Mathematical Portfolio Theory
Module - 03: Mean-Variance Portfolio Theory
Lecture 12: Portfolio performance analysis

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Hello viewers, welcome to this next lecture on the MOOC course on Mathematical Portfolio Theory. You would recall that so far in the mean variance framework we will talked about how to optimize a portfolio and we talked about efficient frontier and we dwelled upon 2 important lines namely the capital market line and the security market line which is also known as CAPM. So, in today's lecture we will conclude our discussion on the mean variance framework with a particular topic namely the performance of the portfolio and how to analyse that. So, accordingly we will look at various measures or ways or the classical ways of measuring how a portfolio is performing, and then we will see that how they are sort of related to each other and can be obtained as transformations of each other under some specified circumstances.

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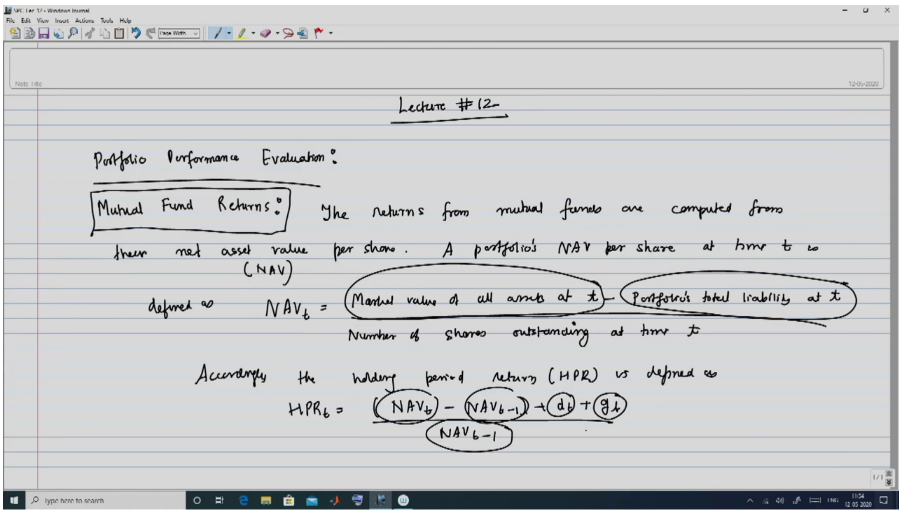

So, let us start off our discussion on portfolio performance evaluation. Now, in order to motivate this let us just talk about mutual fund returns. So, mutual fund is a sort of the most commonly used term or the most familiar term that you can get to a portfolio.

So, accordingly I start off with this and state the following that the returns from mutual funds are computed from their net asset value per share, and the Net Asset Value is abbreviated as NAV another term you might be familiar with and a portfolios. So, I have to define what is the NAV. So, a portfolio's NAV per share of the mutual fund at time t is defined as

$$NAV_t = \frac{\text{all assets in the mutual fund-portfolios total liability at time t}}{\text{the total number of shares outstanding}}$$

So, this is a very simple concept that, the NAV is essentially the current valuation per share of the mutual fund. So, when you buy mutual fund you basically buy the number of units of the mutual fund.

So, then it is given by the market value of all the assets; that means, the actual value that you will get if we decide to liquidate all the assets that are in the mutual fund and minus the portfolios total liabilities.

So, whatever liability you have a time t you subtract them to the total assets that are being held in the portfolio and then this means that this is the net amount of money that the mutual fund actually has in case it is liquidated and or in case you decide to sell of the assets. And this is basically the current valuation at time t and then this valuation; that means, this is the value of the assets and then this is the these value of the assets is essentially in the ownership of all the people who have purchase units of the mutual fund.

So, that means, for per unit of the mutual fund you have to take this total amount and divide it by the number of mutual funds that are still outstanding or being held by investors like small investors, financial institutions and so on. So, this will give you basically the valuation of the mutual fund per unit of that particular mutual fund, ok.

So, now, accordingly the holding period. So, remember that at the end of the day we will make the evaluation of the portfolio in terms of how it is performing driven by the basic motivation that an investor invests in risky assets with the intent of maximizing their terminal wealth. So, accordingly for the holding period return. So, the holding period return which is often called HPR is defined as

$$HPR_t = \frac{NAV_t - NAT_{t-1} + d_t + g_t}{NAV_{t-1}}$$

However, we need to account for certain other things some other income streams they can actually come in between time $t-1$ and time t. So, for that we have 2 terms d_t and g_t and I will explain what this terms are.

