Mathematical Portfolio Theory
Module - 03: Mean-Variance Portfolio Theory
Lecture 11: Capital Asset Pricing Model and Single index model

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Hello viewers, welcome to this next lecture on this MOOC course on Mathematical Portfolio Theory. You would recall that so far we have been looking at the mean variance framework due to Markowitz and we have looked at optimization of portfolio in terms of minimizing the variance, as an indicator of risk and then we looked at what is an opportunity set and the efficient frontier. And we made an important statement that any portfolio on the efficient frontier can be obtained as a combination of two portfolios on the efficient frontier. And I looked at the case when the portfolio comprised of only risky assets and then we extended this to the case when a portfolio comprised of a risk free asset in addition to a collection of risky assets. And in the last class, we have talked about something which is known as the capital market line. So, in this class we will go ahead and talk a little bit more in that framework and in particular; we will look at what is known as the capital asset pricing model and we will talk about what is known as the single index model, which is sort of an alternative way of modelling the return of the return of a particular asset or a portfolio in terms of the market portfolio. So accordingly, we begin our lecture with a discussion on the Capital Asset Pricing Model or CAPM.

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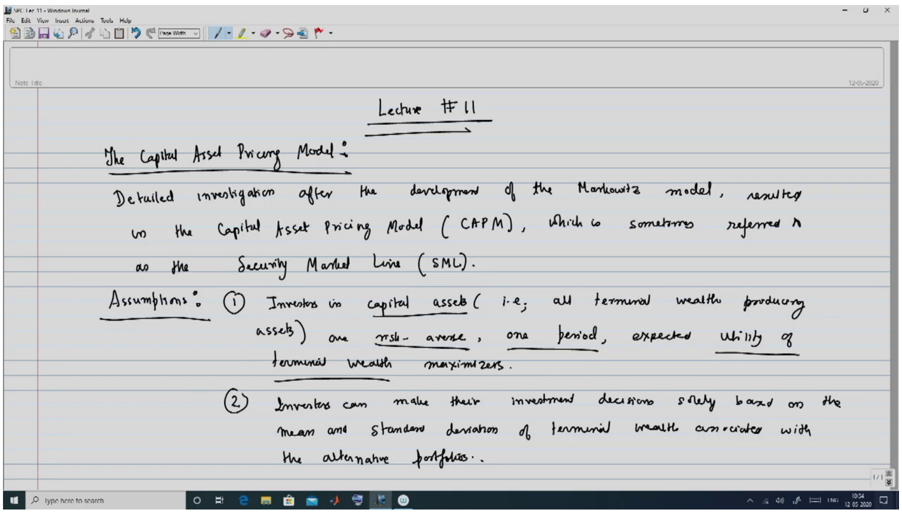

Alright, so let us begin with the brief motivation of the capital asset pricing model. So, we can say that the detailed investigation after the development of the Markowitz model. So, Markowitz model turned out to be a very important milestone in financial engineering; so, obviously there was a lot of work that was done after the Markowitz model was presented.

So, this detailed investigation that were a follow up to the establishment of the Markowitz model, this resulted in the Capital Asset Pricing Model which is commonly known in the parlance of finance as CAPM and which is sometimes referred to as; what is known as, the Security Market Line or SML, ok.

Now, this framework of CAPM is based on certain elaborate list of assumptions and we will enumerate them one by one.

So, the first assumption is that investors invest in capital assets; that is all terminal wealth producing assets; that means, you make an initial investment and you basically then make ah; a receive a final amount of payment. So, all the investors who go for capital assets are assumed to be risk averse and we will deal with this concept of risk aversion in more detail, when we talk about utility theory. And this set up is for a one period model and the expected utility of terminal wealth maximizers.

So, this is the framework that all investors in the capital assets are assumed to be driven by three consideration. First of all they are risk averse, secondly they are always looking at their portfolio strategies over a one period or that is the holding period and they are driven by; the optimization requirements of maximizing the expected utility of the terminal wealth.

So, essentially you have terminal wealth and you calculate the utility of the terminal wealth and this concept of utility will be introduced in subsequent classes and then since the utility of the terminal wealth is a random variable; so you need to take into account what is the expectation of that. And since it is all driven by since these are investors of capital assets. So, essentially they are going to be driven by efforts or a portfolio strategy to maximize the expected terminal wealth that is going to be available at the end of the holding period.

Now, second assumption is that investors can make their investment decisions solely based on the mean and standard deviation of terminal wealth associated with the alternative portfolios. So, this means that amongst the different portfolios that are available for consideration; the primary driver has already been noted is the terminal wealth and maximization of expected utility. So, this decision of maximization of expected utility of the terminal wealth; the distinction between them or a preferential setup is driven from the point of view of the investors with the underlying driver being the mean and variance. So, essentially the CAPM assumes that the entire exercise of maximizing in terms of the terminal wealth will be in the mean variance framework due to Markowitz.

