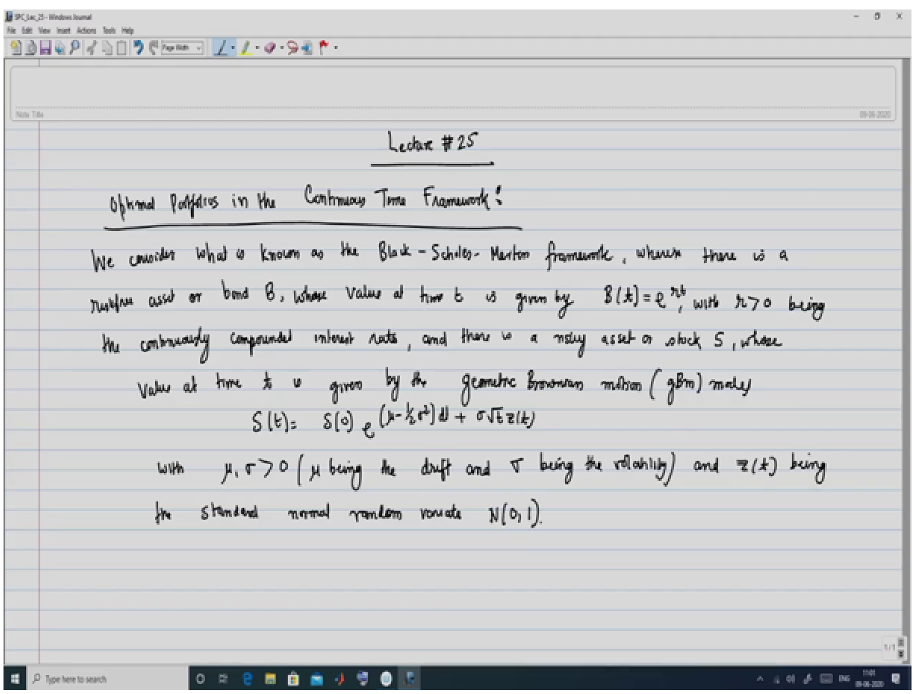
Mathematical Portfolio Theory

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Module 05: Optimal Portfolio and Consumption
Lecture 04: Continuous time model: Hamilton-Jacobi-Bellman PDE

Hello viewers, welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You would recall that until the previous class we had looked at the discrete time setup. And for that purpose we looked at a model wherein we had investment in the bond; as well as a stock with the stock price process being modeled by the binomial model. And then what we did is that, we looked at a single period portfolio optimization and then recognizing that the adaptation of the single period portfolio optimization to the multi period setup is not as straightforward. And for which the dynamic programming principle was proposed. And the dynamic programming principle enabled the multi-period optimization problem being divided into optimization problem over each of the individual single periods that are that constitute this multi period setup. And then we looked at a couple of applications of the dynamic programming principle with specific utility functions. So, accordingly in todays class we are going to now move on from discrete time setup in terms of portfolio optimization to the Continuous time setup with an emphasis on Hamilton Jacobi Bellman equation.

(Refer Slide Time: 01:49)

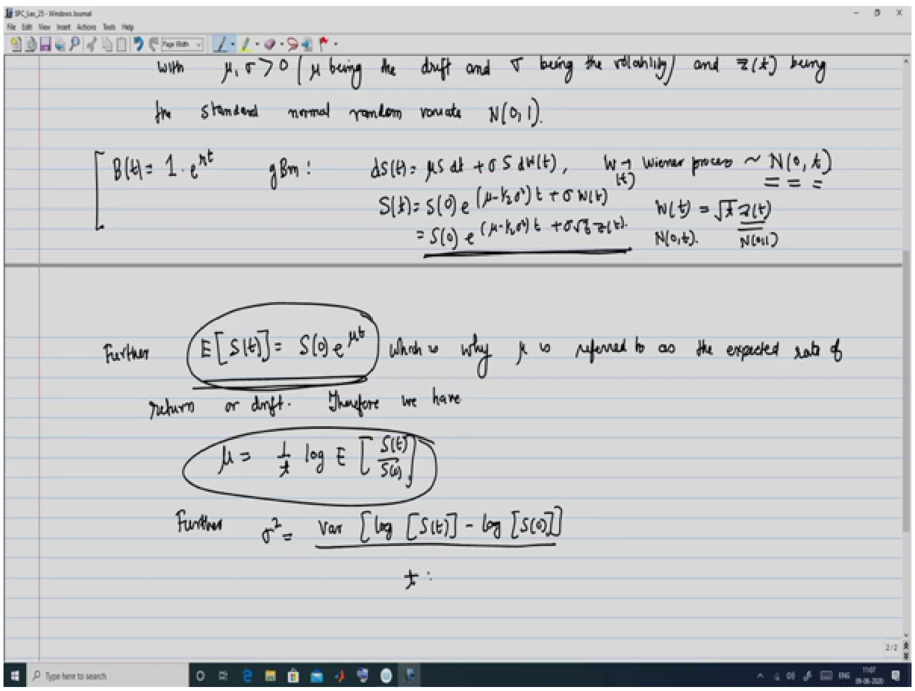


So, accordingly we begin todays lecture on a optimal portfolios in the continuous time framework. So, we begin by saying that we consider; what is known as the Black Scholes Merton framework. Wherein, there is a risk free asset or bond B; whose value at time t is given by and we assume that the initial investment on the bond is 1. So, it is value at time t is given by B t is equal to e raised to r t with r greater than 0; being the continuously compounded interest rate. And there is a risky asset or stock S whose value at time t is given by the geometric Brownian motion or g B m model.