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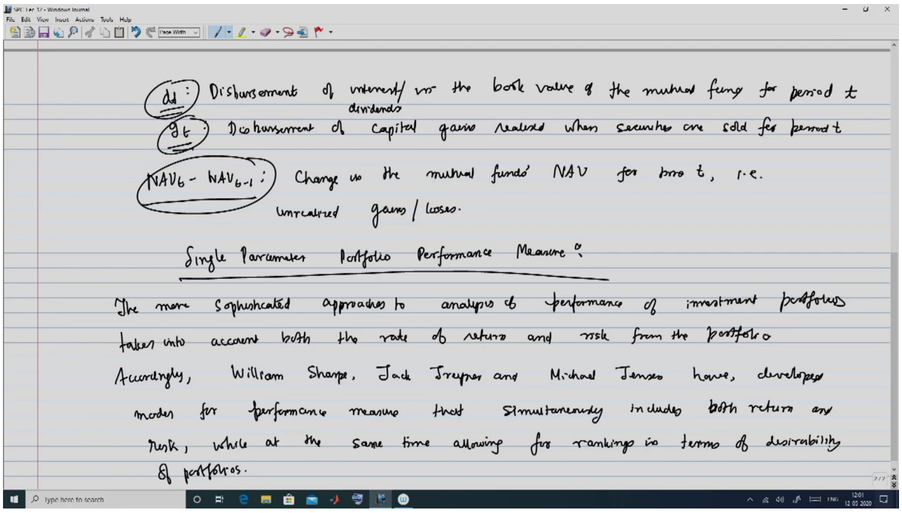

So, here your d_t is disbursement; that means, the payout by the mutual fund in the. So, disbursement of interest in the book value of the mutual fund.

g_t is essentially disbursement of capital gains realized when securities are sold for period t.

$NAV_t - NAV_{t-1}$ is the change in the mutual funds NAV for time t. So, this is the unrealized gains or losses.

So, just to go through this again. See $NAV_t - NAV_{t-1}$ it is going to be the change in the mutual funds NAV in the intervening period t and $t-1$. And this difference of valuation comes as a result of the assets that are that were held at time $t-1$ and are still being held at time t. Now, between time $t-1$ and t, there are 2 types of incomes that can come. One is that the dividends being paid on the assets that are being held by the mutual fund in case it is a stock or interest in case is a bond.

So, they are going to pass this on to their to their investors in the mutual fund and that is what is known as the disbursement of the interest or dividends. And this g_t is the disbursement of capitals gains as a result of selling of some of the assets between time 0 and t minus between time t minus 1 and t.

So, this means that between time t minus 1 and t, three things can happen either you will get some interest or dividends which is reflected in the term dt or some assets are sold in which case you have a capital gains or loss which is indicated by the term gt and then of course, assets that are still being held and then the difference between these two as a result of the market movement between time t minus 1 and time t is given by $NAV_t - NAV_{t-1}$.

Alright, so now we come to the portfolio performance analysis measures and we will essentially look at primarily 3 measures. So, let me start off with the these measures which are known as single parameter portfolio performance measure ok. So, I will begin with a little bit of a narrative on this. So, the more sophisticated approaches to analysis of performance of investment portfolios takes into account both the rate of return and risk from the portfolio.

So, this means that you know this portfolio performance measures again are presented in the paradigm of the mean variance framework namely that you look at what is the rate of return and the risk and reconcile them to find an indicator which will give you a mechanism of choosing the performance of different funds or portfolios and make a judicious decision as far as the investment is concerned.

So, accordingly driven by this mean variance framework, 3 individuals namely William Sharpe, Jack Treynor and Michael Jensen have developed models for performance measure that simultaneously includes both the pillars of the Markowitz framework namely return and risk while at the same time allowing for rankings in terms of desirability of portfolios. So, this means that these 3 individuals came upon with certain ways of measuring the performance in portfolio. This is based on the return and the risk and in this case the risk could mean both the standard deviation or the beta as the case might be and we will see you know the corresponding cases for σ_p and β_p and how it is being used.

