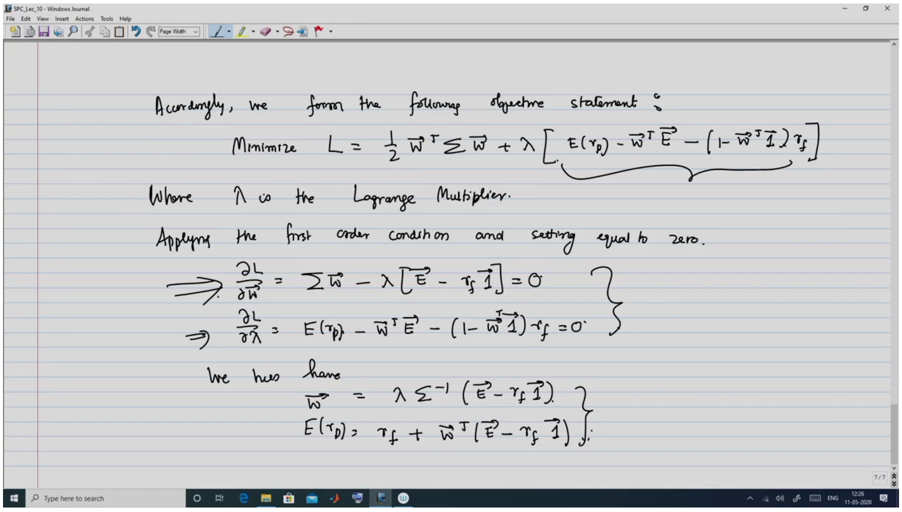

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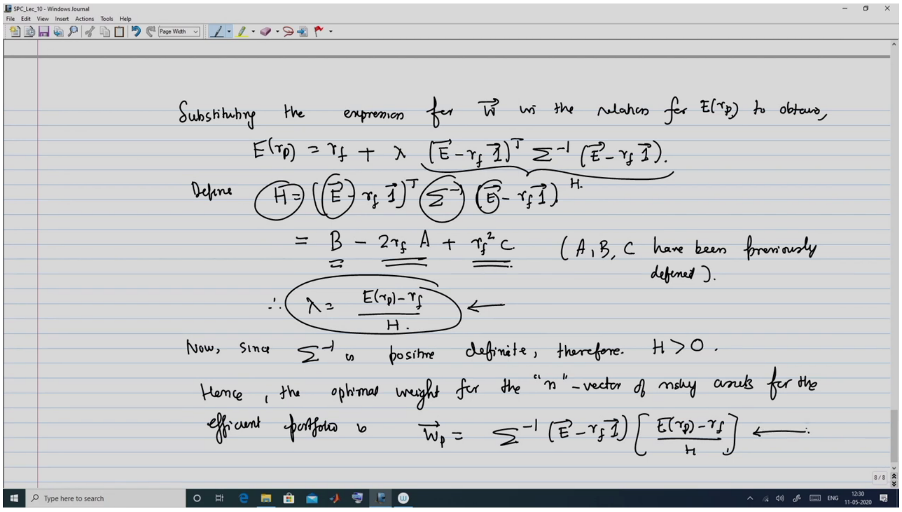

Solution of the optimization problem is given above.
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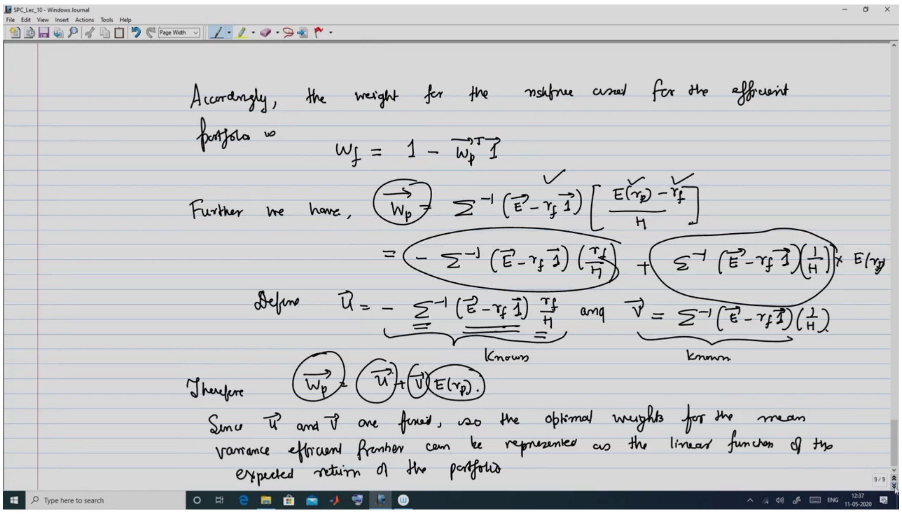