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)dt + \sigma\sqrt{t}z(t)}.$$

with $\mu,\ \sigma\ > 0$, μ being the drift and σ being the volatility. And $z(t)$ being the standard normal random variate $N(0, 1)$.

(Refer Slide Time: 05:31)



So, let me elaborate a little more on this. So, if I the bond price at time t this is on the assumption that you start off with an initial amount of 1 and end up with 1 into e raised to r t at time t. Now, as for the geometric Brownian motion you would recall that this was given by

$$dS(t) = \mu S dt + \sigma S dW(t),$$

where your W was the winner process, which follows $N(0, t)$ distribution. That means, a standard a normal distribution with mean 0 and variance t. So, the solution for this is S of t is equal to

$$S(0)e^{(\mu - \frac{1}{2}\sigma^2)t} + \sigma w(t).$$

Now, we can replace w t by square root of t so just a correction square root of t into z of t. So, if z of t follows $N(0, 1)$ distribution. So, it automatically follows that w follows $N(0, t)$ distribution. So, that is how we get this relation. So, that is this is equal to

$$S(0)e^{(\mu - \frac{1}{2}\sigma^2)t} + \sigma\sqrt{t}z(t)$$

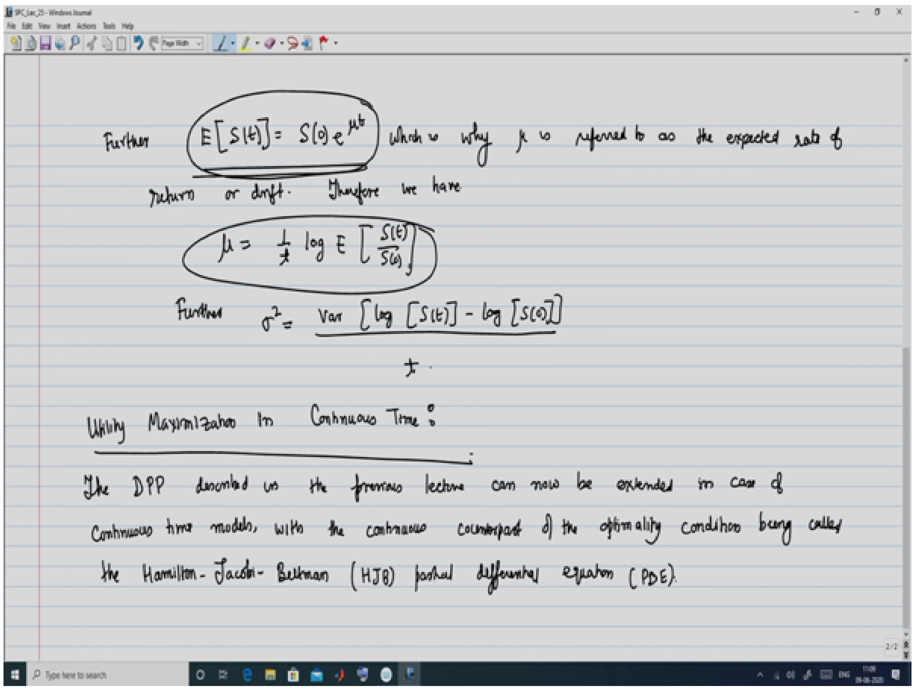
So, further we can show that the expected value of S of t is going to be

$$S(0)e^{\mu t}$$

which is why μ is referred to as the expected rate of return or drift. So, if we look at this relation here you can show that mu is going to be nothing but 1 over t into log of expected value of $\frac{S(t)}{S(0)}$. So, from this relation you can obtain this relation and by taking log on both sides by we can obtain that mu is equal to 1 over t into log of the expected value of S t over S 0. And likewise, you can also obtain that

$$\sigma^2 = \frac{Var[\log(S(t)) - log(S(0)]}{t}.$$

(Refer Slide Time: 08:39)



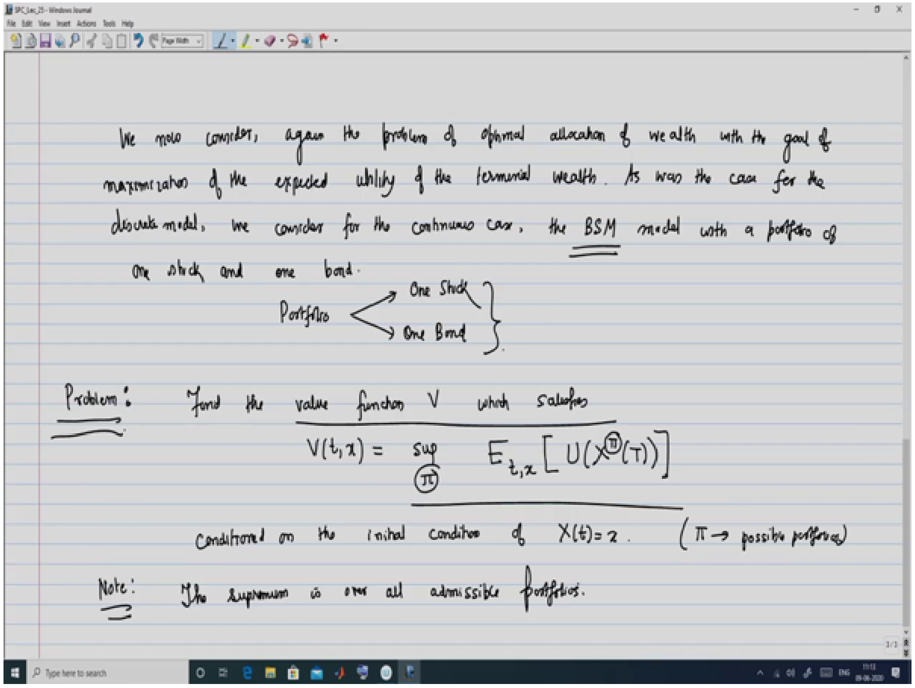
We now move on to the main problem in hand that is the utility maximization in continuous time. So, we begin with a motivation by pointing out that the dynamic programming principle described in the previous lecture; can now be extended in case of continuous time models, with the continuous counterpart of the optimality condition being called the Hamilton Jacobi Bellman abbreviated as HJB Partial Differential Equation PDE.