And the reason for doing this is that it will basically gives a way of our single point indicator of evaluating the performance of the portfolio with the intent of ranking them in terms of desirability from the point of view of investment.

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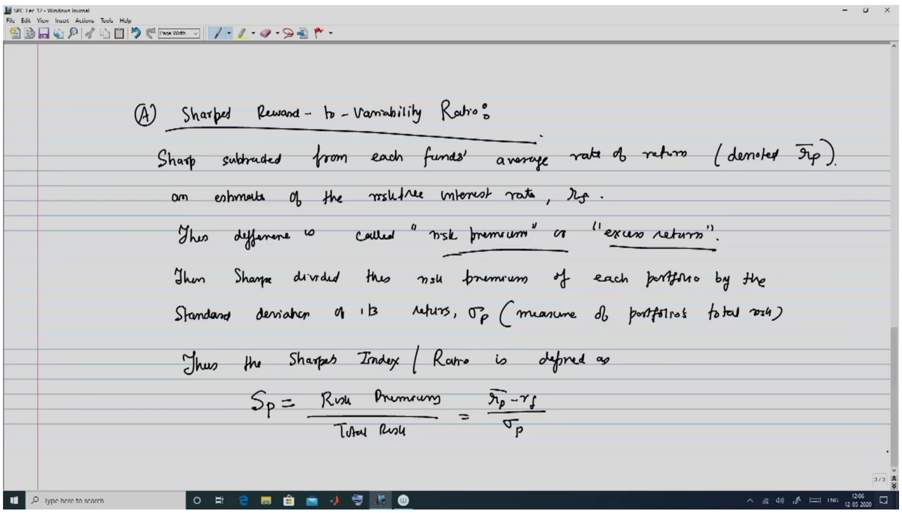

So, we start off with the first of this due to William Sharpe and this is what is known as Sharpe's Reward to Variability Ratio ok. So, what the motivation for this is that the Sharpe subtracted from each funds average or the expected rate of return denoted by Er_p or in this case we will just write $\bar{r_p}$ this is the expected return on the portfolio and estimate of the risk free interest rate which you have already defined as r_f.

Now, this difference is called risk premium or excess return. So, as already highlighted this difference of that expected return on the portfolio which is actually risky portfolio as compared to a risk free return that you would have gone by just investing in a completely risk free bond such as a government bond. These additional difference is an indication of the gain that you expect to make because as an investor you have chosen to take the risk instead of choosing a more safe path of making your investment in a risk free asset.

So, however, coming to the drawback that prompted Markowitz to introduce the mean variance framework that this difference is not exclusively enough to indicate how the portfolio is performing. Because lot of times it might happen that this difference; that means, the excess returns that you are getting might be accompanied by a commensurate amount of a high level of risk. And that is the reason why all these measures are driven by the mean variance framework which takes into account both the return and the risk.

So, we now have to look at a certain as a little bit of an extension of this and not just confined ourselves to the risk premium or the excess return and accordingly what Sharpe did was then he divided this risk premium of each portfolio by the standard deviation of it is returns, namely, σ_p. So, this σ_p is based essentially the risk measure of portfolio's total risk. So, thus the Sharpe's index or sometimes is called the Sharpe's ratio is defined as. So, motivated by this it will be

