Mathematical Portfolio Theory

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Module 04: Non-Mean-Variance Portfolio Theory
Lecture 09: Second order stochastic dominance and Third order stochastic dominance

Hello viewers, welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You recall that in the past few lectures we have been talking about the Non Mean Variance framework. And in the last lecture we started talking about stochastic dominance. And we identified that we are going to look at 3 different criteria, namely, first, second and third order stochastic dominance. And we had discussed in detail the criteria in explicit terms in terms of utility functions in the case of the first order stochastic dominance. And in todays class we conclude the discussion on this topic by looking at second order stochastic dominance as well as third order stochastic dominance.

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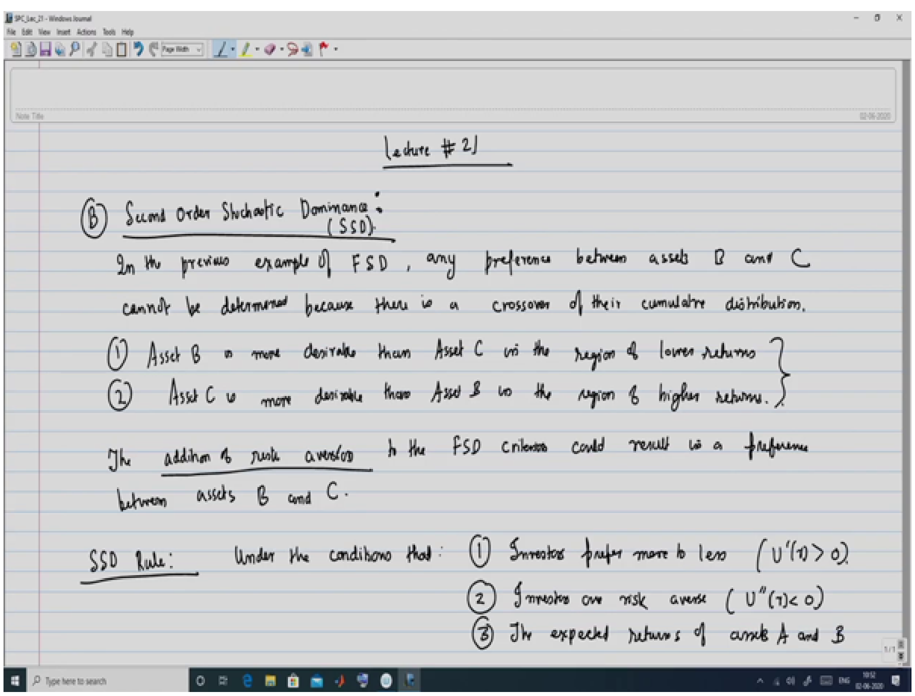

So, according to we start this lecture. So, we first start off with the topic of second order stochastic dominance. So, in the previous example of first order stochastic dominance, any preference using FSD between assets B and C cannot be determined, because there is a crossover of their cumulative distribution of returns. So, just to be a little more elaborate we observe that in that example asset B is more desirable than asset C in the region of lower values of r or lower returns. And the second observation is that asset C is more desirable than asset B in the region of higher returns. Now, what you can do is that the addition of risk aversion. So, we try to address this issue that we cannot determine which of B or C is preferable. In general, we can say that this is something that makes FSD unusable in case of cumulative distributions crossing each other. That means, in some region one of them is at a higher level than the other and this dominance reverses in some other range of the returns and so accordingly the addition of risk aversion to the FSD criteria could result in a preference between asset B and C. So, this means that while the first order stochastic dominance criteria is not applicable in this case. However, even in this case a preferential choice can be made based on a criteria under the assumption that we add one more condition to the basic assumption that were associated with FSD and that is going to be the criteria of risk aversion. So, it is the addition of risk aversion that enables us to move from first order stochastic dominance to second order stochastic dominance. So, for second order Stochastic Dominance we will denote this by SSD all right. So, let us now look at specifically the SSD rule by taking into account that we have added the risk aversion. So, we identify that under the conditions, that first of all investors prefer more to less. So, this is

$$U'(r) > 0.$$

So, this was already there and now on top of it we add another condition that investors are risk averse. So, risk averse means that we have

$$U''(r) < 0.$$

So, you remember the 3 cases we had talked about risk attitude of investors and thirdly the expected return of assets A and B this they are equal, that is

$$E_A(r) = E_B(r).$$

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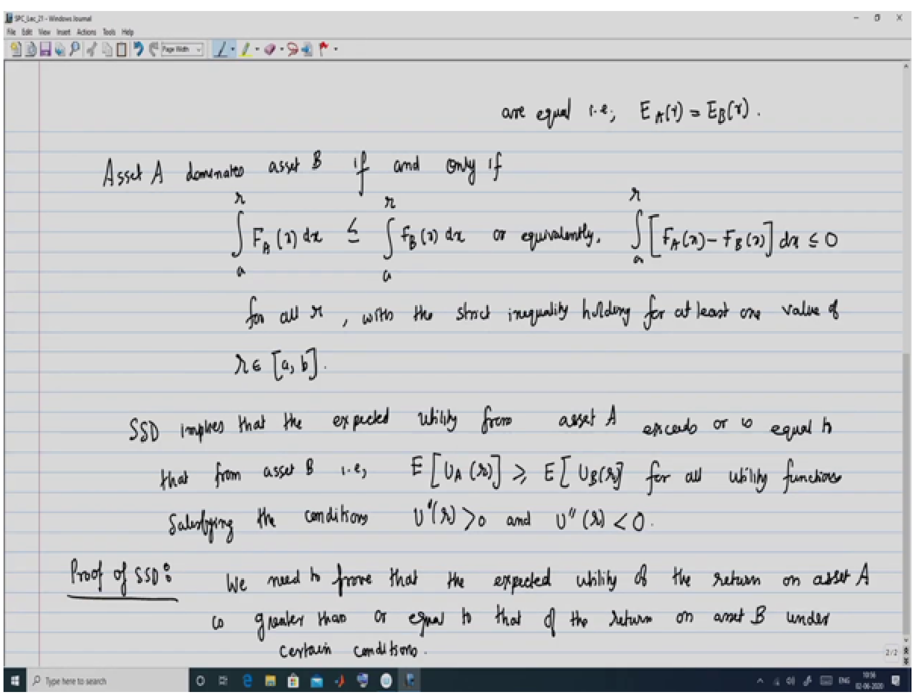