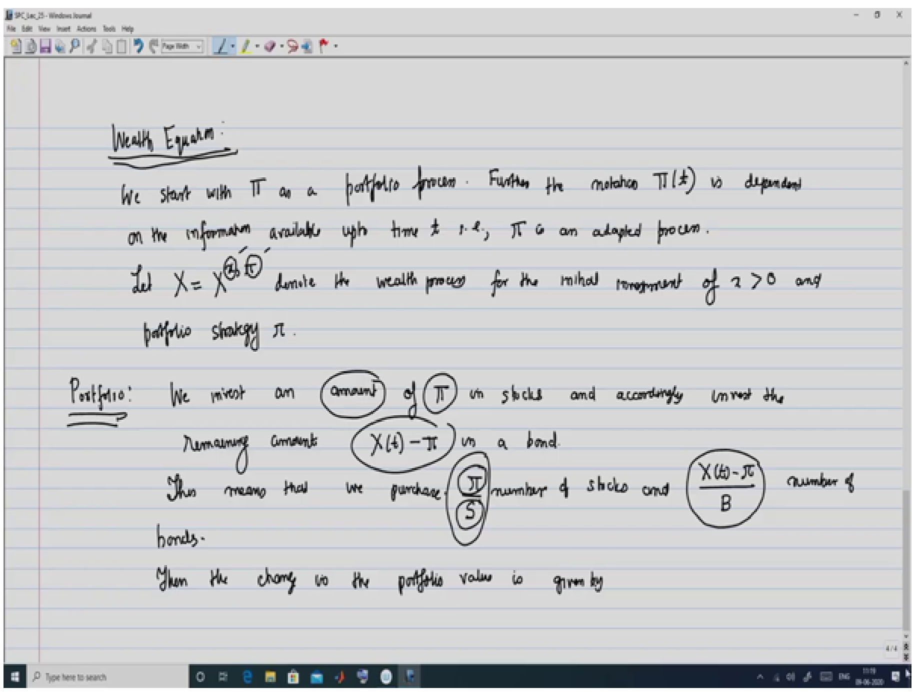
(Refer Slide Time: 10:32)

So, we now consider; again the problem of optimal allocation of wealth with the goal of maximization of the expected utility of the terminal wealth. As was the case for the discrete model, we consider for the continuous case. The black Scholes BSM will be the abbreviation for Black Scholes Merton model with a portfolio of one stock and one bond. So; that means, your portfolio will have one stock and one bond. So, let us state the problem in more precise mathematical terms. And the problem is find the value function V which satisfies V of t x. Remember that we had talked about the value function in the discrete case and this is going to be the supremum of the utility of the final wealth. So, final wealth is $X(T)$ and this utility is $U(X(T))$ and this is a random variable. So, I need to consider it is expectation condition that at time t the wealth level available is x and this has to be run over all the possible portfolio spy that can be held. So, this is so in summary this means that we need to find the supremum of this quantity. And the conditional expectation indicates that this is conditioned on the initial condition of X of t is equal to x and here pi is basically all possible portfolios. Now, here note that; the supremum is over all admissible portfolios.

(Refer Slide Time: 14:19)