$$S_p = \frac{\bar{r_p} - r_f}{\sigma_p}$$

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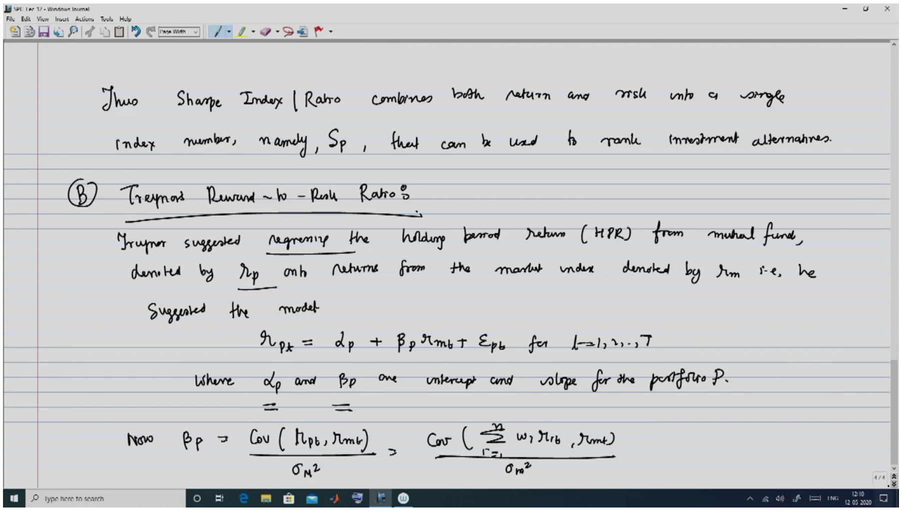

So, thus to interpret this. So, thus the Sharpe index or ratio. So, we will use the term index or ratio interchangeably. It combines both return and risk into a single index number namely S subscript p which can be used to rank investment alternatives.

So, this is means that what you can do is that you can look at various investment alternatives; that means, a collection of portfolios and for each case we can actually calculate the Sharpe ratio and then use the Sharpe ratio to rank the desirability of the portfolios and then accordingly make an investment in a portfolio or a mutual fund that is more desirable alright. So, now, let us move on to the next index due to Treynor. So, this is Treynor's reward to risk ratio. So, what Treynor did was he suggested regressing the holding period return.

So, he suggested that you regress r_p onto returns from the market index whose return is denoted by r_m. So, this is the same framework as the single index model; that is, he suggested the model. So, it's a single index model now I am just modifying this for a portfolio p instead of an asset i. So,

$$r_{pt} = \alpha_p + \beta_p r_{mt} + \varepsilon_{pt}$$

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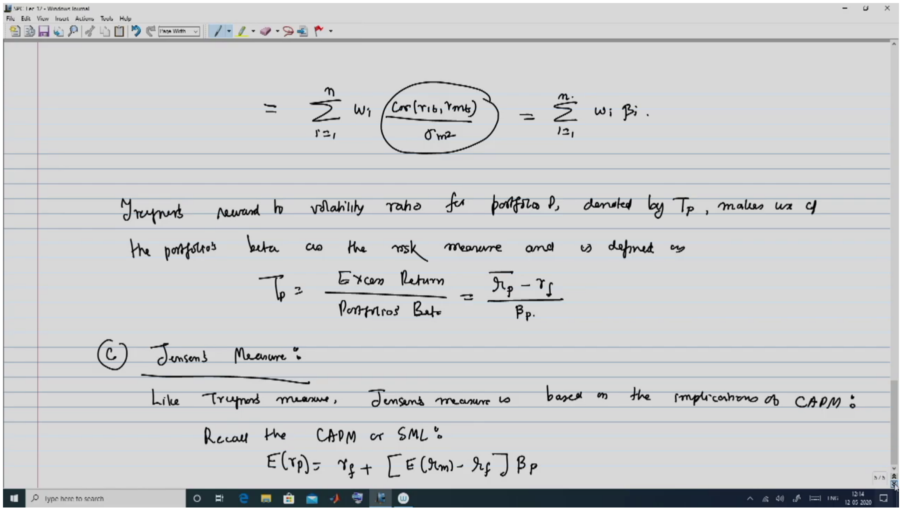

So, this essentially gives you a way of calculating what is going to be the beta of the portfolio once you have made an assessment and determine what is going to be the beta of each of the individual assets, ok. So, coming back to Treynors reward to volatility ratio for portfolio P denoted:

$$T_p = \frac{\bar{r_p} - r_f}{\beta_p}$$

So, like Treynor's measure Jensen's measure is based on the implications of CAPM. So, accordingly recall the expression for CAPM or SML; you would recall that this is the expected return of portfolio. So, I am looking at this in the context of portfolio is equal:

$$E(r_p) = r_f + [E(r_m) - r_f]\beta_p$$

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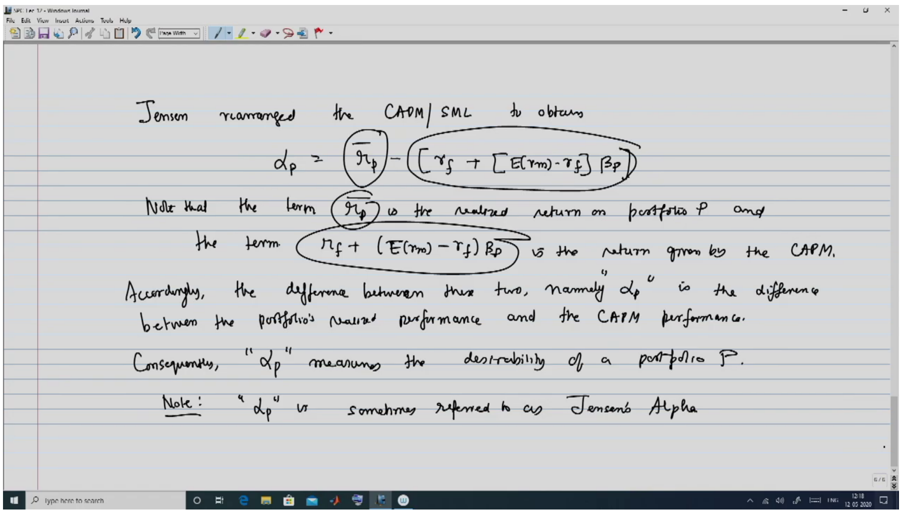

So, what Jensen did was to look at this CAPM and make some rearrangement. So, Jensen rearranged the CAPM or the SML to obtain the following. So, what Jensen did was actually looked at the expected return of the portfolio $\bar{r_p}$ and subtracted the expected return given by CAPM which is r_f; that means,

$$\alpha_p = \bar{r_p} - [r_f + (E(r_m) - r_f)\beta_p]$$

So, he ideally you want your α_p to be positive and as high as possible. So, higher the value of alpha, the more desirable is the investment in the corresponding portfolio for which this α_p has been determined, ok. Now, that we have identified this single 3 single parameter measures in order to ascertain the performance of the portfolio, let us now try to dwell a little bit on how they are correlated to each other and see if there exists some sort of an equivalence behaviour in terms of the qualitative assessment of the portfolio performance.

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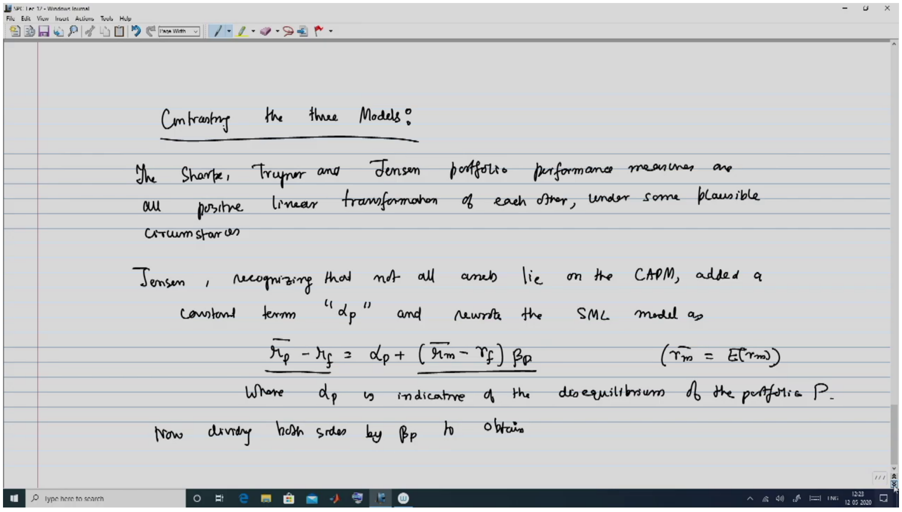

So, accordingly we begin with the topic of contrasting the three models and by this that. So, I will use the word models, ratio and index interchangeably. So, the Sharpe, Treynor and Jensen portfolio performance measure as defined above are all positive linear transformation of each other. So, basically their transformation of each other taken two at a time, under some plausible; that means it can happen, circumstances, alright.

