Mathematical Portfolio Theory
Module 04: Non-Mean-Variance Portfolio Theory
Lecture 18: Kataoka's Safety-First Criterion and Telser's Safety-First Criterion-06

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Hello viewers, welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. I would recall that in the last lecture we had started talking about the Safety First Criteria, and we have done one of the safety criterias, safety first criteria, namely the Roy's criteria, which involves the minimization of the return of the portfolio being less than or equal to certain pre-determined threshold level.

So, in today's class, we will continue our discussion on the safety first criteria, and we will look at two new criterias. And we will discuss those in detail and illustrate that through one example each to further our discussion on Non-Mean Various Portfolio Theory.

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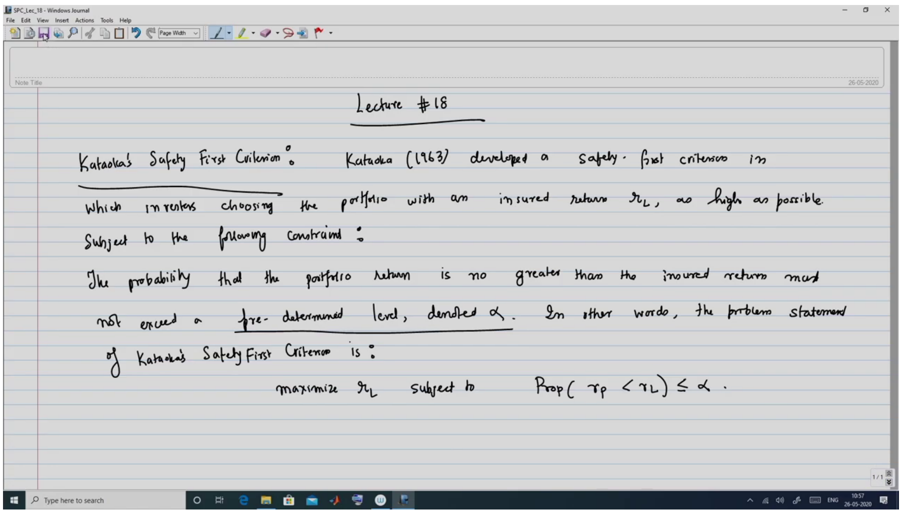
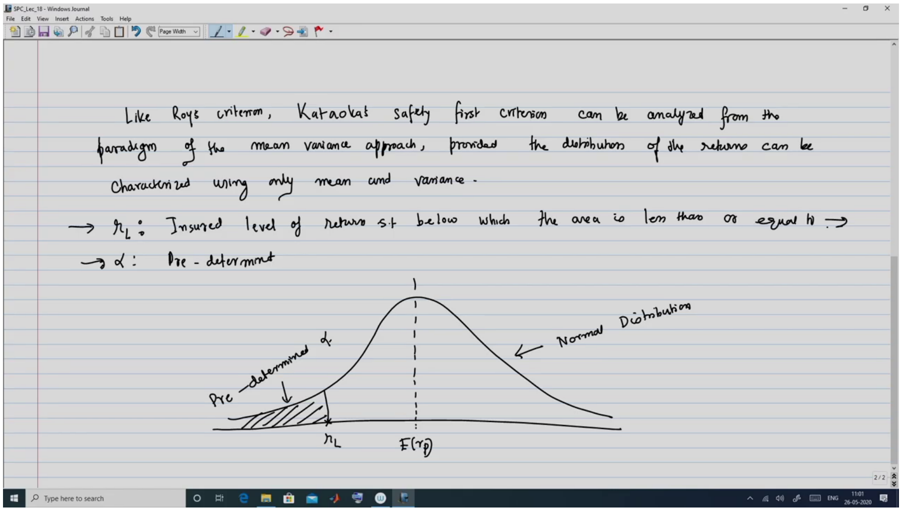


So, accordingly we start this lecture with the new safety cross criteria that is known as Kataoka's a Safety First Criteria. So, Kataoka, in 1963, developed a safety first criteria and this involves investors choosing the portfolio with an insured return r_L as high as possible subject that to the following constraint. So, this means that they want to as large as possible an r_L subject to the following constraint. And the constraint is the following that the probability that the portfolio return, that means, the portfolio that is been actively managed is no greater than the insured return must not exceed a pre-determined level denoted α.

So, in other words, so let me be offer a little bit of clarity in exactly how this translates to mathematically. So, in other words, the problem statement of Kataoka's safety first criteria is the following that he want to maximize r_L subject to the following probability. And the probability is that the probability that your r_P being less than your r_L is going to be less than or equal to some pre-determined level denoted as alpha ok. So, I will explain this in a little more detail as you move along. (Refer Slide Time: 04:39)



So, let us try to reconcile this with the Roy's criteria. So, like Roy's criteria, the Kataoka's safety first criteria can be analyzed from the paradigm of the mean variance approach provided the distribution of the returns can be characterized using only mean and variance ok.

So, let us now introduce the notation. So, first of all, let me talk about the notation r_L. So, this is used to define the insured or denote the insured level of return alright, and you have alpha which is pre-determined. So, what do you do is that we essentially look at the return of the portfolios and suppose that you know it is of this form say it is normally distributed, and I am considering that this is normally distributed for the sake of better clarity.

So, here what we have, so it is on the x-axis we have r_P. So, this is going to be E of r_P, and we have a pre-determined alpha and that is given by this shaded area. And accordingly this value on the x-axis is going to be denoted by r_L. So, I can come back to my statement that here this r_L level is the insured level of return such that a below which the area is less than or equal to this alpha alright, ok.