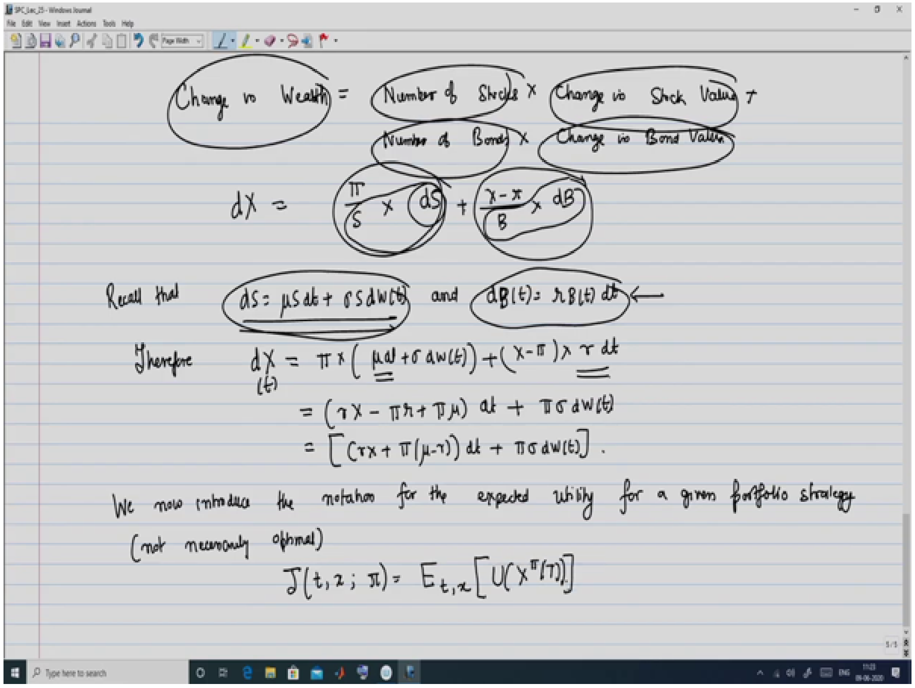
Now, let us move on to the wealth equation just like we had done in the discrete case. So, this will give us a more precise structure of the portfolio. So, accordingly we start with pi as a portfolio process. Further the notation and we will introduce the notation of pi as being dependent on time t. This notation is dependent on the information available up to time little t. That is pi is an adapted process which is driven by what happens over time. Further, we let X which is the wealth variable be denoted by X raised





to superscript little x and pi. And this notation denotes the wealth process which specified x and pi for the initial investment of x being greater than 0 and portfolio strategy π. So, let us now be a little more specific about what our portfolio is going to be and what is this pi. So, the portfolio set up as follows that we invest an amount of pi. So, this is the absolute amount unlike delta which was the number of stocks in the in the discrete case so we invest this amount of a pi in stocks. And accordingly invest the remaining amount what is going to be the remaining amount? So, remaining amount will be X of t minus the amount pi invested in stocks and this amount is invested in a bond. This means that we purchase so if you have if we invest π in stocks. So, that means that we will purchase pi over S where S is the price of the stock. So; that means, the number of stocks that you can purchase is the total amount invested in stock divided by S which will denote the price of the stock. So, we will have pi over S number of stocks and will have $X(t) - \pi$ amount invested in bond divided by B that is the price of the bond. So, this ratio is going to denote the number of bonds. So, then the change in the portfolio value is given by the following principle.

(Refer Slide Time: 18:19)



And the principle is the following that change in wealth is equal to number of stocks multiplied by change in stock value plus number of bonds into change in bond value. So, the change in wealth this is going to be d X, change in stock value this is going to be d S, change in bond value is going to be d B. So, d X will be equal to what is the number of stocks? We have the number of stocks to be pi over S. So,

$$dX = \frac{\pi}{S} \times dS + \frac{X - \pi}{B} \times dB.$$

Now, we recall that

$$dS = \mu S dt + \sigma S dW(t)$$

and

$$dB(t) = rB(t)dt.$$

So, therefore, d X so I am going to replace the value of d S as given by this relation actually I am going to replace d S over S and this value of d B over B will be replaced by the value of d B over B given from here. So, accordingly

$$dX = \pi(\mu dt + \sigma dW(t)) + (x - \pi)r dt.$$

So, I collate the terms which have d t in them. So, this becomes r X minus pi r plus pi mu into d t plus pi sigma into d w of t. And this is nothing, but

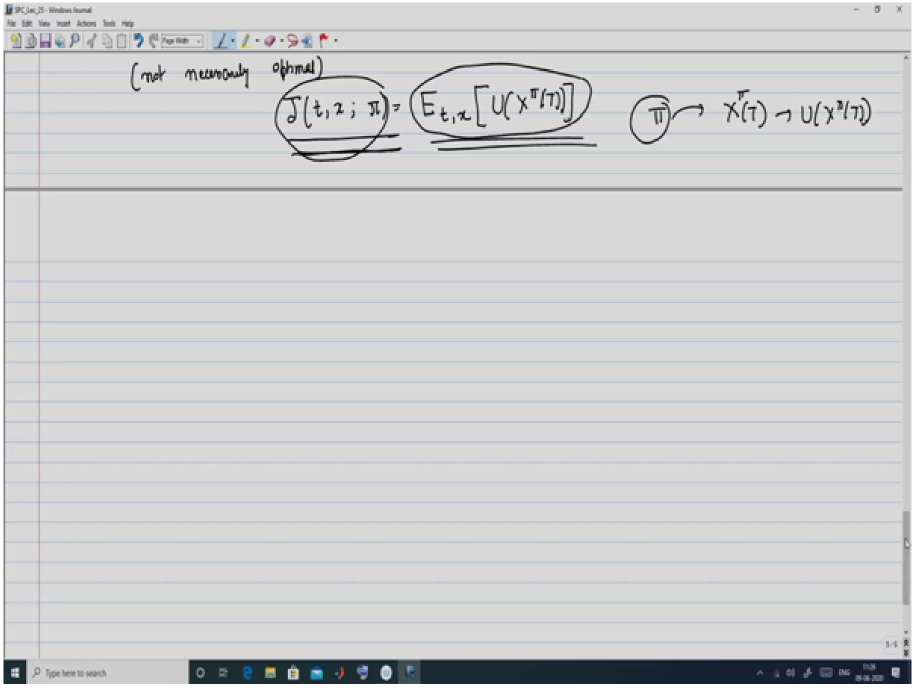
$$[(rX + \pi(\mu - r))dt + \pi\sigma dw(t)].$$

So, we next introduce the notation for the expected utility for a given portfolio strategy not necessarily optimal. So, accordingly we have this notation J t comma x semicolon pi. And this will be given as the conditional expectation conditioned on wealth at time t being equal to X of the utility of the terminal wealth.