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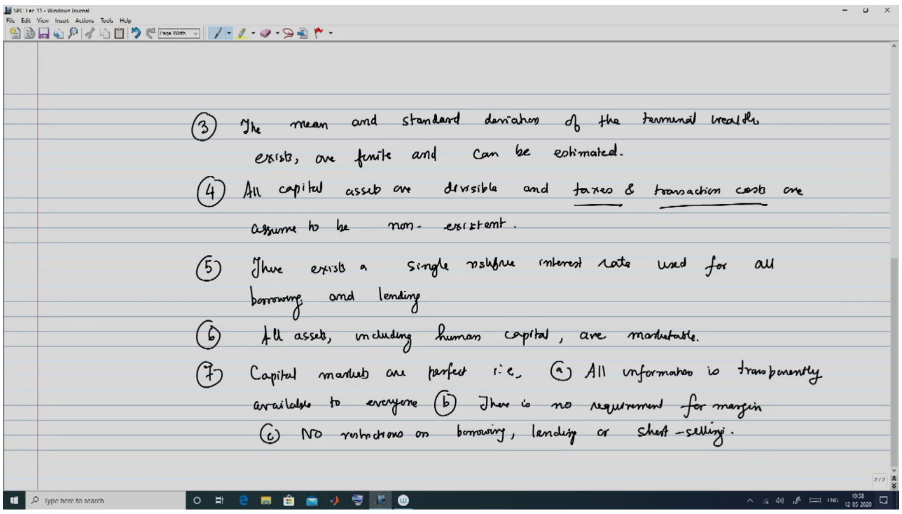

The third assumption is that the mean and standard deviation. So, since we have said that the mean and standard deviations will be the driver of the decision. So, accordingly, this mean and standard deviation of the terminal wealth this exists, are finite and can be estimated; which is the reason why you can make an investment decision based on these estimations.

The next point is all capital assets are divisible; that means, you can buy or sell a fraction of assets like stocks and bonds and all the taxes and transaction costs are assumed to be non-existent. And please remember that this assumption of non existence of taxes and transaction cost is in the context of the CAPM, the discussion of on CAPM. In reality of course, you know they have to be taken into account in which case the analysis becomes a considerably more challenging.

Fifth assumption: There exists a single risk free interest rate. It means at the risk-free rate r_f is used for all borrowing and lending.

Six, all assets including human capital are marketable. Seven, The capital markets are perfect; so this term perfect means the following: as the first thing it means is that all information is transparently available to everyone; that means, in all the investors have identical information. Then there is no requirement for margin and see there are no restrictions on either borrowing or lending or short selling.

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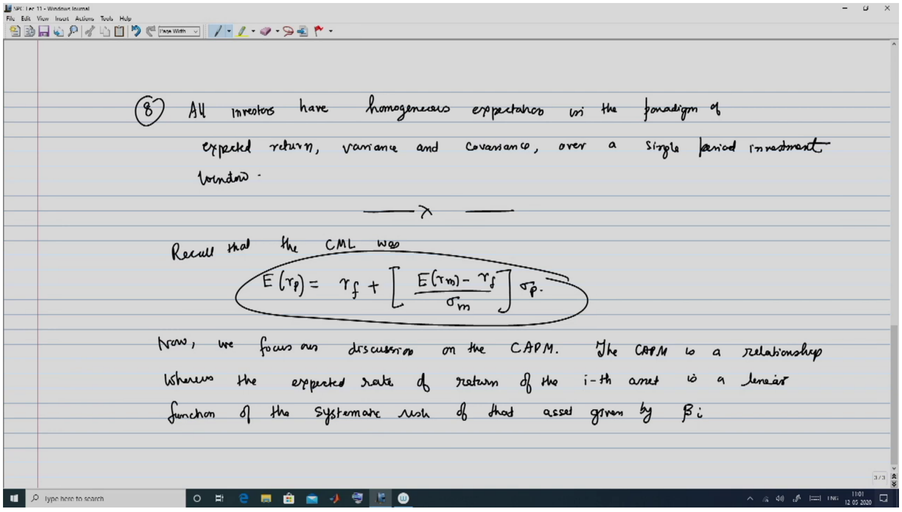

And finally, the last assumption is that all investors have homogeneous expectation in the paradigm of expected return. So that means, they calculate the expected return, variance and covariance in an identical manner over a single period investment window alright.

So now that we have a set up; the framework for the discourse on CAPM; so we can now get into the equation that is driving that. So, accordingly we will start by recalling the CML; the equation for CML that we have done in the previous class. So, recall that; the CML or the capital market line was given by

$$E(r_p) = r_f + [\frac{E(r_m) - r_f}{\sigma_m}]$$

Now, that we have recalled this; we can now focus our discussion on the CAPM. Now, what is the CAPM? The CAPM is a relationship where in the expected rate of return of the i-th asset is a linear function of the systematic risk of that asset given by β_i. So, here β_i essentially means that it is going to be the covariance of the return of the i-th asset with the return of the market portfolio divided by the variance of the market portfolio.