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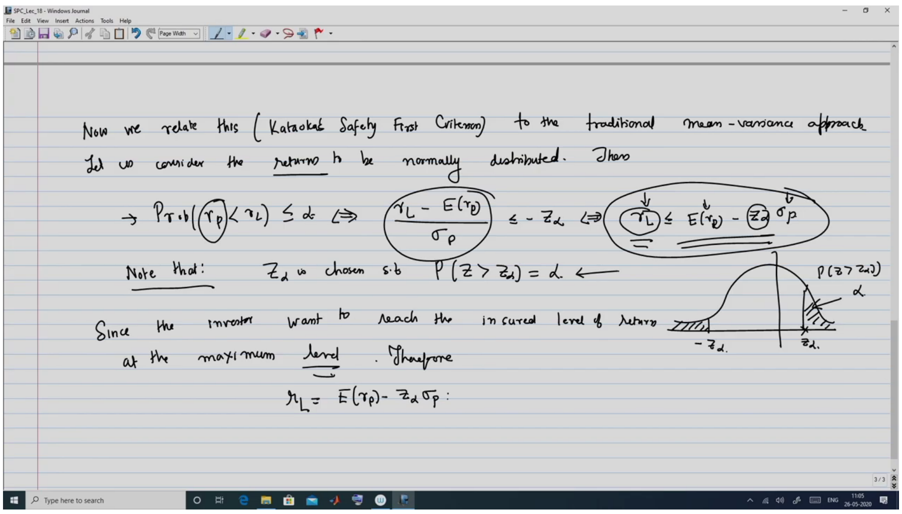

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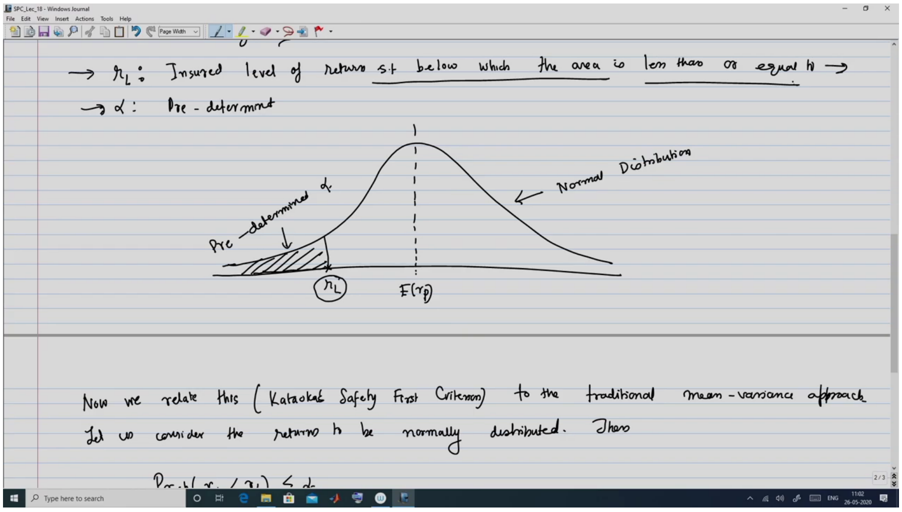

So, let me delve a little dig into this. So, what do you do is that now we relate this, that means, the Kataoka's Safety First Criteria to the traditional mean variance approach, so accordingly let us consider the returns to be normally distributed. So, we have already considered this in this figure that they are normally distributed. So, then what happens, accordingly we can observe that, so we are interested in the probability of the return of the portfolio being less than the insured level, we want this probability to be less than or equal to α.

So, coming back to this normal distribution, so this means that r_L. So, essentially the insured level r_L is that level such that below which so beyond the left of it the area under the curve is going to be less than or equal to alpha ok. So, now, this coming back to this $P(r_P < r_L) \leq \alpha$, what is this mean?

This means that now if I since I have assumed that the returns are normally distributed that is your r_P is normally distributed. So, this condition then reduces to

$$P(r_P < r_L) \leq \alpha \iff \frac{r_L - E(r_P)}{\sigma_P} \leq -Z_\alpha \iff r_L \leq E(r_P) - Z_\alpha \sigma_P$$

So, if you look at this distribution and you have this area under the curve, so, this area if this is the area is alpha, then this is nothing but $P(Z > Z_\alpha) = \alpha$ where this point is going to be equal to Z alpha; and accordingly symmetrically this point here is going to be minus Z alpha. So, that is the reason why we take this $r_P - r_L$ we can use this expression, and this is going to be less than or equal to minus $-Z_\alpha$, alright. So, this is the definition of what is going to be my Z_α.

Now, since the investors want to reach the insured level of return at the maximum level right, remember r_L is less than equal to this, so they want to obviously, you know they want to have the insured level of return to be maximum possible. And since this is the we have the upper bound of that, so therefore, from the investors point of view the insured level of return

$$r_L = E(r_P) - Z_\alpha \sigma_P$$

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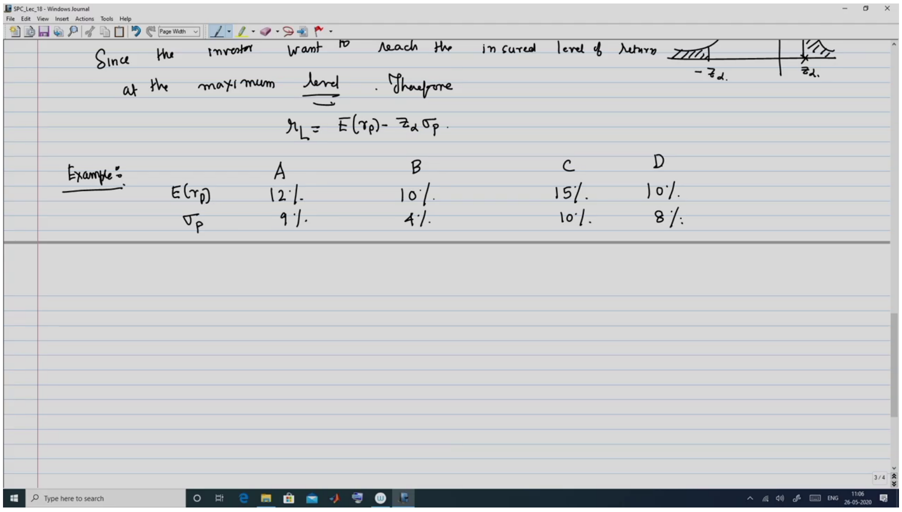

So, now let us look at this through an example. So, we recall the table where we had this portfolios A, B, C and D, and you remember that the expected return $E(r_P)$ was. So, this is the table I had introduced in the last class. So, for A – the expected return is 12 percent; for B, it is 10 percent; for C, it is 15 percent; and for D this is 10 percent. And the corresponding standard deviation sigma P is going to be 9 percent, 4 percent, 10 percent, and 8 percent.