(Refer Slide Time: 22:23)

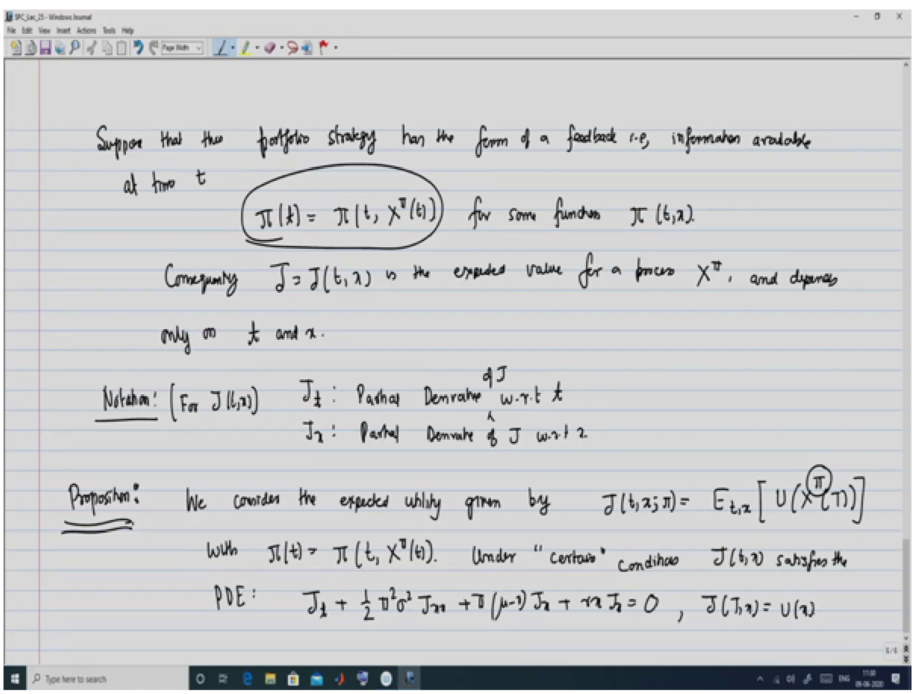
So observe here that this $J(t, x; \pi)$ is going to be the expected utility of the terminal wealth. So, here you observe carefully that for every possible pi and there are many possibilities for π; you will have a different terminal wealth. That means, different terminal wealth depending on what strategy pi you adopt and that terminal wealth I will denote by

$$X^{\pi}(T) = U(X^{\pi}(T)).$$



So, $J(t, x; \pi)$ is basically going to give you a series of values of this conditional expectation depending on what specific π that you have chosen.

(Refer Slide Time: 23:03)

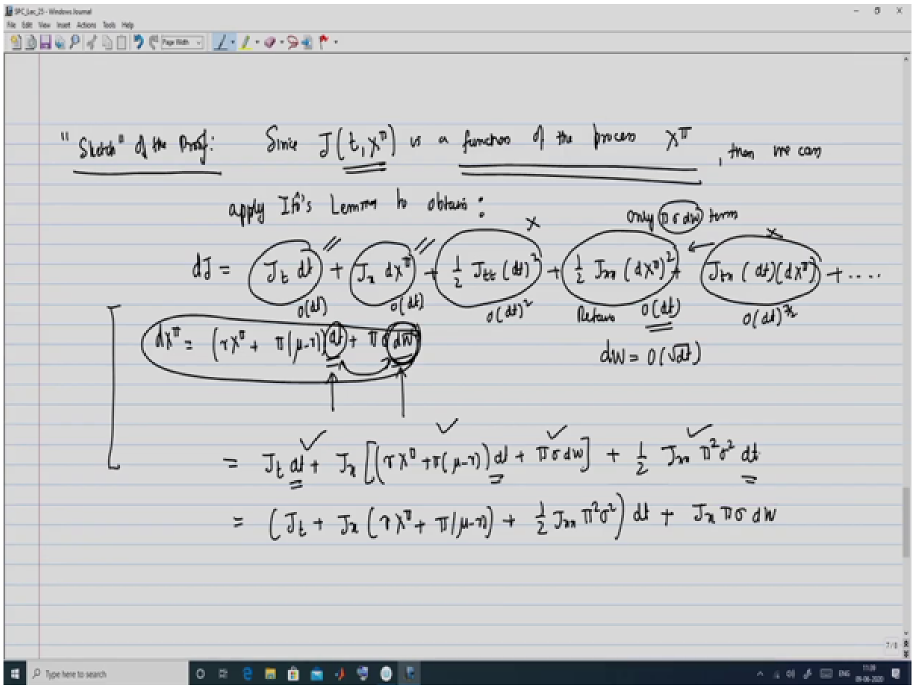


Now we suppose that this portfolio strategy has the form of a feedback that is information available at time t. So, this is going to be

$$\pi(t) = \pi(t, X^\pi(t))$$

for some function pi of t x. So, consequently J will become just $J(t, x)$ because I have taken this to be of the feedback form; is the expected value for a process X pi and depends only on t and x. Next we have a couple of notations and these notations are for $J(t, x)$. So, here J subscript t will be used to denote the partial derivative with respect to t and J subscript x will be the partial derivative of J with respect to x. So, this brings us to a proposition. So, proposition is the following that we consider the expected utility given by $J(t, x)$ semicolon pi is equal to remember this is conditional expectation with respect to t and x of the utility of the final wealth assuming that you have pursued the portfolio pi. Now we need this with pi of t and we assume that this particular pi is feedback. So, this is pi is pi of t into x pi of t. Now this is a very strong result so, we will just say that under certain conditions $J(t, x)$ satisfies the PDE. And what is the PDE that satisfies? The PDE is going to be J subscript t plus half pi square sigma square J x x plus pi into mu minus r J x plus r x J x equal to 0 subject to the condition that J t x is going to be U x.