So, this is the same beta that we had determined in the case of the linear regression which you had done in one of our earlier lectures. So accordingly, I will just first state the CAPM or the security market line the equation for that and then we will look at its derivation and some of the interpretation of that.

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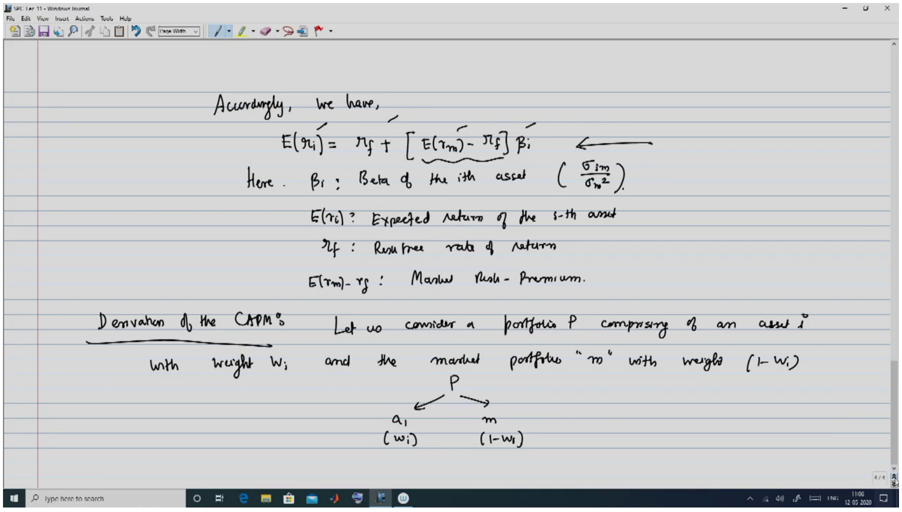

So, accordingly we have

$$E(r_i) = r_f + [E(r_m) - r_f]\beta_i$$

So, this means that it is the additional amount of money that you make over the risk free rate r_f; as a result of your decision to go ahead and invest in the market portfolio as compared to the risk free asset. So, since this is an additional benefit or potentially loss that you are going to get or incur as a result of taking up a risky position, that is the reason why it is known as the market risk premium ok. So now, once I have set up this equation; so let me now talk about the derivation of the CAPM.

So, let us consider a portfolio P and this portfolio comprises of an asset i with weight W_i and the market portfolio m with weight $1 - W_i$. So, this means the portfolio P has this asset a_i and the market portfolio with the respective words being W_i and $1 - W_i$ alright.

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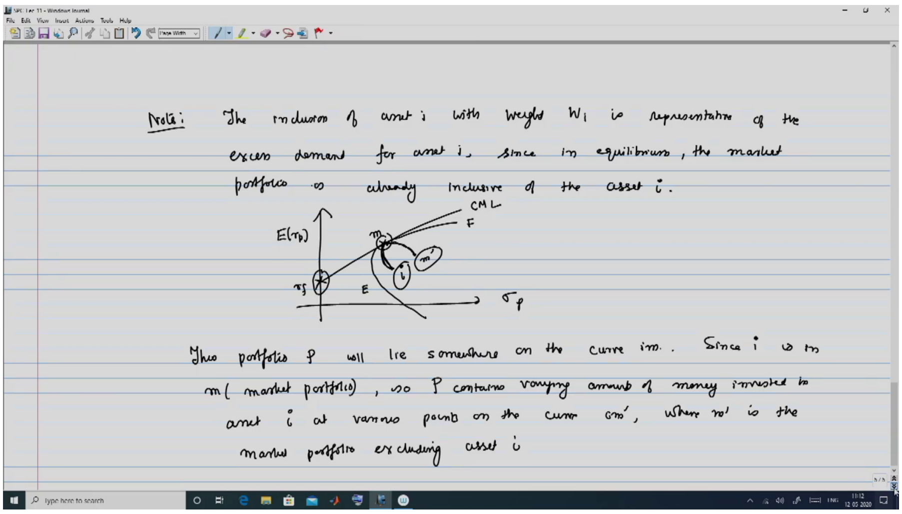

So what you can do is; so now we can make an observation here; and make a note that the inclusion of asset i with weight W_i, what does it imply? So, this implies or is representative of the excess demand for asset i and why I say that this is excess demand? That is because in equilibrium; that means, in market equilibrium; the market portfolio is already inclusive of the asset i.

So, ideally if we remember the market portfolio is the point of tangency that we have on the efficient frontier; so it is enough to actually consider the; the market portfolio as your investment. And; obviously, since the market portfolio includes all assets that are available in the market; so, obviously it also includes a certain amount of investment in the i-th asset.

Now, if instead of exclusively investing in the market portfolio; you have decided that you want to make also an investment in a particular asset; that means, that in the new portfolio P, the amount of your investment in the i-th asset is going to have a weight that is not just related to or in proportion to the weight of that asset in the market portfolio, but there will be also some additional amount of money in their particular asset. So, this means that amongst all the assets that are in the market portfolio; you have chosen to put in an additional amount of investment in just one asset; namely the i-th asset and this is because that there is an extra demand on your part to make an investment in the i-th asset ok. So, graphically this can; this is going to look something like this. So, I look at the $[\sigma_p, Er_p]$ plane and this is r_f and this is going to be my point of tangency which is the market portfolio m and this is the capital market line.