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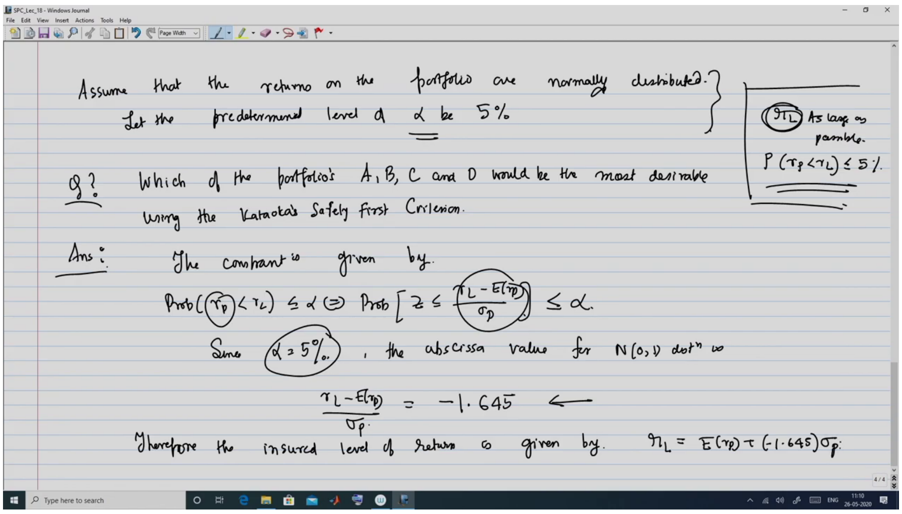

So, now what you do is we assume that, and again this is for the illustrative purpose and this argument can be extended to any distribution that can be specified using mean and variance. So, assume that the returns on the portfolio are normally distributed ok.

So, now let the pre-determined level of alpha be 5 percent. So, this means that you want to find your r_L to be as large as possible, while at the same time ensuring that probability of your return of the portfolio being less than this level r_L this should be obviously, less than or equal to just 5 percent ok.

So, now we have set up this example or with this four portfolios, and I have specified what is going to be my predetermine level of alpha. So, now, what you can do, so accordingly, I want to pose the following question is that amongst this four portfolio A, B, and C, and D, which of the portfolios A, B, C, and D, would be the most desirable to an investor who is using the Kataoka's Safety First Criteria? So, let us try to answer this question. So, let us revisit the constraints.

So, remember that we have to maximize r_L subject to this constraint. So, accordingly the constraint is given by

$$P(r_P < r_L) \leq \alpha \iff P[Z \leq \frac{r_L - E(r_P)}{\sigma_P}]$$

Since r_P is normally distributed, so obviously, $\frac{r_L - E(r_P)}{\sigma_P}$ is normally distributed. And I want this to be less than or equal to alpha now a since alpha equal to 5 percent right, so the abscissa value or the x-axis value for N(0, 1) distribution is given by what this is equal to minus 1.645. And this is going to be equal to the $\frac{r_L - E(r_P)}{\sigma_P}$.

So, therefore, the insured level of return, so, from this relation, we can get the insured level of return is given by the relation. So, from here I get r L is equal to E r P plus 1 minus of 1.645 into sigma P alright.

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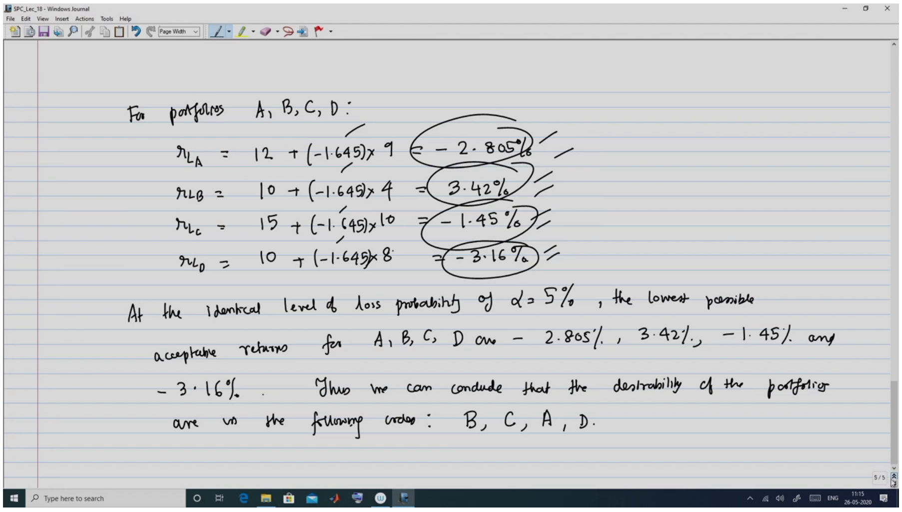

So, now, for portfolios A, B, C, and D, what do we have? We will have, we need to calculate what is $r_{L_A}, r_{L_B}, r_{L_C}$ and r_{L_D}. So, what are these going to be?

These are going to be $E(r_P)$. So, what are the $E(r_P)$? So, you go back to the original table, the $E(r_P)$ were 12 percent, 10 percent, 15 percent and 10 percent ok. So, we will write those values here. So, this is going to be 12, 10, 15, and 10.