So, then what happens is that then we say that asset A dominates asset B if and only if

$$\int_a^r F_A(x)dx \leq \int_a^r F_B(x)dx.$$

or equivalently you can write this as

$$\int_a^r F_A(x)dx - \int_a^r F_B(x)dx \leq 0.$$

For all are with the strict inequality holding for at least one value of r in the range of the possible values of r. So, the second order stochastic dominance this implies that the expected utility from asset A exceeds or is equal to that from asset B. That is in mathematical terms the condition would be expected value of U A of r is greater than or equal to expected value of U B of r for all utility functions satisfying the conditions U prime of r greater than 0 which was already there and in addition for SSD we have U double prime of r is less than 0 all right. So, next what do you do so we next do the proof of SSD. So, what we do is we need to prove that the expected utility of the return on asset A is greater than or equal to that of the return on asset B under certain conditions.

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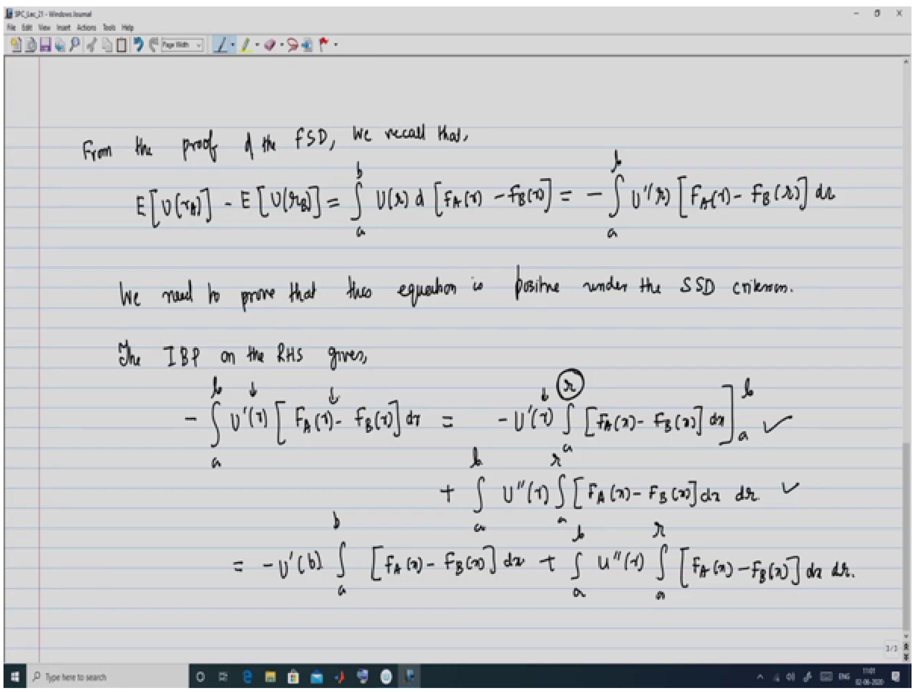

So, now proceeding with the proof we start off with where we had left in the proof for FSD. So, from the proof of the FSD the first order Stochastic Dominance, we recall that the expected utility on the first asset minus the expected utility on the second asset this is

$$E[U(r_A)] - E[U(r_B)] = \int_a^b U(r)d[F_A(r) - F_B(r)] = -\int_a^b U'(r)[F_A(r) - F_B(r)]dr.$$

So, what we need to prove? That we now need to prove that this equation is positive under the SSD criteria. So, in order to achieve that so what you do is that that the integration by parts which I will denote by IBP on the RHS of the above equation, this gives that we have this as integral minus of integral a to b U prime of r F A of r minus F B of r dr. This turns out to be so I will take this as the first function and this as the second function. So, this is going to be a first function into integral of second in the range a to b minus integral of derivative of first function. So, this minus and minus becomes plus U double prime of r into integral of second function that is integral from a to r of F A of x minus F B of x dx and then integral with respect to dr. Now, you observe carefully, so for the first term we will replace r equal to b. So, this gives us

$$-\int_a^b U'(r)[F_A(r) - F_B(r)]dr = -U'(b)\int_a^b [F_A(x) - F_B(x)]dx + \int_a^b U''(r)\int_a^r [F_A(x) - F_B(x)]dxdr.$$