And I will take this curve to be EF to indicate that this is the efficient frontier and what I do is that I will have this curve, where I will indicate this asset i to be this point and this is some portfolio m prime which I will specify as I discuss the pattern of this curve im prime alright. So now we can say that this portfolio P which I have constructed with the i-th asset in the market portfolio will lie somewhere on the curve im. Now, since the asset i is in m; remember that m is the market portfolio; so it also always has to be inside that. So, the portfolio P contains varying amounts of money invested in asset i at various points on the curve im prime, where m prime is the market portfolio excluding asset i.

So, let me explain this; see if you observe carefully at the CML, the; the line joining r_f with m. So, when you have r_f; this basically means that you are invested exclusively in the risk free asset and this means that you are invested in the market portfolio. So, this liner_f and m; in analogous way when I to consider a portfolio P; then that portfolio is going to lie on this line.

Now, if we are invested in the i-th asset then you are here. So, if you are invested in a combination of the i-th asset in the market portfolio, then you are essentially lying on this line and here on this curve; not the line, but rather the curve and here this m prime that you have; so this i is an exclusive investment in the i-th asset and this m prime is a portfolio which is the same as the market portfolio except that it does not include the i-th asset.

So that means, that if we take the market portfolio and remove the i-th asset and proportionately increase or decrease the weights of the market portfolio that will become the new portfolio m prime which is almost identical to the market portfolio; except that it will not have the i-th asset alright. So let us now look at a bit of an analysis of this curve im prime; so accordingly we will look at what is the expected return and risk.

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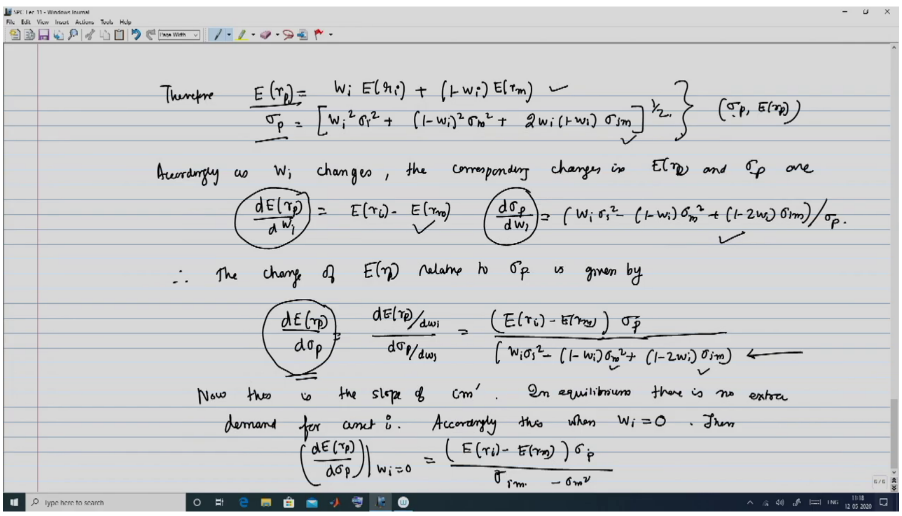

Derivation of the slope is described above.

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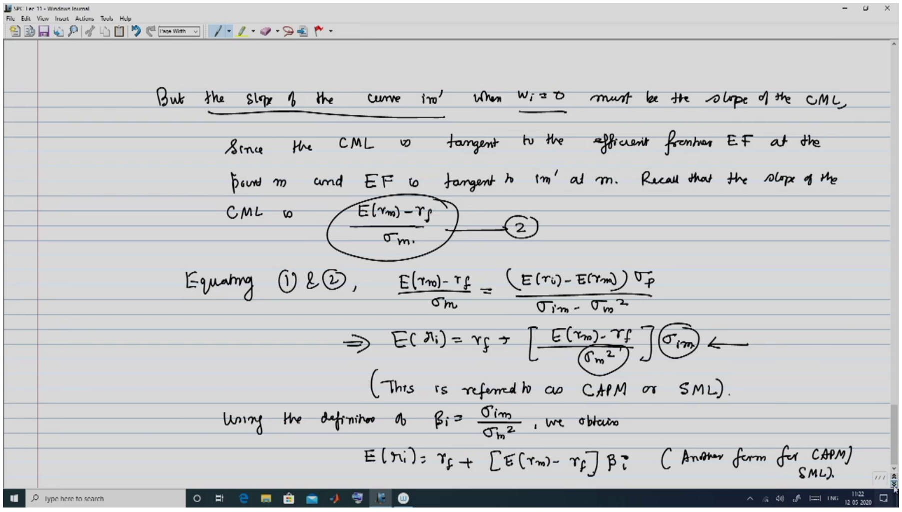

Now we make an observation that the slope of the curve im' where $W_i = 0$. So, remember that this has to be reconciled with the fact that when $W_i = 0$. Then your portfolio P does not have any extra investment in the i-th asset, so it has the same investment as basically the market portfolio. So, reconciling that with the slope being evaluated at $W_i = 0$; we can now make the following observation. And the observation is that the slope of the curve when W_i equal to 0; this must be the slope of the capital market line.