And then we have sigma P. So, what are the σ_P values, the σ_P values are going to be 9, 4, 10, and 8. So, this is going to be 9, 4, 10, and 8. So, I have substituted the E_{r_P} and the σ_P value, and then I multiply this with minus 1.645. So, each of those terms I factor in the minus 1.645. So, this was actually obtained from this value of minus 1.645 was obtained from the table for standard normal distribution.

So, this effectively turns out to be minus 2.805 percent, this is 3.24 percent, this is minus 1.45 percent, and this is minus 3.16 percent alright. So, then so that means what? So that means, that since all of them you consider the identical loss of probabilities. So, at the identical level of loss probability of alpha equal to 5 percent, and I am considering they are identical because I have minus 1.645 in all the cases, the lowest possible acceptable returns for A, B, C, D are minus 2.805 percent, 3.24 percent, minus 1.45 percent, and minus 3.16 percent.

So, thus we can conclude that the desirability of the portfolios are in the following order. So, first I will have B because this is the highest value; then next I will have C which is minus 1.45 percent; then I will have A which is a minus 2.805 percent; and then finally, we have D which is minus 3.16 percent.

So, that means, I am basically choosing in terms of starting with the highest that we are highest r_L that we get insure level of return, and then coming down all the way to the lowest value which is minus 3.16 percent ok.

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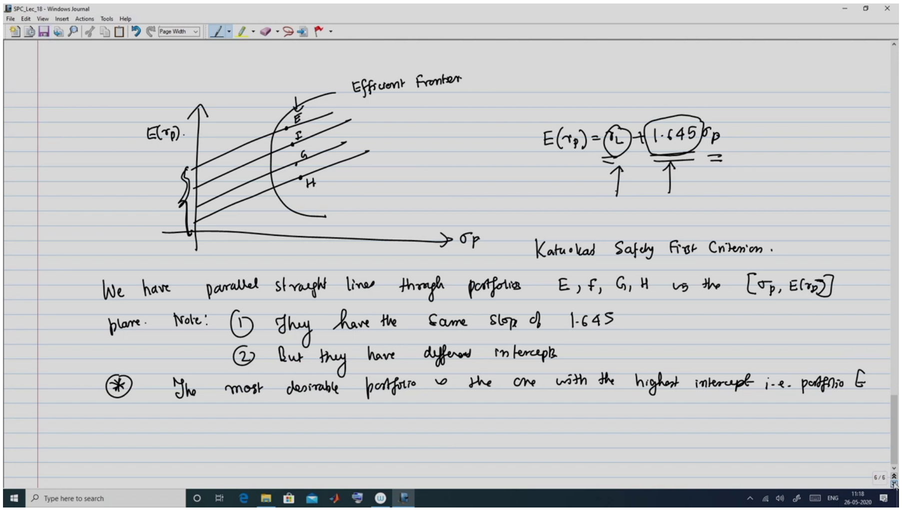

Now, let us see how does this turn out to be graphically. So, so graphically one can view this like that. So, suppose now I have this relation. So, with alpha equal to 5 percent, I have the relation

$$E(r_P) = r_L + 1.645\sigma_P$$

So, if we take the $[\sigma_P, E(r_P)]$ plane, so these are going to be the some straight lines. And then what we have is we have the efficient frontier. So, what you have is now that suppose we call this portfolio, say, E, F, G, H. So, all these portfolios have an identical the slope of 1.645, but the r_L which is the intercept that is going to be highest in case of this portfolio E, because this is got the highest intercept. And this is just the manifestation of a graphical representation of the Kataoka's Safety First Criteria, ok.

So, just to sort of elaborate on this, so we just write that we have parallel straight lines through portfolios E, F, G, H in the $[\sigma_P, E(r_P)]$ plane. So, a note the following that the first thing you note that they have the same slope of 1.645, but they have different intercepts. And the punch line is that as I have explained just now the most desirable portfolio is the one with the highest intercept, namely that is portfolio E which is the largest intercept amongst all the 4 portfolios, alright.

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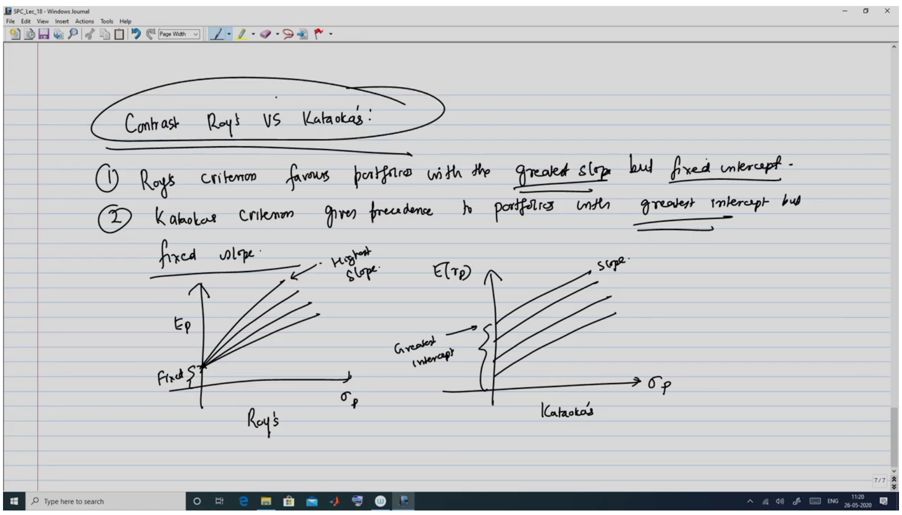

So, now that we have done this Kataoka's and Roy's criteria. So, we are to we want to now contrast Roy's criteria with Kataoka's criterion, and we make the following observation. First of all remember that Roy's criteria favors portfolios with the greatest slope, but fixed intercept. So, recall this from the previous class, and from the discussion that we have completed just now the Kataoka's criteria gives precedence to portfolios with greatest intercept but fixed slope.