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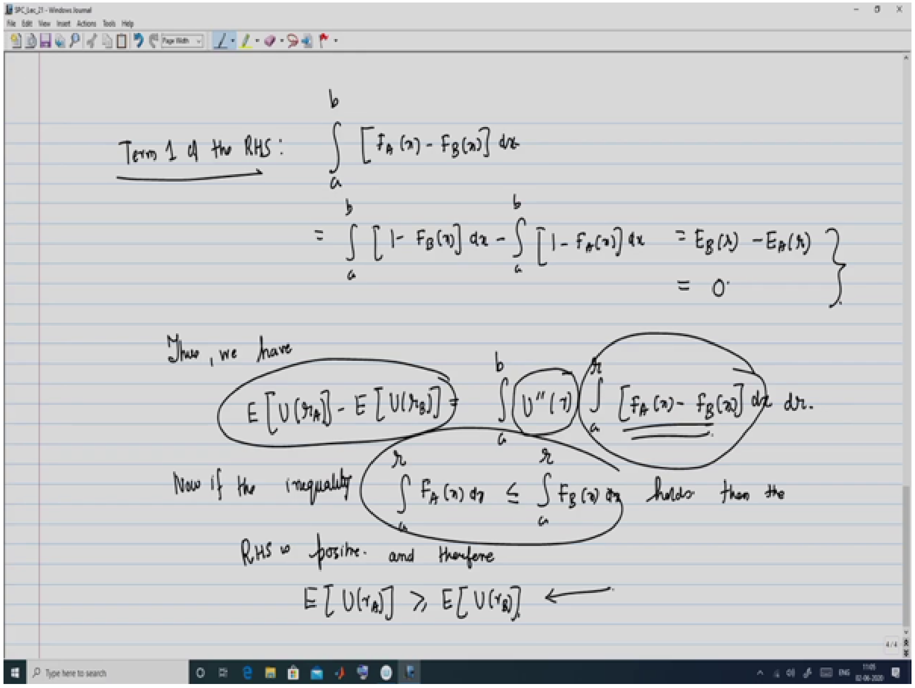
So, now what we do is we look at term 1 of the RHS. What was the term 1 on the RHS? It was this term. So, here in the term one of the RHS we just consider the integral that is integral a to b F A of x minus F B



of x dx. So, what does this become? This becomes integral a to b 1 minus FB of x into dx minus integral a to b 1 minus F A of x dx. And this is nothing but E B of r minus E A of r and this is equal to 0, because we assume that both the expectations are identical. So, thus the first term above vanishes and we only have the second term as given here. So, therefore we get

$$E[U(r_A)] - E[U(r_B)] = \int_a^b U''(r) \int_a^r [F_A(x) - F_B(x)]dxdr.$$

Now, if the inequality of integral a to r F A of x dx is less than or equal to integral of a to r F B of x dx holds, then this term that we have here. So, then the RHS is positive and therefore

$$E[U(r_A)] \geq E[U(r_B)].$$

So, if we have this relation holds then this particular integral is going to become negative and U double prime of r this is negative by the fact that the investor is risk averse. So, accordingly the entire expression is going to be positive. So therefore, this difference is going to be positive and hence we arrive at the final conclusion that the expected utility of returns on asset A is going to be greater than or equal to the expected utility on return B on return on asset B. And that is how thus the condition for second order stochastic dominance is proved.

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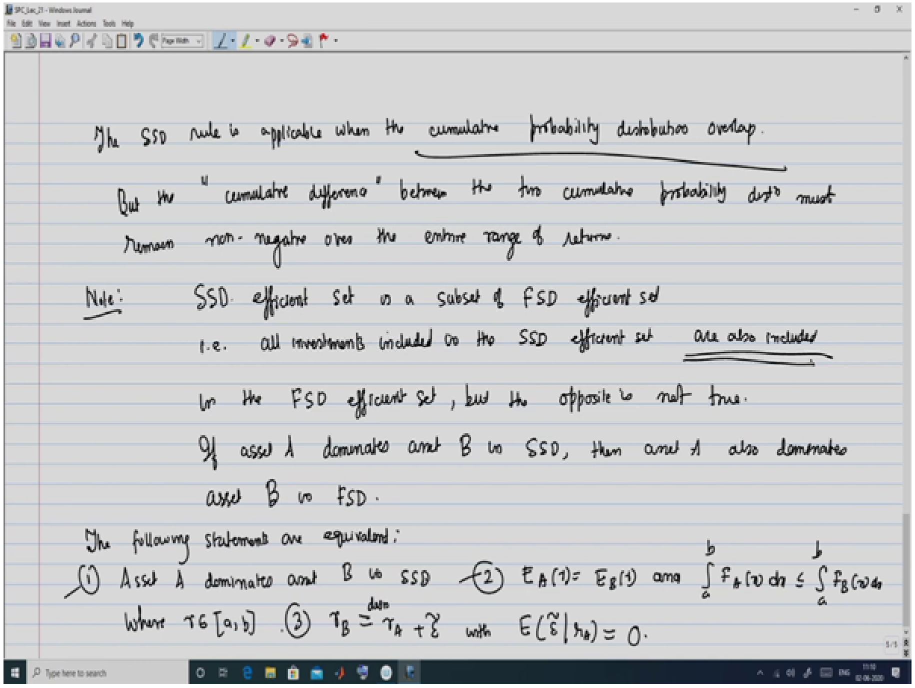
So, now a few observations. So, the second order stochastic dominance rule is applicable when the cumulative probability distribution overlap. So, remember that the reason why we moved on to second order stochastic dominance by adding the condition of U double prime being less than 0, is that the first order stochastic dominance cannot be applied whenever the cumulative probability distribution they overlap or they cross each other. However the second order stochastic dominance rule remedies that, of course you know with the additional assumption of the investor being risk averse and it is applicable in the case of scenarios where the cumulative distributions actually overlap. Having said so I just want to make one more observation and that is that but the cumulative difference. So even though it is applicable in the case when the cumulative probability distributions overlap, but we still need that the cumulative difference between the two cumulative. So, it is basically the cumulative difference of the two cumulative probability distribution must remain non negative over the entire range of returns. So, now just we note the following that now that we have the set up for SSD completely in place. So, which you can now make a comparative analysis in