Now, we have got the optimal weights for the *n* risky assets. So obviously, accordingly, the weight for the risk free asset for the efficient portfolio is W_f.

Now, as I have already observed that, since the vectors u and v these are fixed, so the optimal weights that means, this W_p these optimal weights for the mean variance efficient frontier can be represented as the linear function that means, $u+v$ as the linear function of the expected return of the portfolio that means, linear function of $E(r_p)$.

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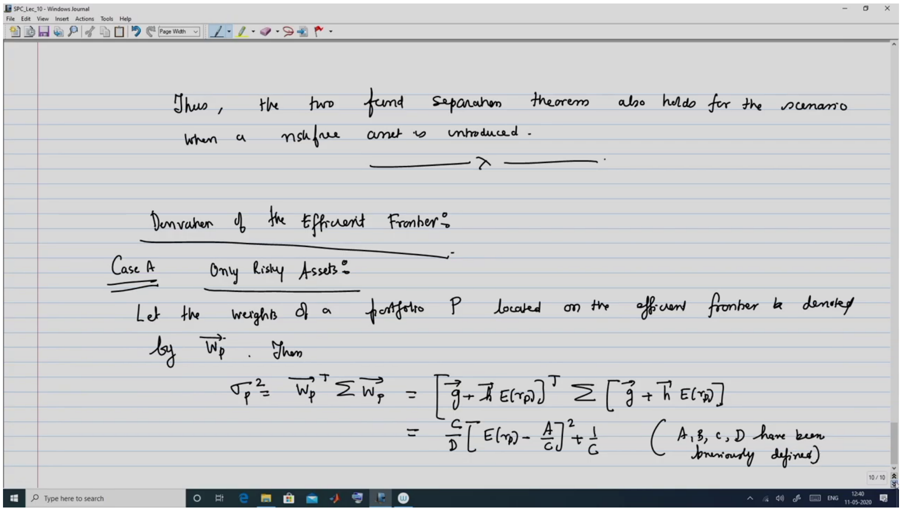

Thus, so, this immediately has the fall out. Thus, the two fund separation theorem that we had seen earlier in case of n risky asset also holds for the scenario where when a risk free asset is introduced.

So, now, we come to another topic that is derivation of the efficient frontier alright. So, we look at the two cases. So, let us first begin with the case A, where we have only risky assets, remember that this was the efficient frontier where you had the curve. So, let the weights of a portfolio P located on the efficient frontier be denoted by W_P.

Then remember, we had

$$\sigma_P^2 = \vec{W}^T \Sigma \vec{W} = \frac{C}{D}[E(r_P) - \frac{A}{C}]^2 + \frac{1}{C}$$