So, in case of Roy's criteria, if you have $[\sigma_P, E(r_P)]$ you have fixed intercept, but you will pick up the portfolio that has the highest slope. So, he get a slope, but the intercept is fixed. So, this is the crux of the Roy's criteria. However, in case of the Kataoka's criteria, if we look at the $[\sigma_P, E(r_P)]$ plane as we have seen just now, you will have a fixed slope, so that means, you will have identical slope and then you will pick the one with the greatest intercept namely this one alright.

So, here the slope is fixed, so here the intercept is fixed, and here the slope is fixed. So, you basically pick up the one with the highest slope. And here you pick the one with the greatest intercept, so that is the primary contrast between the Roy's and the Kataoka's criteria ok.

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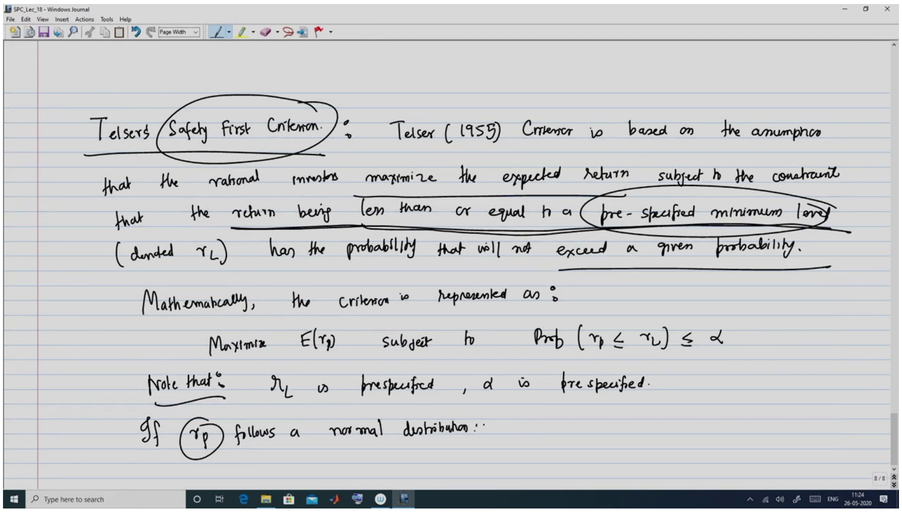

So, let us now come to the last of the three criterias, and that is Telser's Safety First Criteria. So, Telser in 1955 had proposed this criteria which is based on the assumption that the rational investors maximize.

So, it brings us to again the basic motivation for rational investors that they want to maximize the expected return, but since it is a safety first criterion, so they will maximize the expected return subject to the constraint that the return being less than or equal to a pre-specified minimum level which will denote by as before r_L has the probability a sort of this means that this is the probability of return being less than or equal to the pre-specified level of r_L. This probability will not exceed a given probability.

So, it so mathematically how do I present this? So, the criteria is represented as maximize $E(r_P)$. Now, what do you want? We want to maximize the expected return, but we need the condition that subject to the probability that the returnr_P of the portfolio being less than or equal to r_L, that means, this probability of the portfolio being less than or equal to r_L this is does not exceed some pre-defined given probability of alpha.

So, here note that first of all your you have a pre-specified minimum level. So, here your r_L is pre-specified, and it will not exceed a given probability. So, this indicates that alpha is also pre-specified ok. Now, let us say, you know as we have done in the other two cases. So, we focus on a situation that if r follows a normal distribution that r_P that means, the return of the portfolio it follows a normal distribution, then what happens?

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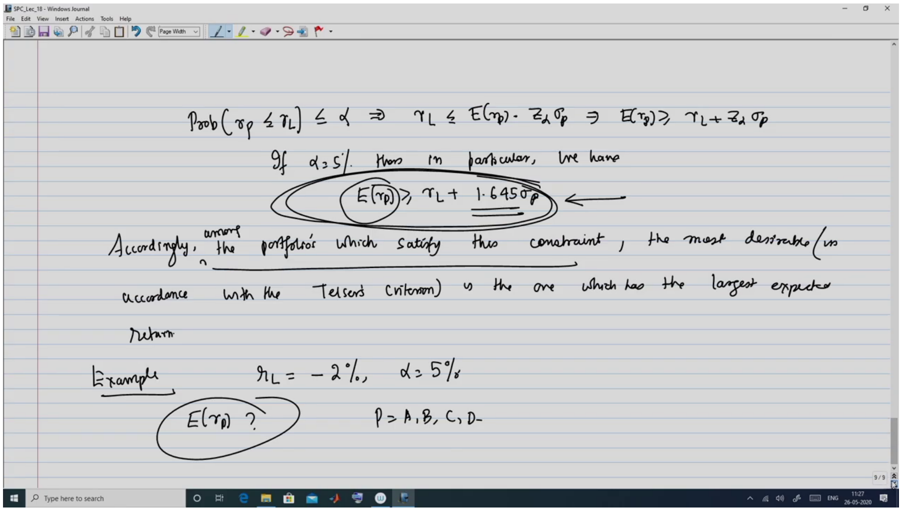

Then we have, so what do we need the constraint is probability that return of the portfolio r P less than or equal to n, this must be less than is less than equal to pre-specified r subscript L, this being less than or equal to a pre-specified level alpha. This implies and that you have r L being less than or equal to E r P minus Z alpha sigma P which implies that E r P is greater than or equal to Z l sorry so the greater than equal to r L plus Z alpha into sigma P.

Now, if you are if alpha is equal to 5 percent, then in particular for this alpha what we have is E r P is greater than or equal to r L plus 1.645 sigma P. So, accordingly, once I have this condition as, so we can make that accordingly the portfolios which satisfy this constraint the most. So, amongst the portfolio whose satisfy this constraint the most desirable in accordance or as driven by the Telser's criteria is the one which has the largest expected return.