terms of the efficient set. So, it is observed that the SSD efficient set is a subset of FSD efficient set. That is this means that all investments included in the SSD efficient set are also included in the FSD efficient set. But the opposite is not true. So, this means that in other words if asset A dominates asset B in the sense of SSD, then asset A also dominates asset B in FSD. So finally, we have the following statement as we have done in the case of FSD. So, the following statements are equivalent. So, this is similar to the equivalent statements we had given in the case of FSD. So, the first statement is that asset A dominates asset B in SSD. Secondly,

$$E_A(r) = E_B(r)$$

and

$$\int_a^b F_A(x)dx \leq \int_a^b F_B(x)dx, \text{ where } r \in [a,b],$$

and the third statement is that

$$r_B = r_A + \tilde{\epsilon} \text{ with } E(\tilde{\epsilon}|r_B) = 0.$$

So, these three statements are equivalent to each other, all right.

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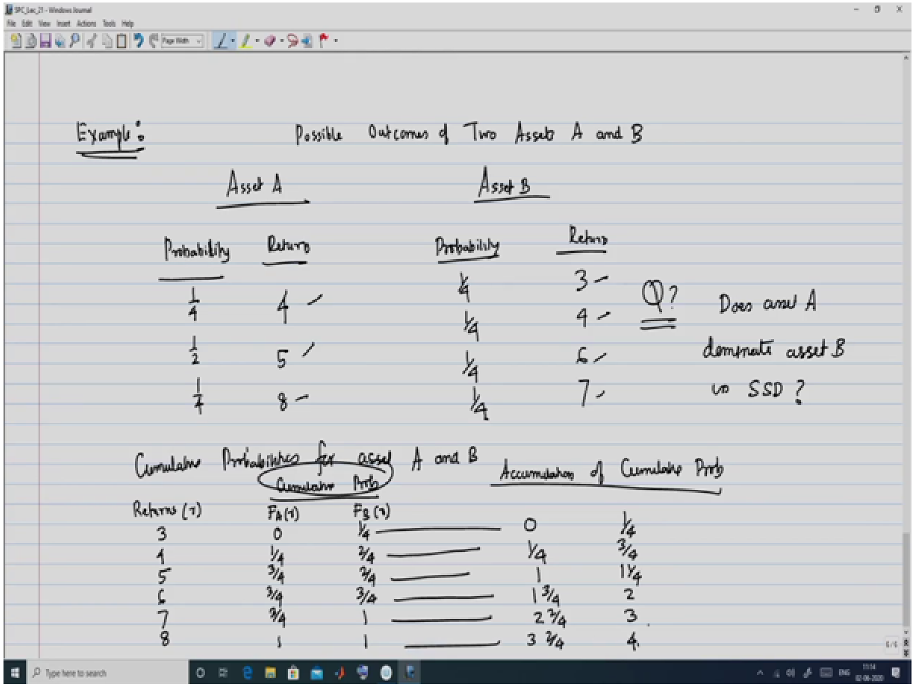
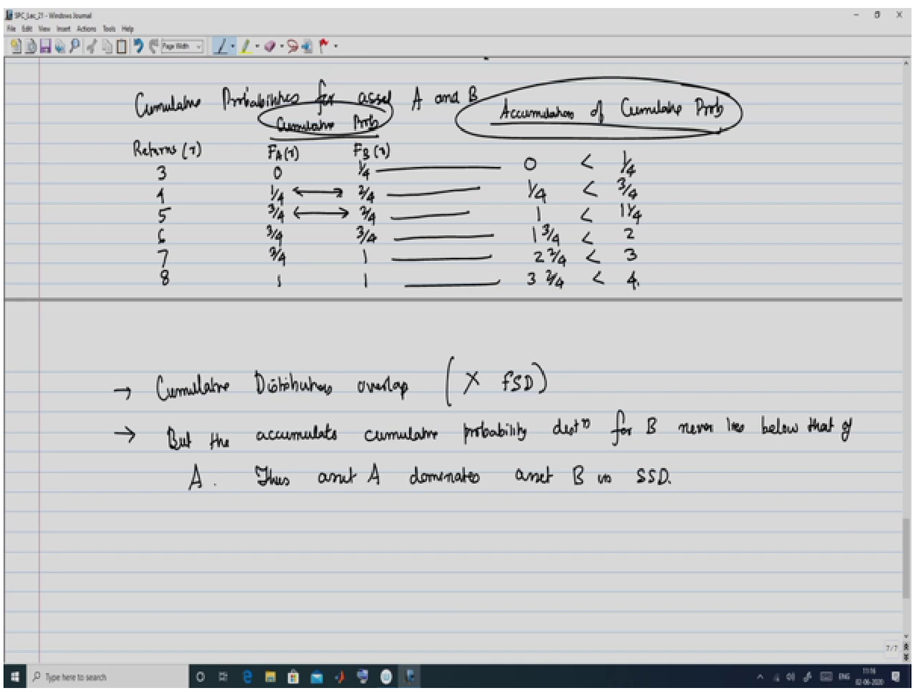
So, now it is time to look at an example to ascertain the desirability of portfolios in the SSD framework. So, we consider this example of possible outcomes of two assets A and B. So, first we enumerate the distribution of returns of asset A. So, asset A takes the returns of 4 5 and 8 with the respective probabilities of 1 by 4 half and 1 by 4. And similarly in the case of asset B we look at the returns 3 4 6 and 7 with respective probabilities of 1 by 4 in each of the cases. And the question that I want to answer in the context of this discussion is that does asset A dominate asset B in the paradigm of SSD. So, this is the question that I want to answer. So this so for this what we need is that we need to look at cumulative probabilities for asset A and B. So, what are the returns? So, let me first enumerate the returns. So, the returns are 3 4 5 6 7 and 8. So, the returns are 3 4 5 6 7 and 8. Now, what is going to be the cumulative probabilities? So, cumulative probabilities that means my FA or r and FB of r. So, this is going to be FA of r is going to be 0, 1 by 4, 3 by 4, 4 by 4, 3 by 4 and 1 and the cumulative probabilities of B this is 1 by 4, 2 by 4, 2 by 4, 3 by 4 1 and 1. And then what you do is that we will take an accumulation of these cumulative probabilities. And what is the accumulation going to be? The accumulations are going to be 0 and 1 by 4 1 by 4 and 3 by 4 1 and 1 1 by 4 1 3 by 4 and 2 2 2 by 4 that is 2 and half 3 and 3 and half and 4.