Since, the capital market line is tangent to the efficient frontier which I have identified as EF at the point m and also we recognize the fact that EF is tangent toim' at m ok. So, now that we have made this reconciliation; so let us just recall what is the slope of the capital market line? So, recall that the slope of the CML is:

$$\frac{E(r_m) - r_f}{\sigma_m}$$

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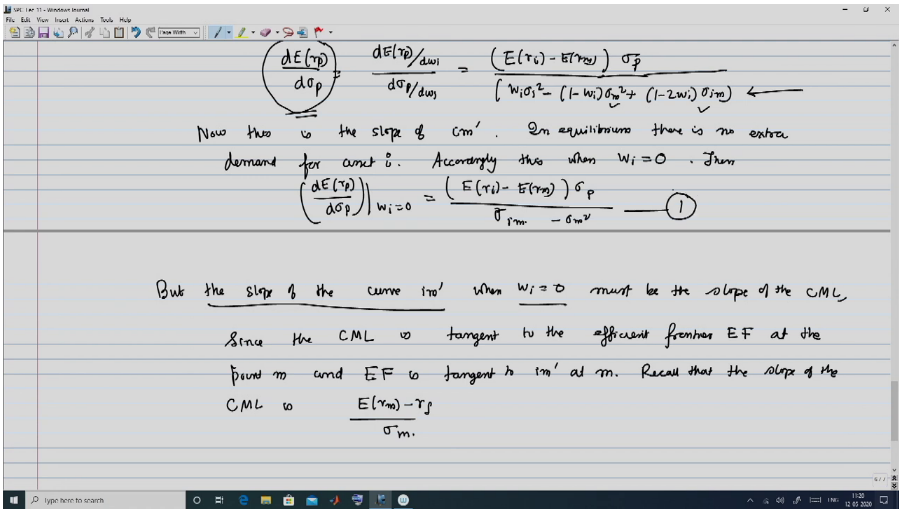

So, I identify this slope at $W_i = 0$ and this slope as 2. So, this expression I will call as 1 and this expression I will call as 2. So, accordingly since both the slopes have to be identical; so, equating 1 and 2; by this I mean the expressions, what we will get?. This is referred to as the CAPM or the Security Market Line or SML.

Now, I have actually put this CAPM; when I began the discussion in a slightly different form. So, in order to obtain that form which involves the beta of the i-th asset; we take into note the definition of beta. So, using the definition of β_i which is the covariance of the asset with the market portfolio divided by the variance of the market of portfolio; we obtain, from this relation we will get that E of; so this is actually Er_i; please make this correction.

So, we will get $E(r_i)$ is equal to r_f plus $E(r_m) - r_f$. So, this is the marquetry premium into this term σ_{im} over σ_m^2; I will represent this with β_i which is another form of; for CAPM or the security market line ok. Now, I have used the term capital asset pricing model and what I have done here is essentially I have instead got what is going to be the expected return on an asset, in terms of the beta of that particular asset and the risk premium and the risk free asset r_f.

So, in order to justify this term that this has some relation with the pricing of an asset, what I am going to do now; is I am going to talk about the scenario of and a qualitative discussion of; how this is related to assets being priced correctly. And for that we will look at the scenarios of when the asset is over-priced and when the asset is under priced in the context of the form of CAPM or SML given in terms of β_i; that is the last equation that we have obtained here, alright.

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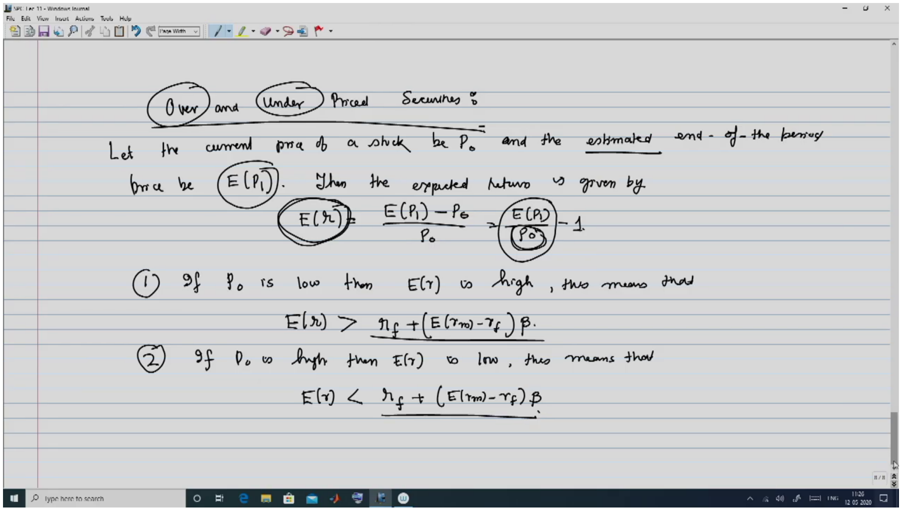

So we discuss now; on over and under priced securities. So, accordingly let us look at a simple example; so let the current price of a stock be denoted by P_0 and remember that CAPM is in a single period framework. So, accordingly the estimated or expected end of the period price be given by $E(P_1)$; remember P_1 is a random variable; so we have to take the expected return.