(Refer Slide Time: 26:50)



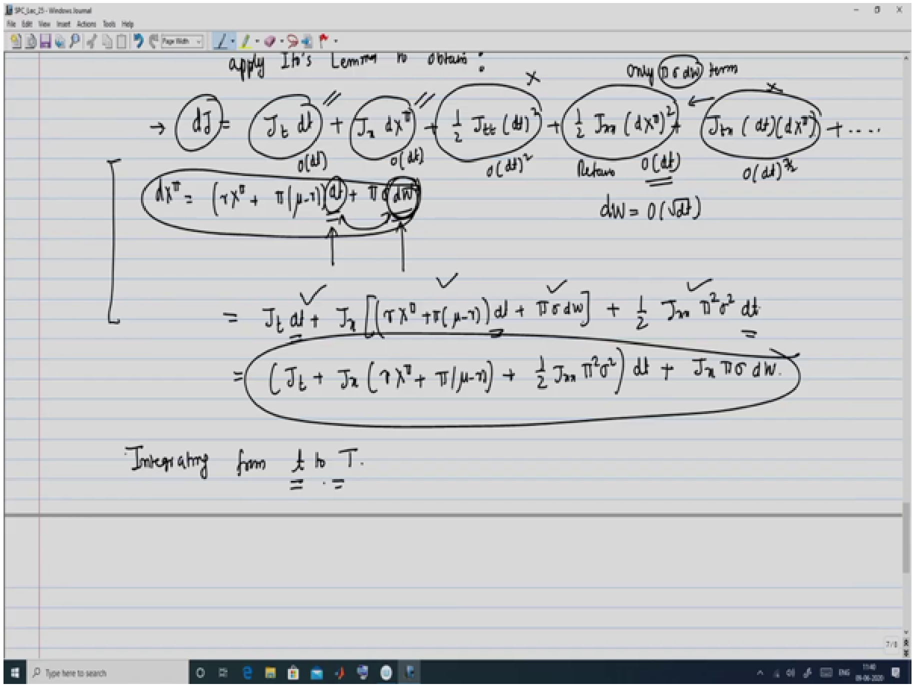
So, we now do a sketch of the proof for the proposition. Now remember that since pi is now of the feedback form. So, this J will now become a function of t and x pi. So, this is a function of the process x pi as I mentioned. Then we can apply Itos lemma to obtain the following and Itos lemma is nothing, but the Taylor series expansion in case of in the stochastic setup. So, accordingly we get

$$dJ = J_t dt + J_x dX^\pi + \frac{1}{2} J_{tt}(dt)^2 + \frac{1}{2} J_{xx}(dX^\pi)^2 + J_{tx}(dt)(dX^\pi) + \cdots .$$

So, just to give an intuition of how we get this so remember that; our d X pi the wealth process was r X pi plus pi into mu minus r into d t plus pi sigma into d W. So, here if you go back and look at the definition of winner process so you will observe that d w is of the order of square root of delta t. So, accordingly this term is; obviously, of the order of d t this term. So, d X pi has a d t and d w which is order of square root of delta t so this is going to be order of d t now here you observe this terms this is going to be order of d t square. Now if you observe this term d X pi square. So, that is if you squared this we will find that here I will have a d t square term here I will get d t term and the cross term will be d t into square root of d t. So, if I get a d t square term out of this I as d t raised to 3 by 2 out of the cross of those two and d w is of the order of d t square root of d t. So, we are only going to retain the term which is going to be order of d t. Because if this is order of square root of d t so that means, the square of this is going to be order of d t plus there are some more terms. And here if you observe carefully this is going to be of the order of d t raised to 3 over 2. And of course, there are some more additional terms. So, what you are going to have is that we are going to retain this term, we are going to retain this term, we are not going to retain this term, we are not going to retain this term and here we are only going to retain pi sigma d w term. So, accordingly what do we have we have then J t d t plus J x d X pi. So, this is going to be r x pi plus a pi into mu minus r into d t plus pi sigma d w. And from here I will only have a term of half J x x pi square pi square sigma square into d t. So, now, I am going to collate the terms which involve d t. So, I will get