So, Jenson recognizing that not all assets lie on the CAPM which is why he looked at the difference of the realized return and the return predicted by CAPM, added a constant term that is the Jensen's alpha, α_p and rewrote the SML model as what? As

$$\bar{r_p} - r_f = \alpha_p + (\bar{r_m} - r_f)\beta_p$$

So, in equilibrium I will have $\bar{r_p} - r_f$ equal to this which is the CAPM.

So, α_p is an additional term that has shown up which has disturbed the equilibrium as given by the CAPM. So, accordingly I can make the statement that the α_p is an indicator or is indicative of the disequilibrium of the portfolio P. So, that means, the extent to which the equilibrium as given by CAPM has been disturbed. Now, what do you do is that now we divide both sides by β_p to obtain.

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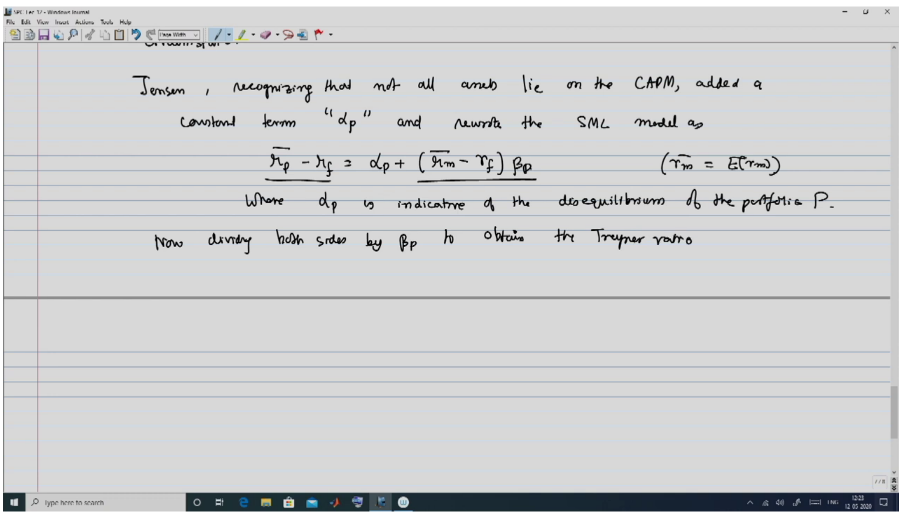

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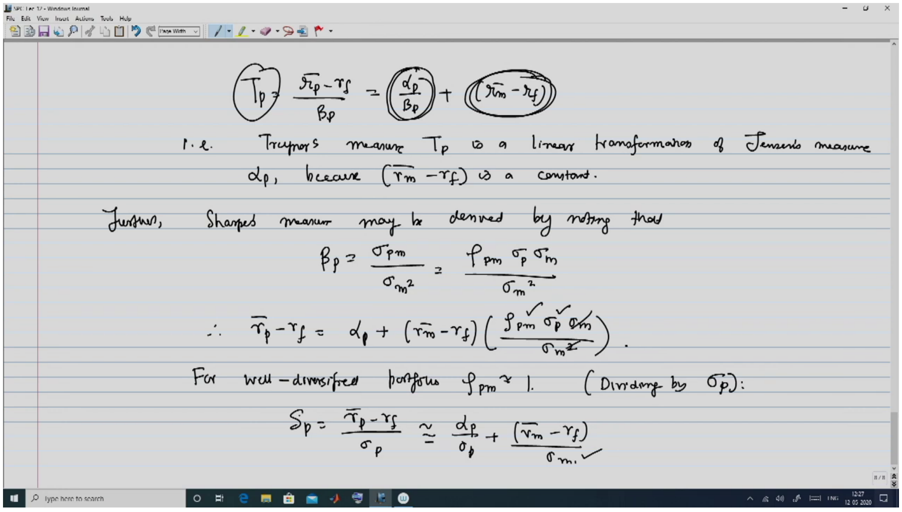

So, remember that once you have β_p. So, we can now relate this to the Treynor ratio to obtain that Treynor ratio which was

$$T_p = \frac{\alpha_p}{\beta_p} + [\bar{r_m} - r_f]$$

So, we can view this at that T_p has this linear transformation $\bar{r_m} - r_f$ and then has a factor of $\frac{\alpha_p}{\beta_p}$. So, this is a linear transformation of Jensen's measure α_p because remember that $\bar{r_m} - r_f$ is a constant.