Then the expected; so this is actually the expected price which you call as estimate and so the expected return is given by; so we will make use of the definition. So, what is expected return?

Now, let us look at the over and under pricing aspect in the context of the CAPM. So, first of all; if P_0 is low, now see if P_0 is small; then this quantity is going to be large. So, if this is low; then consequently $E(r)$ is going to be high because $E(r)$ and P_0 behaves in a inverse manner. So, once I say that this is high; this means that the $E(r)$ of the asset that you get; will be greater than the expected return that is given from the formula for the expected return of the CAPM.

So, this means

$$E(r) > r_f + (E(r_m) - r_f)\beta$$

Now, here I am dropping the subscript i for notational convenience. Now, let us look at the other case that if P_0 is high:

$$E(r) < r_f + (E(r_m) - r_f)\beta$$

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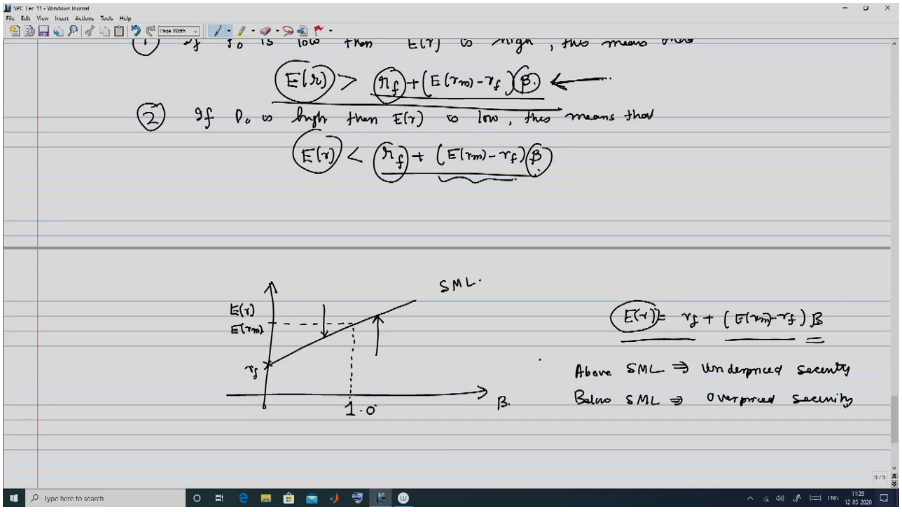

So, what does this mean graphically? So, graphically I can represent this as; so if I look at the expected return against beta and this is my r_f. So this is a straight line; remember that this is a straight line with intercept r_f and slope with a variable beta as the X-axis and $E(r)$ as the Y-axis. So, this equation this is what is known as the SML with a slope being given by this.

So, this may; so here I will have $E(r_m)$. So, this point here; so when $E(r) = E(r_m)$; alright, so that means, from CAPM we get

$$\beta_i = 1$$

Now coming back to the SML, so what happens here is that; if your $E(r)$ is greater than this value which is given by the security market line. So, this means that; if it is higher then it lies above the security market line. So, this expression means that the expected return is lies above the security market line; that means that the price is low and the price has to increase and the market will correct itself to increase it.

Likewise, when I have $E(r) < r_f$; then this means that the price P_0 is too high and then it needs to come down in order to attain the market equilibrium given by CAPM. So, these two statements that I have just now made, then I am going to reconcile them and conclude the following, that if you are above the SML, then this means that; so this is the scenario; that means, that the asset is or the security is under priced. And if you are the below the SML, then this means that what you have is an overpriced security.

So, we now come to the last topic of this lecture that is the single index model and the reason I am introducing the single index model is that we can better appreciate the interpretation of the term β_i and secondly, to introduce the concept of how the risk of a particular asset or a portfolio can be decomposed into two different kinds of risk; namely the systemic risk and the non-systemic risk.