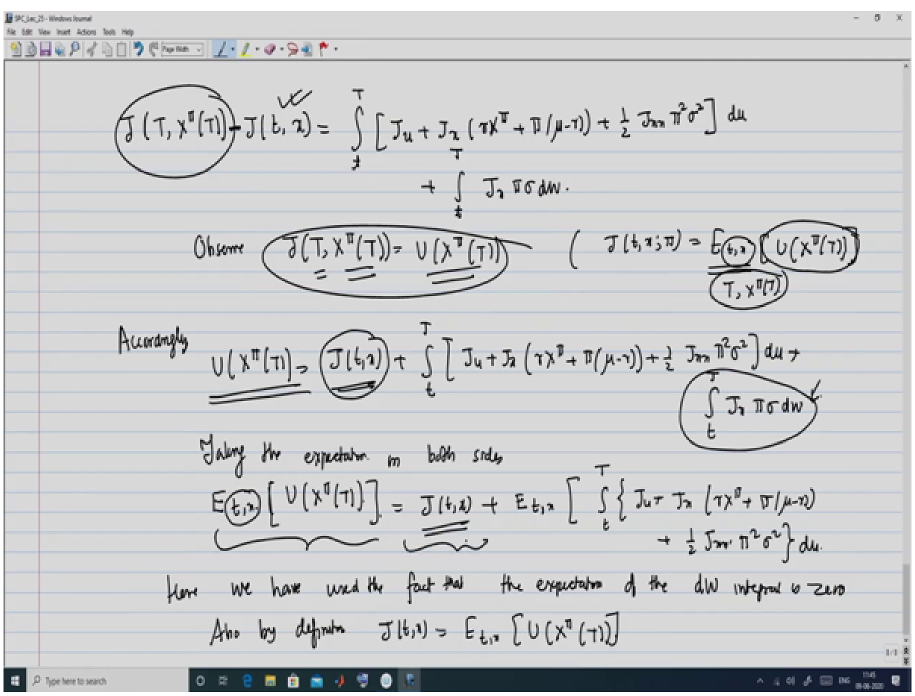
$$[J_t + J_x(rX^\pi + \pi(\mu - x)) + \frac{1}{2}J_{xx}\pi^2\sigma^2]dt + J_x\pi\sigma dW.$$

(Refer Slide Time: 31:44)



So, now I have this relation of d J equal to this expression. So, what I am going to do now is I am going to integrate from t to capital T. So, when I do the integration what will we get. So, I am integrating d J from t to capital T.

(Refer Slide Time: 32:06)



So, this is going to give me

$$J(T, X^\pi(T)) - J(t, x) = \int_t^T [J_u + J_x(rX^\pi + \pi(\mu - r)) + \frac{1}{2}J_{xx}\pi^2\sigma^2]du + \int_t^T J_x\pi\sigma dW.$$

Now, we observed carefully that this value that is J of T X pi T this is going to be nothing, but U of X pi of. And the reason is that if we remember that J t J of t x pi this is nothing, but E t x of U of x pi of t. Now when I look at J t x pi of t then of course, this expectation no longer remains conditional because then this becomes an expectation with the condition given that small t is equal to capital T and x is equal to x pi of T. So, this means that for this value this quantity this argument that we have here this is no longer a random variable and accordingly we get this result. Ok so now, we are going to substitute this value of J T x pi T. So, accordingly we get the following. So, we will get

$$U(X^{\pi}(T)) = J(t,x) + \int_t^T [J_u + J_x(rX^{\pi} + \pi(\mu - r)) + \frac{1}{2}J_{xx}\pi^2\sigma^2]du + \int_t^T J_x \pi \sigma dW.$$

Now, we take the expectation on both the sides we get

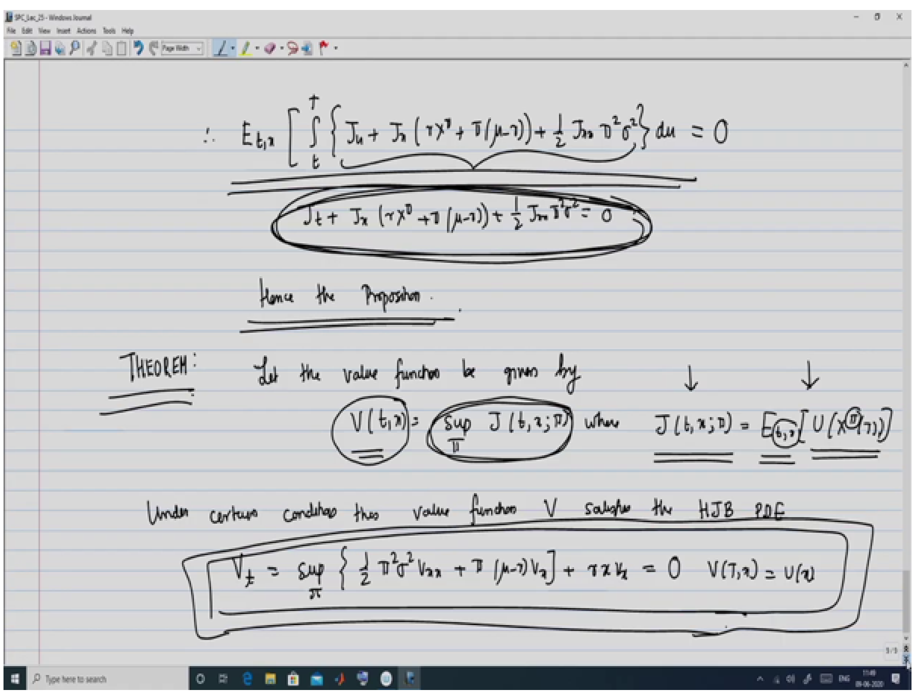
$$E_{t,x}\left[U(X^{\pi}(T))\right] = J(t,x) + E_{t,x}\int_t^T [J_u + J_x(rX^{\pi} + \pi(\mu - r)) + \frac{1}{2}J_{xx}\pi^2\sigma^2]du.$$

Now observe carefully so here I have just straightaway taken the expectation here this expectation will be the same variable because this is conditioned on a little t N x and this is a function of little t N x. And here we have assumed that you know the expectation can be brought inside and the expectation of d w is going to be 0 remember the d w is a standard normal random variate. So, this expectation is going to be 0. So, we just note that here we have used the fact that the expectation of the d w integral is 0. Also by definition

$$J(t,x) = E_{t,x}[U(X^{\pi}(T))]$$

So, that means, this term and this term are identical and hence they cancel out.