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So, here what we observe that there are two observations. So, we have that the cumulative distributions overlap. So, you see here 1 by 4 is less than 3 by 4, but here you observe that 3 by 4 is greater than 2 by 4. So, there is an overlap. So obviously, we cannot work out using we cannot carry out the desirability test using first order stochastic dominance and so accordingly we have to look at the accumulated cumulative distribution. So, then we can say that the accumulated cumulative probability distribution for B this will this never lies below that of A. So, if you observe carefully that the cumulative distribution of B see 1 by 4 here is greater than 0, 3 by 4 is greater than 0 1 1 by 4 is greater than 0 2 is greater than 1 3 by 4 3 is greater than 2 2 by 4 and 4 is greater than 3 2 by 4. So, you can see that the accumulated distribution of b this is never below that of A and thus we conclude that asset A dominates asset B in SSD. So, a very sort of straightforward application is that when you are trying to look at the dominance in FSD if you need to first observe whether there is a crossover, if there is no crossover of course, you can straight away use the FSD.

But if there is a crossover like the example that you have considered, then FSD is no longer applicable and in that case we have to look at the accumulated values of the cumulative distribution. And if it turns out that for each of the returns the accumulated cumulative distribution for one asset is no less than that of another one, then obviously, you can have a clear cut criteria of figuring out which of the two assets is desirable. But, this time in the in the sense of second order stochastic dominance. So, we now move on to the last of the three stochastic dominance criteria that we had begun to explore and this is what is known as the third order stochastic dominance criteria.

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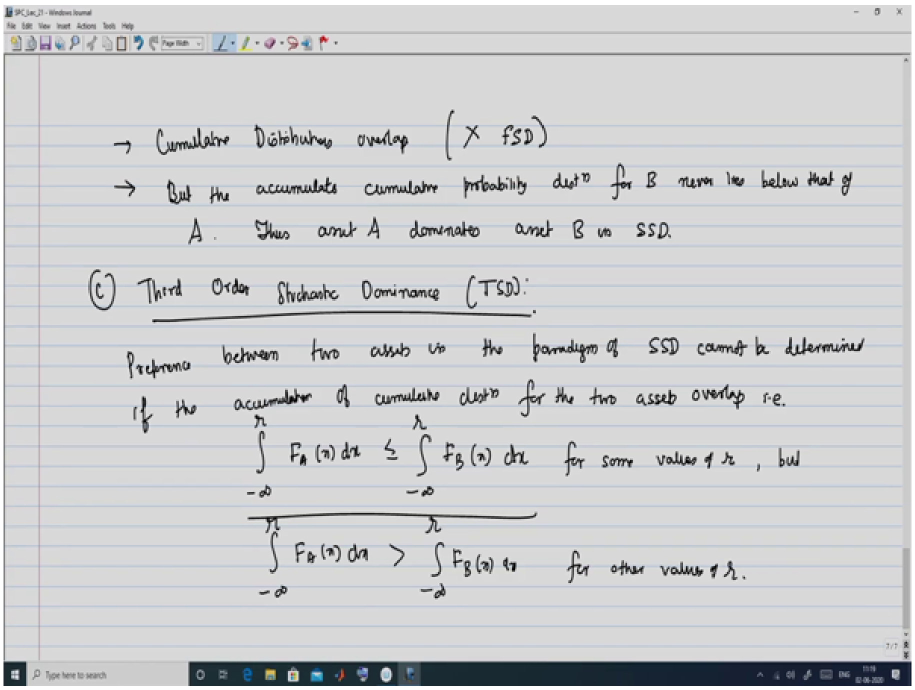

So, accordingly we start our discussion on Third order Stochastic Dominance criteria and we will denote this by TSD. So, let me begin with the motivation for this. So, the preference between two assets in the paradigm of second order stochastic dominance this cannot be determined, if the accumulation of cumulative distribution for the two assets overlap. So, that is so earlier what we had for second order stochastic dominance we had this criteria that from we had

$$\int_{\infty}^{r} F_A(x)dx \leq \int_{\infty}^{r} F_B(x)dx,$$

for all possible values of r. But if it turns out that this is not the case that is there is an overlap in the sense that this inequality holds for some values of r in the range for r, but the opposite holds. That means,

$$\int_{\infty}^{r} F_A(x)dx > \int_{\infty}^{r} F_B(x)dx,$$

for other values of r. Then obviously we are no longer in a position to make use of the SSD criteria. So, it is under this circumstance that we resort to the TSD criteria.