So, this means that basically you want to; you want to have this Telser's criteria being satisfied. And once this is satisfied your goal is essentially to maximize this E of r P. So, amongst all those portfolios you satisfy this criteria, the most desirable one is the one which is the highest E r P because eventually according to the criterion your goal is to maximize the E r P, ok.

So, now, we consider the example again that we have seen earlier. Now, since in this case I need to pre-specify my r L and alpha. So, I take my r L to be equal to say minus 2 percent and alpha equal to 5 percent, so that corresponds to 1.645. So, then what is going to be the E r P in all this cases alright. So, for portfolio A, B, C, and D. So, we have this 4 portfolios. So, what is going to be the E r P?

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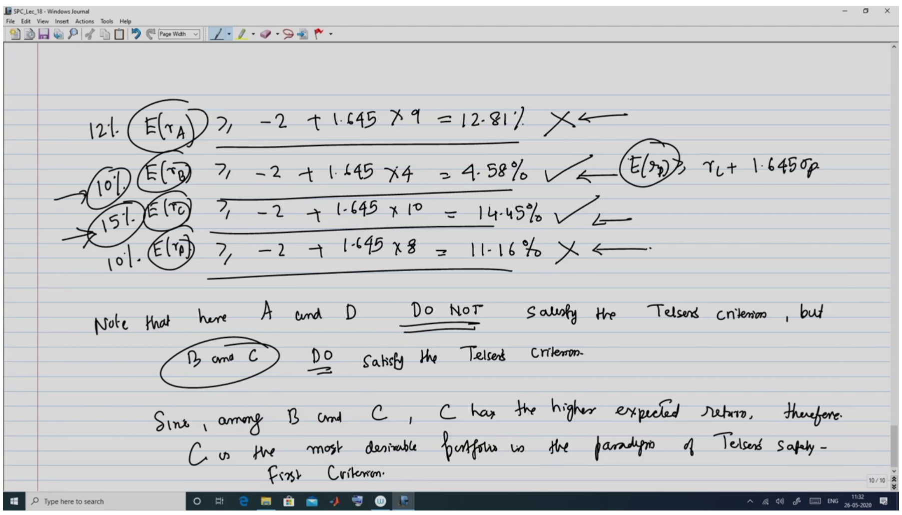

So, accordingly for the first portfolio, so we will enumerate the $E(r_A), E(r_B), E(r_C), E(r_D)$. All this will be greater than or equal to, what did you have, the condition was we had greater than equal to

$r_L + 1.645\sigma_P$. So,

$$E(r_P) \geq r_L + 1.645\sigma_P$$

So, what is the r_L? So, r_L, I have already specified is going to be minus 2 percent. So, this is pre-specified. So, accordingly this is going to be minus 2 percent let me put it minus 2 minus 2, minus 2, minus 2, plus 1.645, plus 1.645, plus 1.645, plus 1.645, and into sigma P. So, the sigma's were 9 percent, 4 percent, at 10 percent, and 8 percent.

So, in this case, this turns out to be equal to 12.81 percent. In the second case, this turns out to be equal to 4.58 percent; and the third case this turns out to be equal to 14.45 percent; and we have the last one to be 11.16 percent.

So, now, observe very carefully. 12 percent is not greater than or equal to 12.81 percent; and 10 percent is not greater than or equal to 11.16 percent; but 10 percent here is greater than 4.58 percent; and 15 percent here is greater than 14.45 percent. So, the even before we can start maximizing the E of r P, we have to check the situations where the Telser's criteria is not satisfied.

So, it turns out to me that that it is not satisfied in case of A, and it is not satisfied in case of D. So, accordingly where we state that here A and D do not satisfy the Telser's criteria, but B and C do satisfy that Telser's criteria.

So, now basically we have to choose one amongst B and C. Now, since among B and C, C has, so it is a 15 percent return which is higher than 10 percent; C has the highest or in this case higher expected return. Therefore, C is the most desirable portfolio in the paradigm of Telser's Safety First Criteria.

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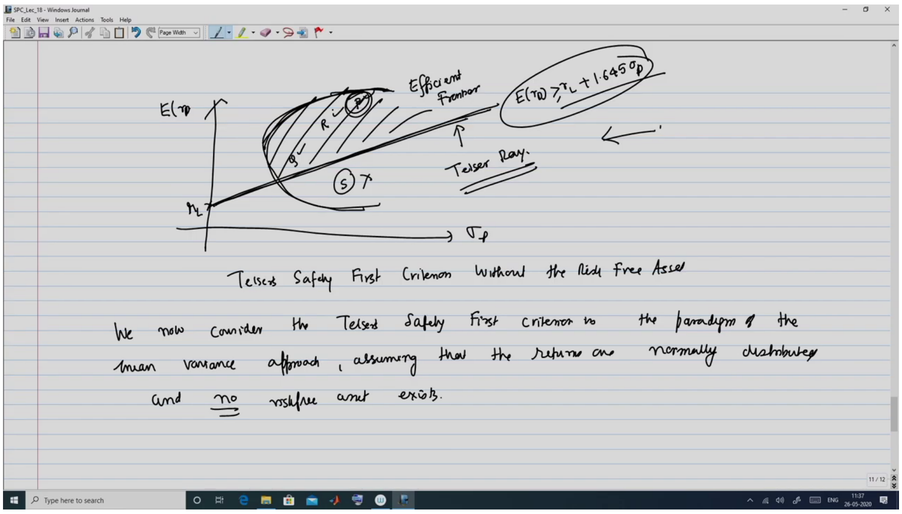

So, graphically you can see that, so let us look at Telser's Safety First Criteria without the risk free asset. So, that means, since there is no risk free asset, so the efficient frontier would look like this in the $[\sigma_P, E(r_P)]$ plane.