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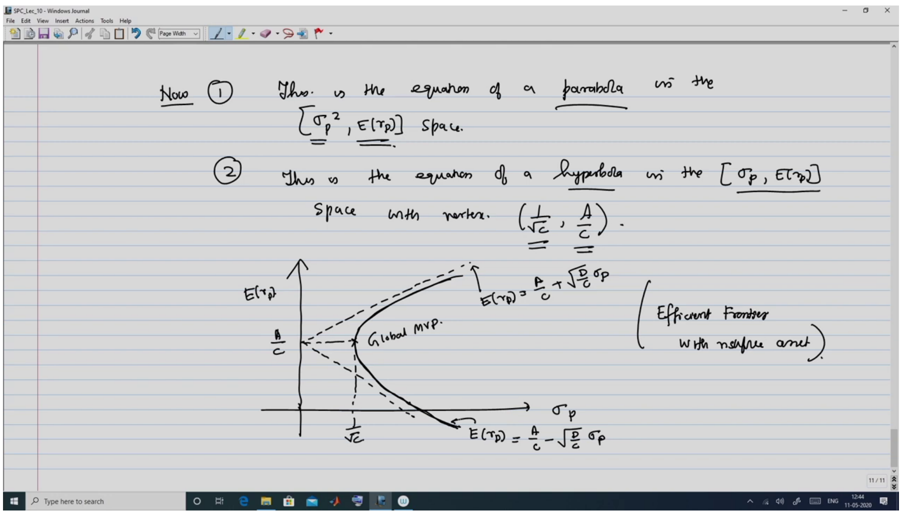

So, I leave this calculation to you. Now in this case so, now, we can make a couple of observations out of this. 1; so, here you observe carefully that I am looking at σ_P square and here I have a Er_P square because there is a square term at the top. So, this can be interpreted in two ways. So, if we so, this is the equation of a parabola in the $[\sigma_P^2, Er_P]$ space. So, this means that, if we take your σ_P^2 square to be y so, so, this is the equation of a parabola.

Remember that this equation of a parabola in the $[\sigma_P^2, Er_P]$ space and if you consider the sigma ErP space then this becomes something like x square and this becomes y minus some constant square. So, accordingly this is the equation of a hyperbola in the $[\sigma_P, Er_P]$ space. So, depending on whether you are plotting σ_P^2 square or σ_P^2 against Er_P, it will either be a parabola or it is going to be a hyperbola. So, that is how we know so, far we have been actually looking at this type of graph and this is now a formal justification of why we had taken our efficient frontier of this particular shape ok.

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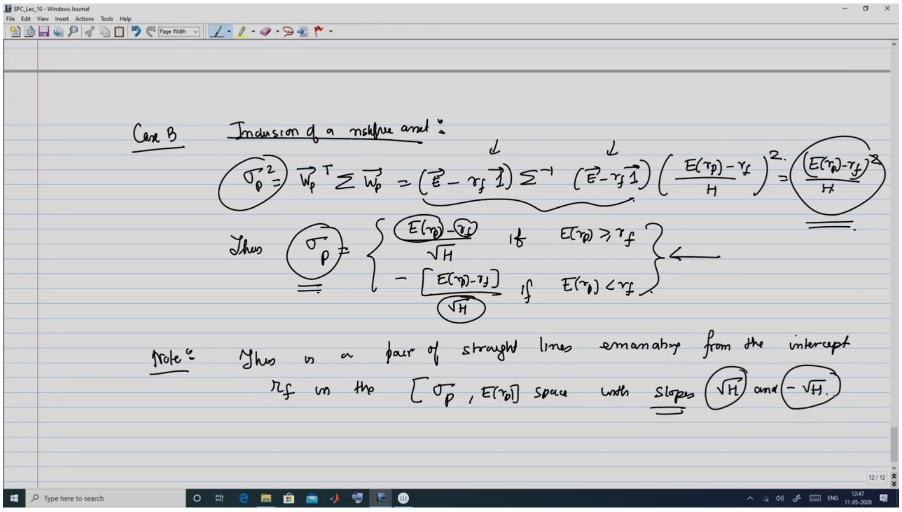

Now it is time for us to look at case B, when we can include a risky asset. So, let look at case B which based, which is on the inclusion of a risk free asset.