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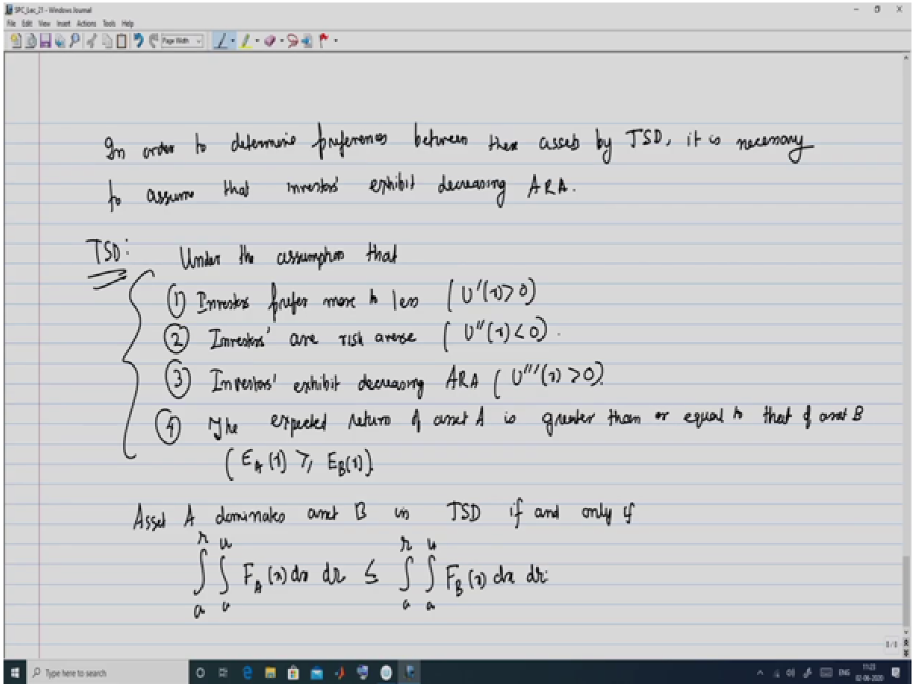
So, in order to determine preferences between these assets by the third order stochastic dominance, it is necessary to make one more assumption that investors exhibit decreasing absolute risk aversion. So, now we are in a position to define what is the third order stochastic dominance. So, under the assumption that firstly investors prefer more to less. So, remember this is U prime of r greater than 0 which you used in FSD and then we bring the risk aversion property, that risk averse investors are risk averse that was introduced for SSD and this was equivalent to U double prime of r less than 0. Thirdly we have the new condition that



is applicable in the case of TSD, that investors exhibit decreasing ARA that is U triple prime of r is greater than 0. And the last condition is that the expected return of asset A is greater than or equal to that of asset B. So that means, EA of r is greater than or equal to EB of r. So, then under this following assumptions we say that asset A dominates asset B in TSD, if and only if we have the following criteria that

$$\int_a^r \int_a^u F_A(x)dxdu \leq \int_a^b \int_a^u F_B(x)dxdu.$$

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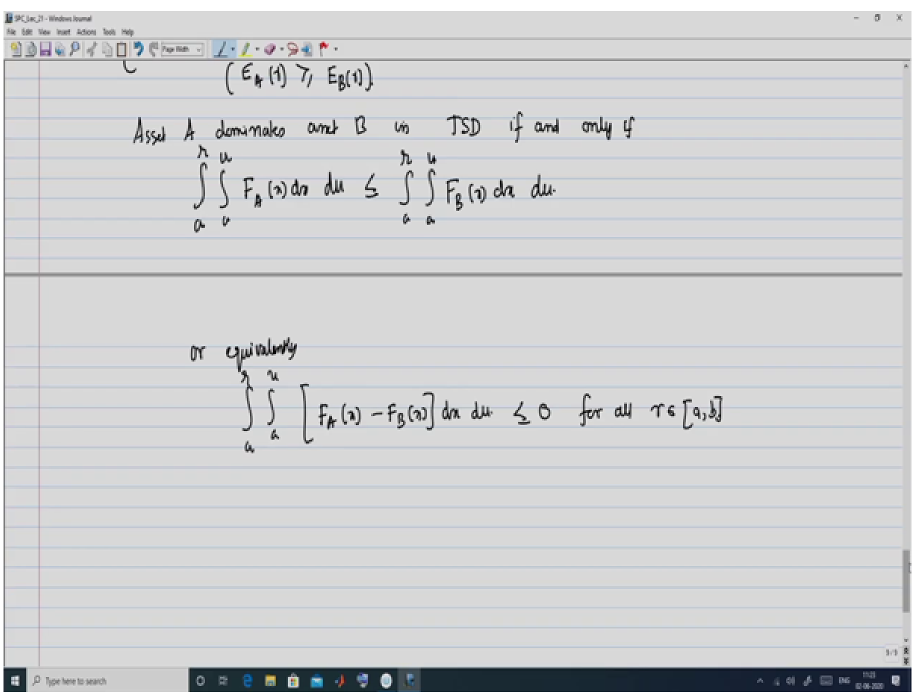

Or equivalently this can be written as

$$\int_a^r \int_a^u [F_A(x) - F_B(x)]dxdu \leq 0, \text{ for all } r \in [a, b].$$