So, if we have some portfolio Q, R, P and there is some portfolio S. So, basically what happens is that we have the efficient frontier. And once we said that bring the Telser ray into the picture, we figure out that S does not satisfy that Telser criteria, and then only Q, R, and P satisfies the Telser criteria. And so you have to the most desirable portfolio amongst Q, R, P is the one that has the highest expected return namely P.

So, we can now summarize this as we consider that Telser's Safety First Criteria in the paradigm of the mean variance approach assuming of course that the returns are normally distributed, and no risk free asset exists. So, I can now make the following statement in the context of this picture that we have here.

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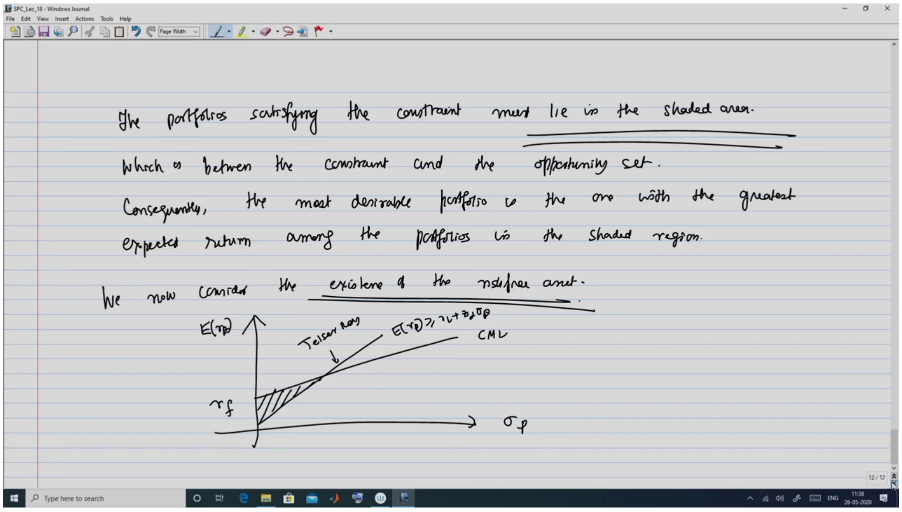

I can make the following statement that the portfolios satisfying the constraint given by the Telser's criteria that must lie in the shaded area right. And the reason I said that must lie in the shaded area because the shade this entire area will give the opportunity set or feasible set; and amongst those you can only choose the ones that lie above this Telser's Ray, so that means, that you can only consider those which are lying in the shaded area as given.

So, since the portfolios they are satisfying the constraint, this must lie on the shaded area, and this shaded area is between the constraint and the opportunity set. And so consequently the most desirable portfolio is the one with the greatest expected return among the portfolios in the shaded region.

So that means that you know that the portfolios in the shaded region, so this means that the portfolios in the shaded region are the ones who satisfy that Telser's constrained and once you have the collection of all the portfolios that satisfy that Telser constraint you then have to look at the objective of the Telser's criteria which is maximization of the expected return. So, amongst those which are lying in the shaded region, the most desirable portfolio is going to be the one which has the highest expected return.

So, accordingly if you go back to the figure, you will you will notice that this is going to be the portfolio. So, you would recall that in that case this is going to be this, so this is going to be the portfolio P which lies at the top alright. So, to conclude this, we now consider the existence of the risk free asset.

So, in this case, what you will have is we will have again you look at $[\sigma_P, E(r_P)]$ plane, now since we have considered now the existence risk free asset, so that means, the efficient frontier now is going to be no longer the previous one, but rather the CML emanating from r F, and then the Telser's Ray would be going to be given by.

So, this is Telser's Ray which is basically given by

$$E(r_P) \geq r_L + Z_\alpha \sigma_P$$

So, that means, the region above the Telser Ray, but below the CML, this is going to be my new shaded region.

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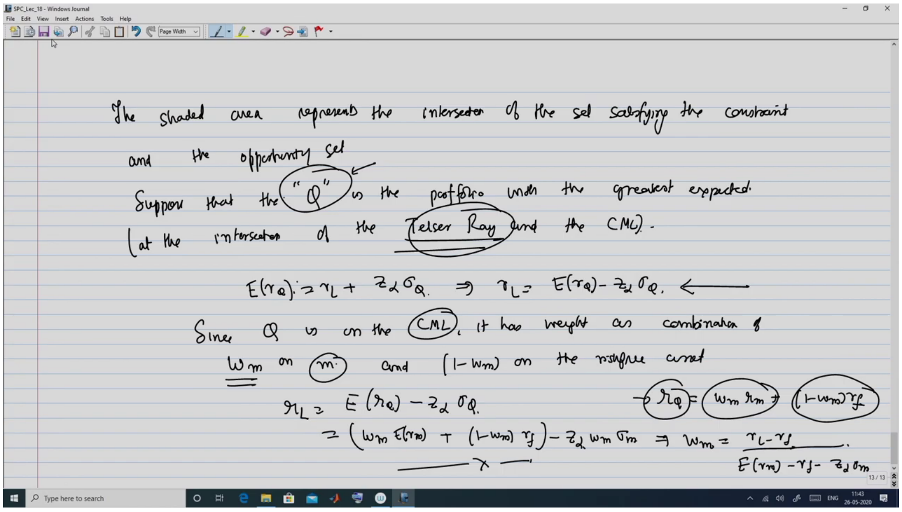

So, therefore, I can say that the shaded area represents the intersection of the set satisfying the constraint, that means, anything that is lying above this Telser Ray and the set and the opportunity set.