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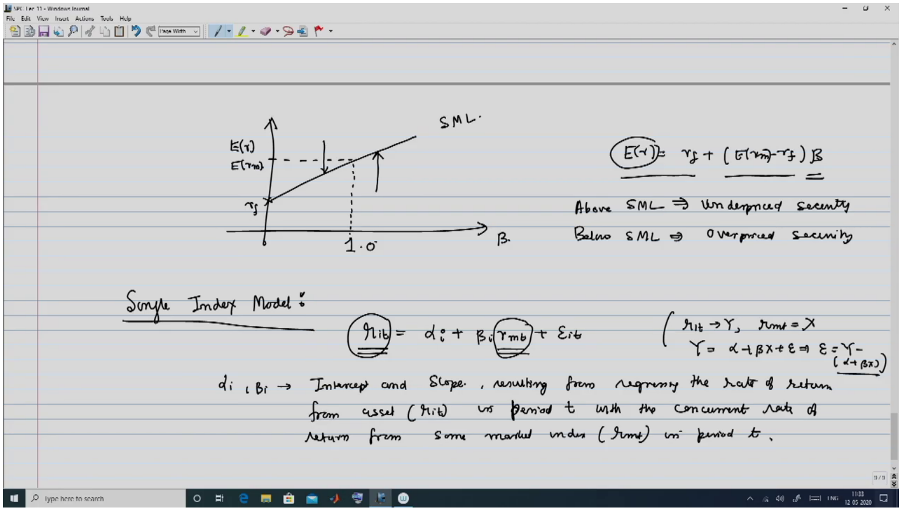

So accordingly, we start the single index model. So, we consider this to be a time dependent model; that means, the return at of an asset a_i at time t; that means, between that in t.

This is:

$$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it}$$

Now, it is called a single index model because the return of the asset at time t is going to be modelled as a function of the return on the market at time t which is equivalent to a market index. So, here the single index refers to the market index whose return will be denoted by r_{mt} and remember that this is a random variable.

So, here what we have is that alpha and β_i; these are the intercept and slope and these are determined as a result of regressing the rate of return from a_i. So; that means, $r_i t$ in period t; that means, from the time point $t-1$ to t with the concurrent rate of return from some market index which are denote as r_{mt} in period t.

So, what it means is that I considered different time intervals 0 to 1, 1 to 2 and so on and I look at what is the; what are going to be the return in those time intervals or those single periods. And then I will denote the period t to be an investment that was done at time t minus 1 and held up to time t. So, according to the corresponding return for the i-th asset is going to be denoted by r_{it} and that of the market is going to be denoted by r_{mt}.

So, what I am going to do; is I am trying to look at, so what I will do is that I will treat my r_i to be some random variable Y and I will treat r_m to be some random variable X. So, this is similar; so I will treat my r_{it} to be some Y and r_{mt} to be random variable X and you want to find a linear relation between them. So, this will be something like

$$Y = \alpha + \beta X + \varepsilon$$

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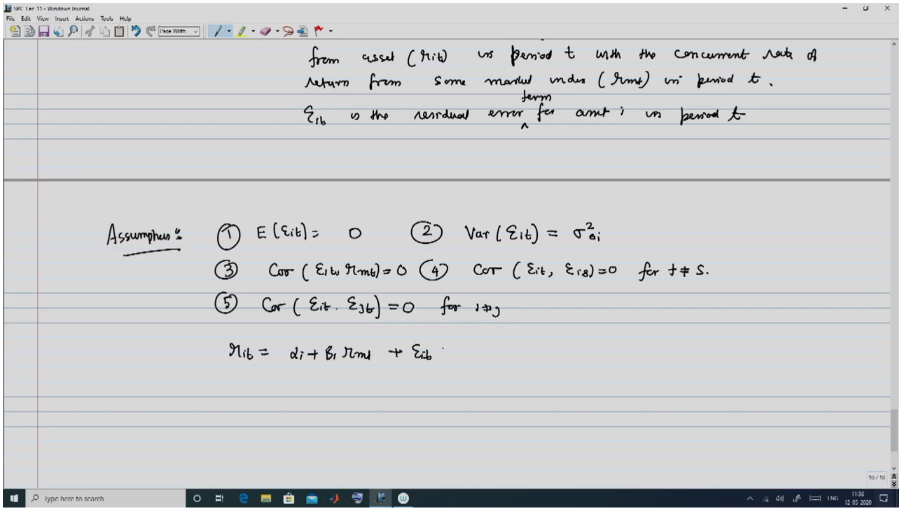

So now in order to carry out this exercise; I need to make certain assumptions and these assumptions are following that the expected value of ε_{it} which is the residual factor. So, remember that here I will take ε_{it} is the residual error for; this is the residual error term for asset i in period t.