It is easy to observe that

$$S_p = \frac{\alpha_p}{\beta_p} + \frac{\bar{r_m} - r_f}{\sigma_m}$$

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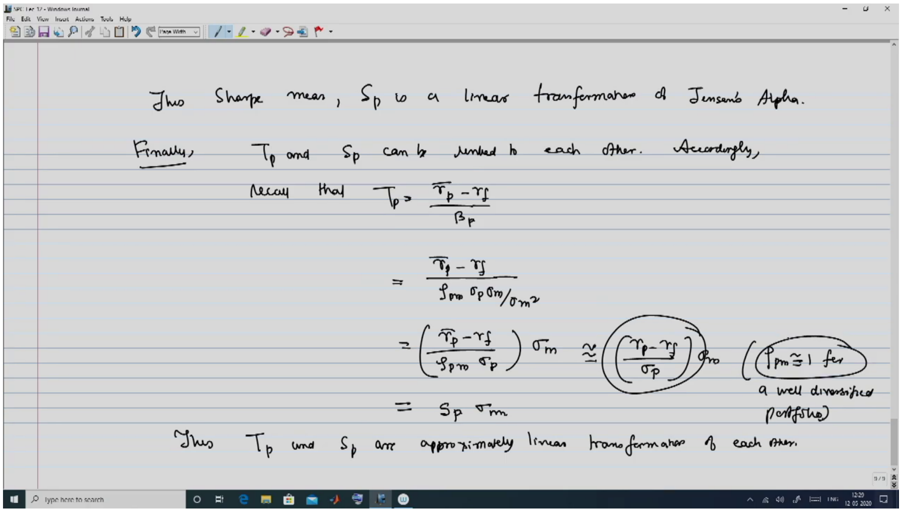

So, from this relation thus Sharpe measure S_p is a linear transformation of Jensen's alpha. So, we have looked at Jensen's alpha and this relation to the Treynor ratio and the Sharpe ratio. So, we are only left with the one combination how to connect the Treynor and the Sharpe ratio. So, finally, T_p and S_p can be linked to each other.

We finally have

$$T_p = S_p \sigma_m$$

So, now what we have is that we have now a collection of all these 3 results where we took at the Treynor, Sharpe Treynor and Sharpe ratios and the Jensen's alpha and then we saw that how each of them is essentially a linear transformation of the other in specific circumstances in particular one circumstance that you particularly identified was that for a well diversified portfolio. The correlation coefficient of the portfolio p and m is approximately equal to 1.

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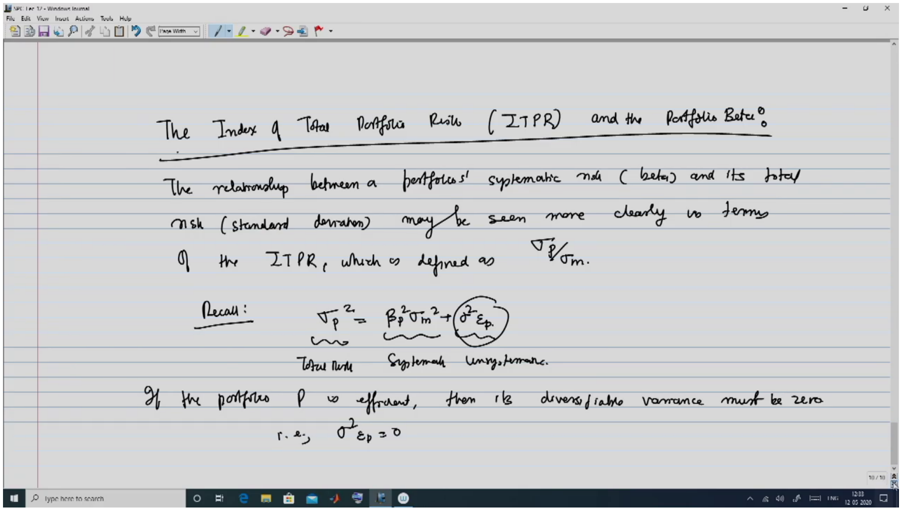

Alright so now, we come to just one last topic and this is what is known as the Index of Total Portfolio Risk or which is ITPR and the portfolio beta ok. So, for this we note that the relationship between a portfolio systematic risk. Remember that we had introduced the term systematic in our systemic risk a beta and its total risk that is a standard deviation may be seen more clearly in terms of the ITPR:

$$ITPR = \frac{\sigma_p}{\sigma_m} = \beta_p$$

So, in some sense it is an indicator of the risk of the portfolio visibly the risk of the market portfolio as given by the standard deviation which is what is known as the total risk or the or you can view this as the unsystematic risk. So, accordingly we recall the expression for the single index model done in the previous class.