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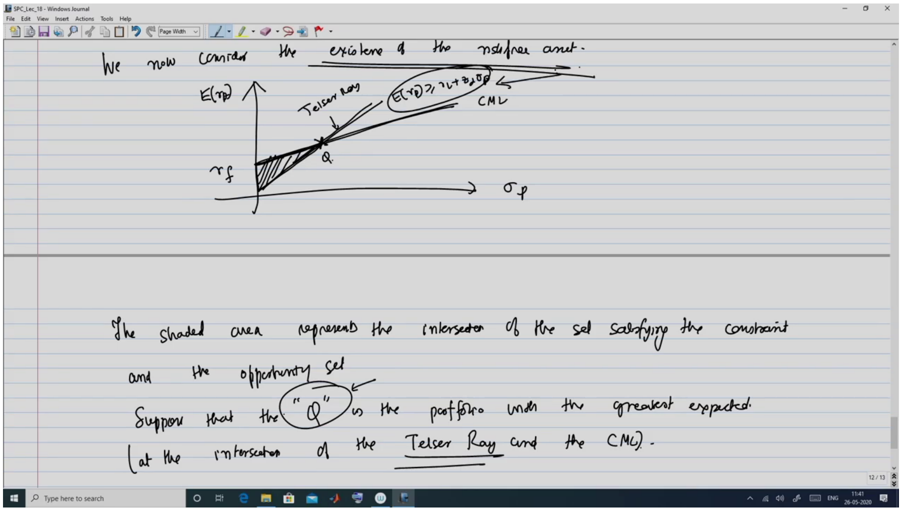

So, it is the intersection of the region above the Telser Ray and the opportunity set and this intersection is given by this shaded region alright. So, suppose that we have the portfolio suppose that what we have now is we now have basically a this shaded region, and suppose I choose that we find that this portfolio.

So, there is some portfolio Q not to be confused to the Q in the previous diagram. So, suppose that Q is the portfolio, so I denote by Q the portfolio with the greatest expected return. And what is this going to be, the portfolio the greatest expected return is going to be one the point of interest. So, amongst all the portfolio is here the one with the greatest return is going to lie here.

So, this portfolio the greatest expected return which is at the intersection of the Telser Ray and the CML, so. So, here I identify this with K. So, we make this observation, now we say that the what, so accordingly if I identify this with Q, so this may then you have to then now since this lies on the Telser Ray, so obviously, it will satisfy the Telser relation.

So, accordingly I can write this as

$$E(r_Q) = r_L + Z_\alpha \sigma_Q$$

So, this can be written as

$$r_L = E(r_Q) - Z_\alpha \sigma_Q$$

Now also remember that as Q not only lies on the Telser Ray, which is why this condition is satisfied, but also since Q is on the CML, it has weights as combination of w m on m. So, remember that any portfolio in the CML can be obtained as a combination of the market portfolio. So, let say there the weight of the market portfolio is w m, and 1 minus w m on the risk free asset.

So, then what do we have, then you will have the appropriate expression of r_L.

Finally, we have

$$w_m = \frac{r_L - r_f}{E(r_m) - r_f - z_\alpha \sigma_m}$$

So, this brings us to the end of today's lecture. So, just to do a recap of what we have done. So, we picked up from where we left up in the last lecture, and where we had done the Roy's criterion. And we introduced the Kataoka's criteria and we compare both of them.

So, in case of Roy's criteria, the intercept was fixed and then we the most desirable portfolio was going to be the one which is the highest slope. And in case of the Kataoka's criteria the slopes were fixed, and then amongst the different portfolios with the identical slopes you ended up picking the most desirable portfolio as the one which has the highest intercept.

And then we talked about the Roy's criteria which man sought to maximize the expected return given a certain constraint which was given by the relation

$$E(r_P) \geq r_L + Z_\alpha \sigma_P$$

And so where the probability that your return of the portfolio, so it is equivalent to the portfolio of the return probability of the return of your portfolio being less than some pre-defined criteria r L and you could not let that probability fall below in alpha beyond a certain level.

And we looked at the graphical illustration of both these criterias that you discussed together in the context of assuming that the returns are normally distributed which can be extended to returns of the distribution of the returns being not only just normally distributed, but also it could be any other distribution that relies only on the mean and variance.

And then you also looked at a graphical representation of the Telsar's criteria once you know in the case where we had the Markowitz framework without any risk free asset, and then in the context of the capital market line when a risk free as. So, in the first case no risk free asset was included, it was only risky assets; and the second case in case of the capital market line we implemented that Telser's criteria, and this had to be done in the context of the capital market line.

So, this concludes our discussion on the safety first criteria which was the one of the first topics that we had talked we had dwelled upon when you started our discussion on the new non-mean variance framework. And then in the next class onwards we will continue our discussion on other non mean variance approaches in order to optimize our portfolio.

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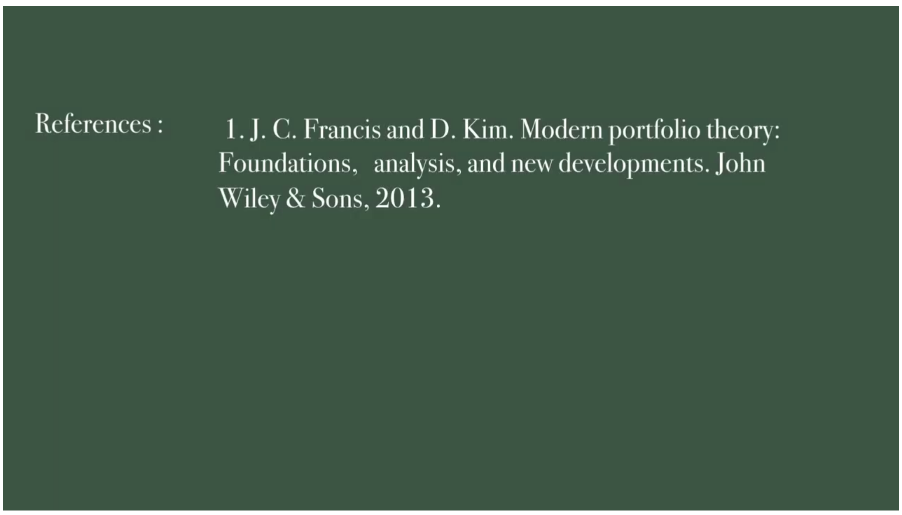

Thank you for watching.