The assumptions:

1. $E(\varepsilon_{it}) = 0$
2. $Var(\varepsilon_{it}) = \sigma_{\varepsilon_i}$
3. $Cov(\varepsilon_{it}, r_{mt}) = 0$
4. $Cov(\varepsilon_{it}, \varepsilon_{is}) = 0$ for $t \neq s$
5. $Cov(\varepsilon_{it}, \varepsilon_{jt}) = 0$ for $i \neq j$

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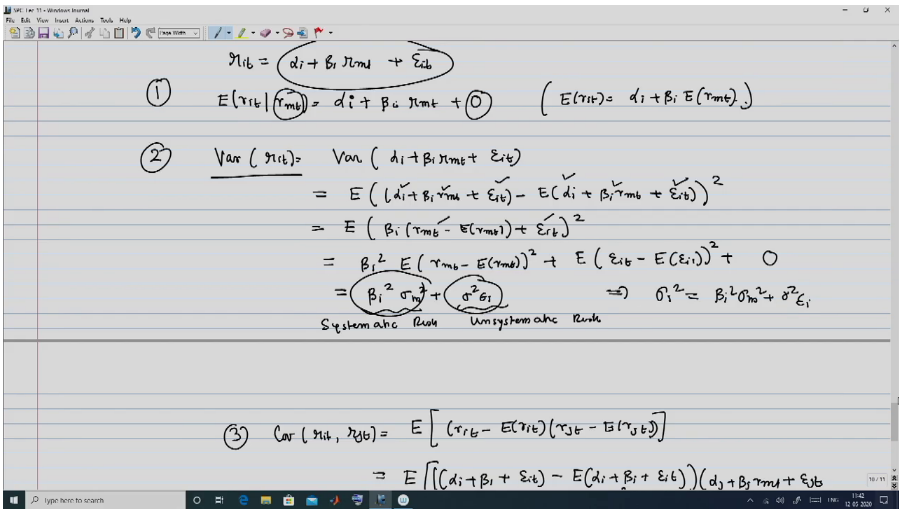

Let us calculate the expectation, variance and covariance of all this. The derivations are shown above.

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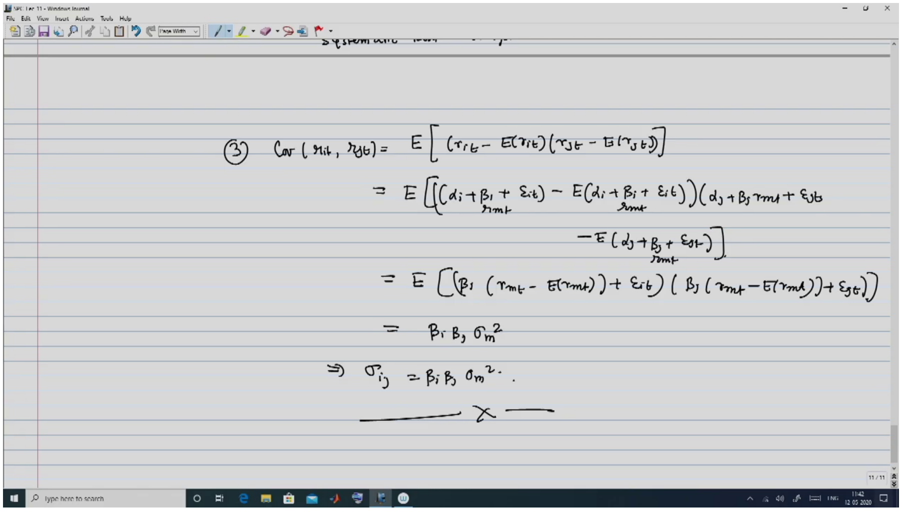

The last derivation is given above.

So this concludes this lecture; just to do a recap of what we have done in this lecture, we essentially looked at an extension of the capital market line and introduced the very important concept of capital asset pricing model.

And we derived the relation for the capital asset pricing model and we also gave an interpretation of the capital asset pricing model in the context of determination of whether the asset is overpriced and under priced and how it relates to the security market line or CAPM. And finally, we talked about what is the single index model and looked at its expectation variance and covariance.

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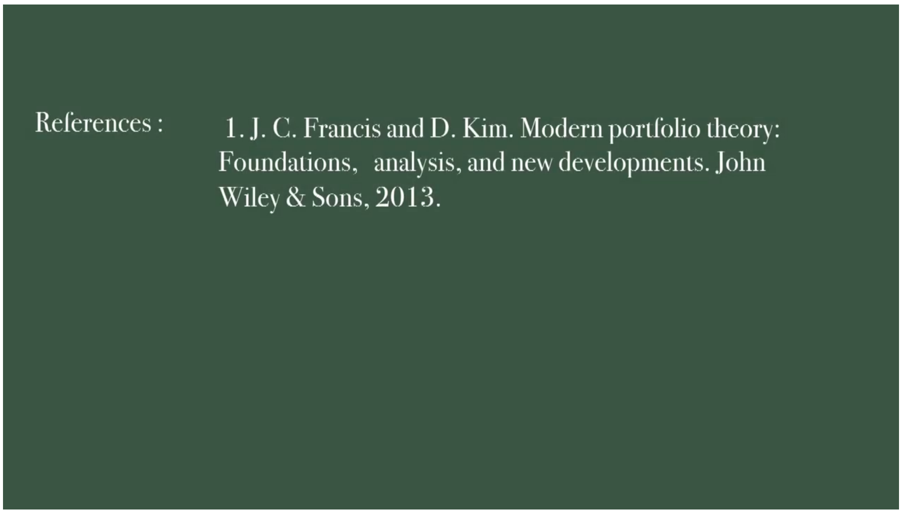

Thank you for watching.