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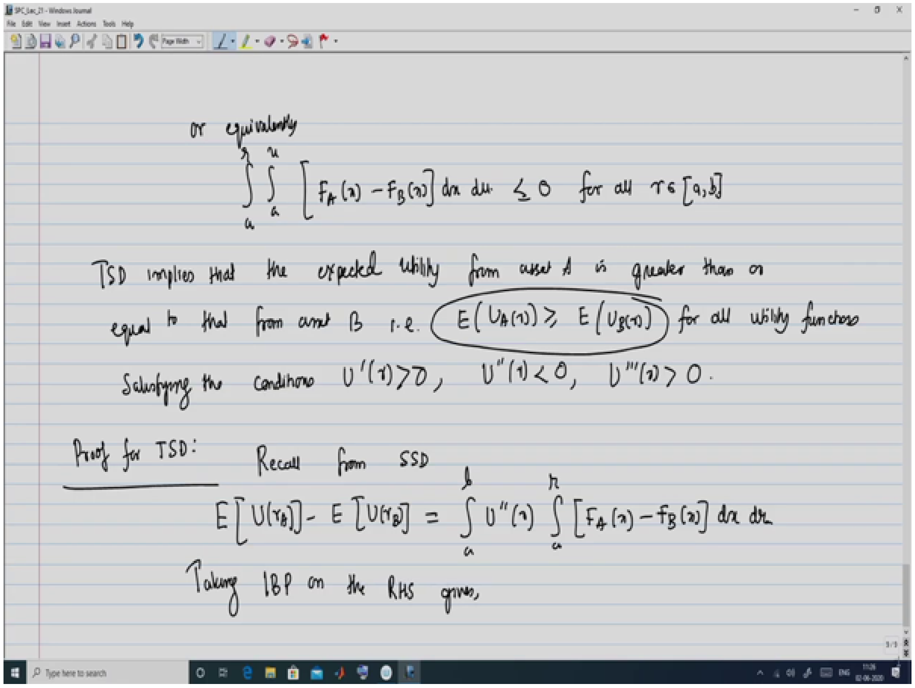

So, what does the TSD do is that it implies that the expected utility from asset A is greater than or equal to that from asset B. That is expected utility of A is greater than or equal to expected utility of B for all utility functions satisfying the condition of U prime of r greater than 0 which you had in the case of FSD. Then, U double prime of r less than 0 which you had in the case of SSD and you triple prime of r greater than 0 which was introduced in the case of TSD. So, let us go with the proof for TSD. Now, since we want to prove that expected utility of A is greater than or equal to B. So, as before we now recall from SSD an expression for the difference of the two. So,

$$E[U(r_A)] - E[U(r_B)] = \int_a^b U''(r) \int_a^r [F_A(x) - F_B(x)] dx dr.$$

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So, taking IBP on the RHS gives

$$\int_a^b U'(r).$$

So, after we take the integration by parts this becomes

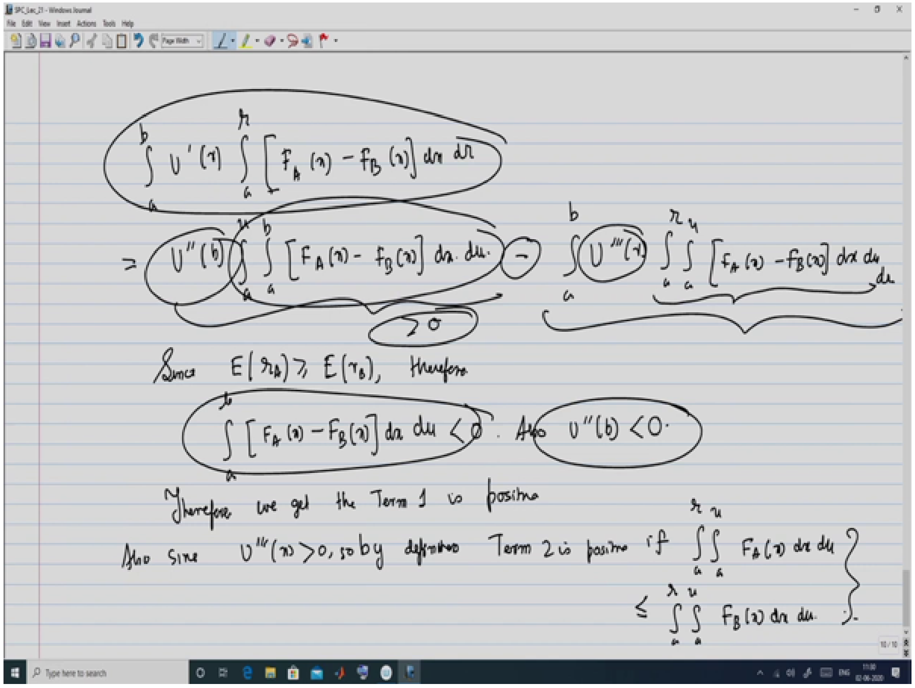
$$\int_a^b U'(r) \int_a^r [F_A(x) - F_B(x)] dx dr.$$

i So, this is the expression on the RHS. So now, we are applying the integration by parts, so this becomes

$$U'(b) \int_a^u \int_a^b [F_A(x) - F_B(x)] dx du - \int_a^b U'''(r) \int_a^r \int_a^u [F_A(x) - F_B(x)] dx du dr.$$

Now, by assumption since the expected return of asset A is greater than or equal to the expected return of asset B. Therefore, what we have is that

$$\int_a^b [F_A(x) - F_B(x)] dx du < 0.$$



Also

$$U''(b) < 0.$$

Therefore combining these two we get that this term is positive. So, this term is greater than 0, all right. So, next now come to the next term. So, also since U triple prime of x is greater than 0. So by definition term 2 that means this term 2 will be positive, if this integral is greater than 0. So that means, that if

$$\int_a^r \int_a^u F_A(x)dxdu \leq \int_a^r \int_a^u F_B(x)dxdu.$$

So that means that it is automatically obtained that this is less than 0. So, this integral is going to be less than 0 and

$$U''(b) < 0.$$

So, this is less than 0, so the entire expression is going to be greater than 0. And then what we have is we want these expression. So, here this is positive and there is a negative outside. So, as long as this quantity is positive, if I have this condition holding.