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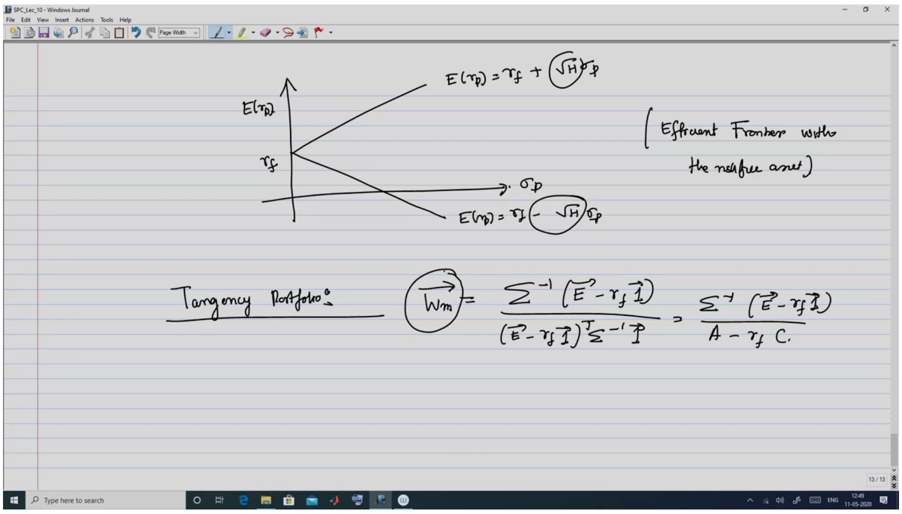

So, graphically this looks something like this. So, it looks like. So, it emanates from r_f and the equations are

$$Er_P = r_f + \sqrt{H}\sigma_P$$

$$Er_P = r_f - \sqrt{H}\sigma_P$$

So, now, we come to the first last point, which is the tangency portfolio. Remember the tangency portfolio was the portfolio designated as m. So, you can work out that the weight of this portfolio so, it is a long derivation.

So, in summary, let me just do a brief recap of whatever we have done in today's class. So, we started off looking at an extension of the previous scenario where we only had portfolio comprising of risky assets and now we have considered the possibility that we can include a risk free asset to the existing portfolio of n number of assets.

And with that, we looked at the determination of what is going to be the if weights of the efficient frontier and we showed that the two fund theorem that earlier was proved in case of a portfolio of only risky assets now, holds in case of this situation where a risk free asset is also included.

And the last topic that we looked at was, looking at the determination of the nature of the curve of the efficient frontier both in the case when its only risky assets and there we see that it is can either be a parabola or a hyperbola depending on whether you take the $[\sigma_P^2, Er_P]$ space or the $[\sigma_P^2, Er_P]$ space.

And then, in case of risky assets being combined with the risk free asset, the efficient frontier turns out to be a pair of straight lines and finally, we just noted what is going to be the tangency portfolio that we had seen up as the point of contact of the efficient frontier in case of a portfolio of all risky the assets and the as the efficient frontier with a straight line in case of a risk free asset being included.

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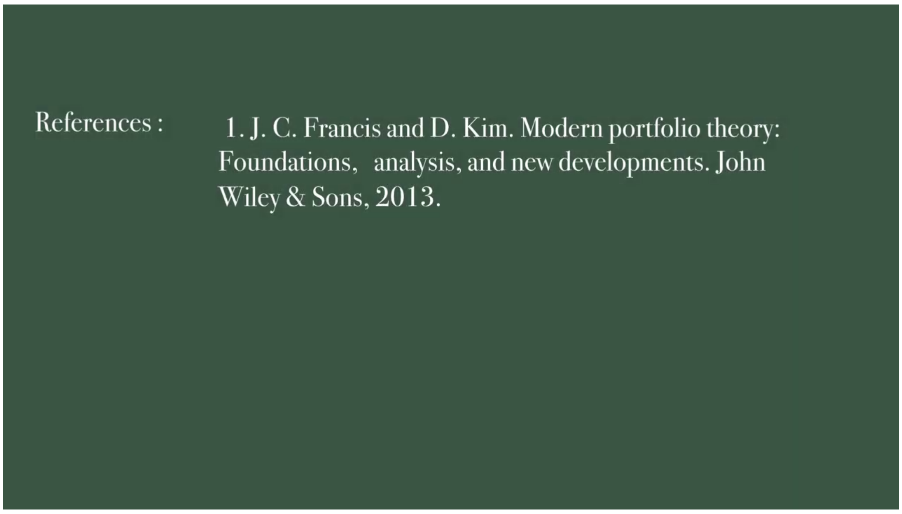

Thank you for watching.