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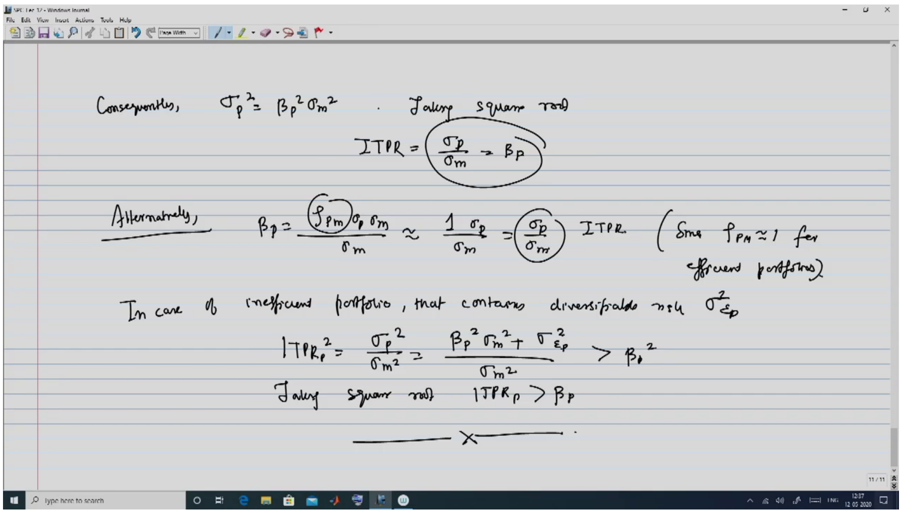

So, consequently what do we have? So, consequently from this relation we have

$$\sigma_p^2 = \beta_p^2 \sigma_m^2$$

As

$$\beta \equiv \frac{\sigma_p}{\sigma_m},$$

we have

$$ITPR^2 > \beta_p^2$$

So, this concludes our discussion on the mean variance portfolio theory and from the next lecture we will start talking about the framework for the non-mean variance portfolio theory.

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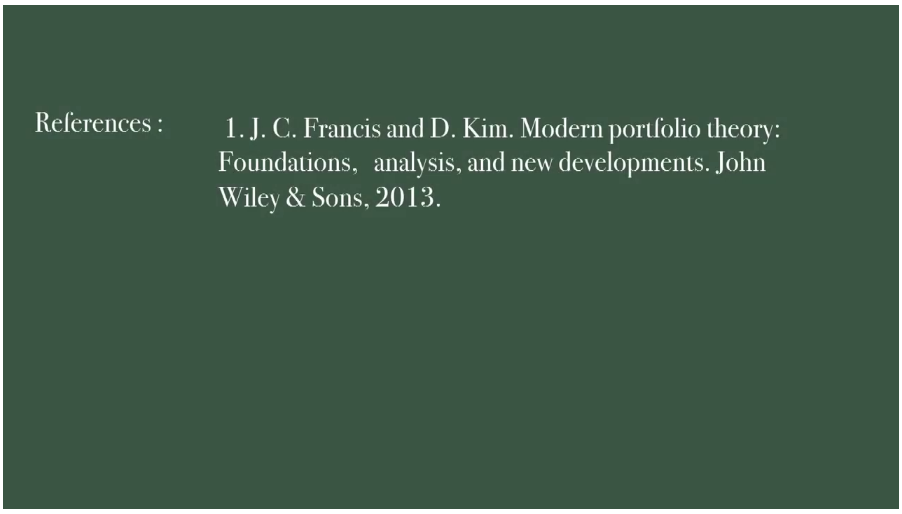

Thank you for watching.