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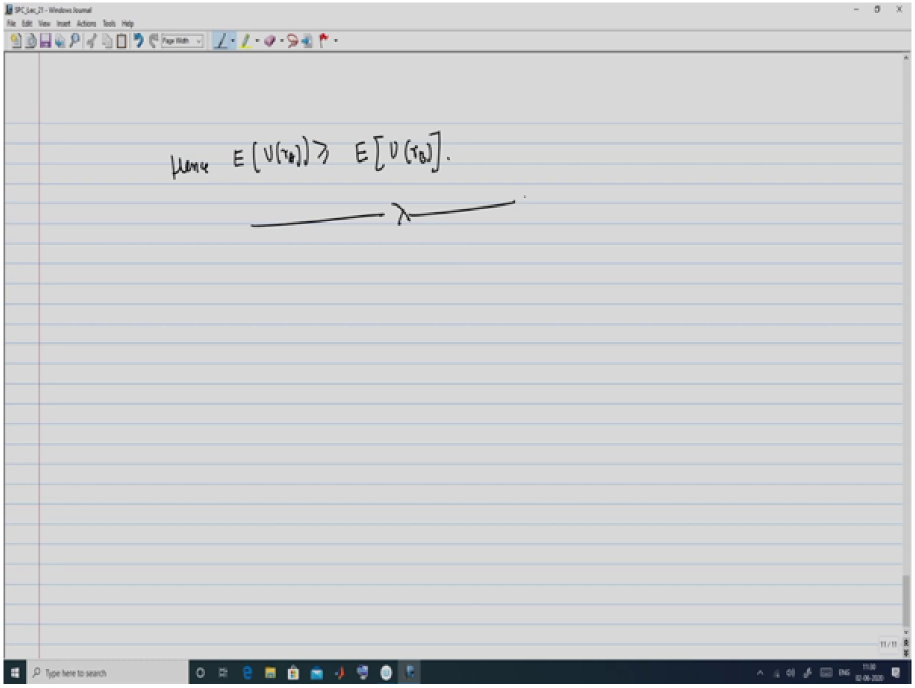
So, hence

$$E[U(r_A)] \geq E[U(r_B)].$$

So, this brings us to the end of these two lectures that dealt with stochastic dominance. One common threat in all of them was that eventually the stochastic dominance will be viewed in terms of the expected utility of one asset being more than that of the another asset. So, in all the cases you observed that an asset A dominates an asset B if the expected utility of A is greater than or equal to expected utility of B. However, the only difference as we move on from each of the criterias from the first order to second to the third order, we have to bring in additional conditions in terms of the utility functions. So, in the first case what we had is that the investors work was want more for their investment. So, you had $U'(r) > 0$. now in case the first order dominance criteria can is not applicable due to the crossover of cumulative distribution. So, in that case we had to move on to second order stochastic dominance, where we had risk aversion as given by $U''(r) < 0$. Now, further if the cumulative distribution also exhibits a crossover, then we have to move on to third order stochastic dominance in which we had to bring in one more criteria that is of absolute risk aversion which was equivalent to $U'''(r) > 0$. So, these 3 criterias are very important criterias in the



paradigm of the norm invariance framework and with this we come to the conclusion of our discussion on the non-mean variance framework and then we will move on to the next topic which will be on the optimization of portfolios in discrete and continuous time.

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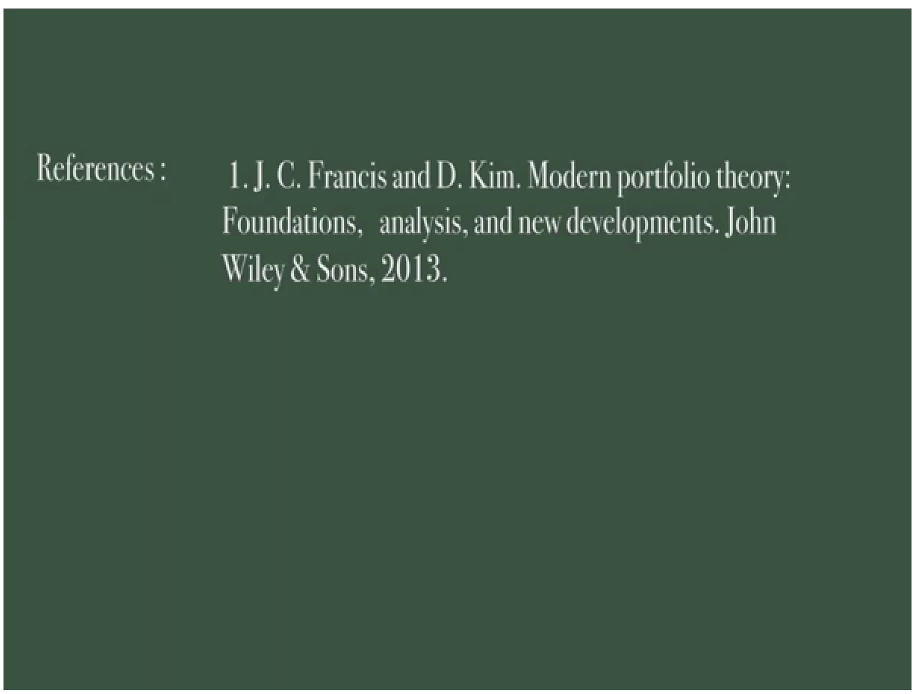

Thank you for watching.