(Refer Slide Time: 37:18)



So therefore, what we get is that

$$E_{t,x}\int_t^T [J_u + J_x(rX^{\pi} + \pi(\mu - r)) + \frac{1}{2}J_{xx}\pi^2\sigma^2]du = 0.$$

And accordingly we can conclude that this particular term that is J u or now I can restore t. So,

$$J_u + J_x(rX^{\pi} + \pi(\mu - r)) + \frac{1}{2}J_{xx}\pi^2\sigma^2 = 0.$$

Hence the proposition. So, now that you are equipped with the proposition we now come to a very important theorem. So, the statement of the theorem goes as follows; let the value function be given by

$$V(t,x) = \sup_{\pi} J(t,x;\pi),$$

where we have already defined

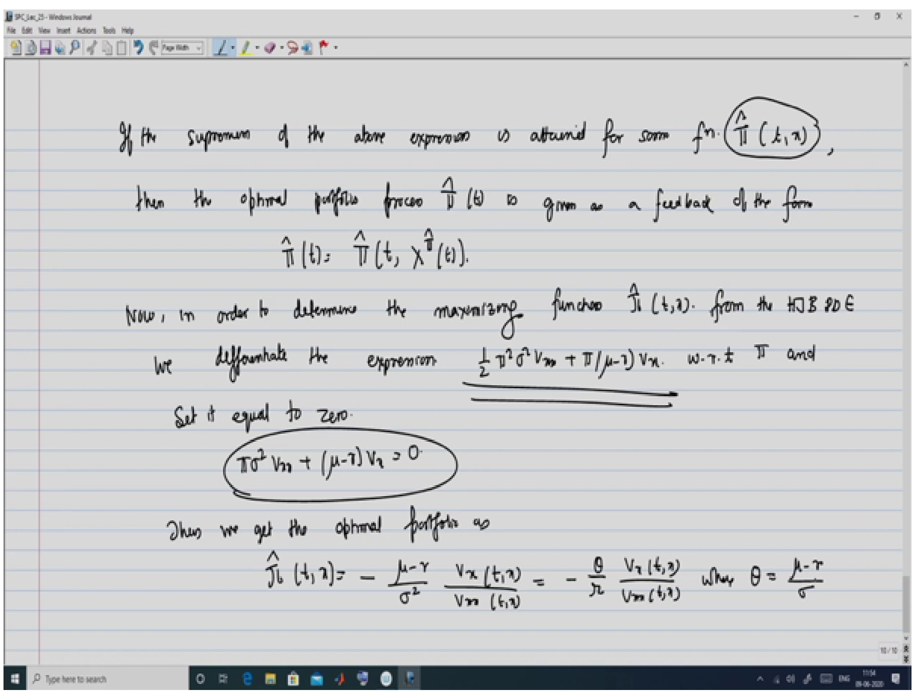
$$J(t,x;\pi) = E_{t,x}[U(X^{\pi}(T))].$$

So, what you are interested in is you are interested in finding out what is the expected value of the terminal wealth starting off at little t and x. And this can take many values depending on what your pi is which you will denote by J t x of pi. And what you want to do is the maximization of this expected utility which is the same as maximization of this J. And that is the reason why this supremum of this J or the maximum value of this J is what we will call as the value function. So, then under certain conditions this value function V satisfies the HJB PDE that is the Hamilton Jacobi Bellman PDE. And what is the PDE this is

$$V_t = \sup_{\pi}\{\frac{1}{2}\pi^2\sigma^2 V_{xx} + \pi(\mu - r)V_x\} + rxV_x = 0, \quad V(T,x) = U(x).$$

So, we can obtain that by basically looking at this expression right. So, what is going to be V T x V T x is going to be supremum of J T x pi. So, you basically take the supremum over this and then you arrive at this particular relation that is the classical HJB equation. So, this is the equivalent the continuous time equivalent for the DBP or the discrete dynamic programming approach.

(Refer Slide Time: 41:00)



So, now, just an observation that; if the supremum of the above expression is attained for some function pi hat. So, this is similar to delta hat that we had done. So, the value of pi or the portfolio pi for which the supremum is attained and we call that to be pi hat of t x. Then the optimal portfolio process pi hat of t is given as a feedback of the form

$$\hat{\pi}(t) = \hat{\pi}(t, X^{\hat{\pi}}(t)).$$

So, now, in order to determine the maximizing function pi hat of t x from the HJB PDE we differentiate the expression. So, which expression are you going to differentiate? it is going to be this expression. So, we differentiate this expression

$$\frac{1}{2}\pi^2\sigma^2 V_{xx} + \pi(\mu - r)V_x.$$

And differentiating with respect to what we want to find out what is our π. So, differentiate with respect to π and set it equal to 0. So, if we differentiate this with respect to π. So, I will get

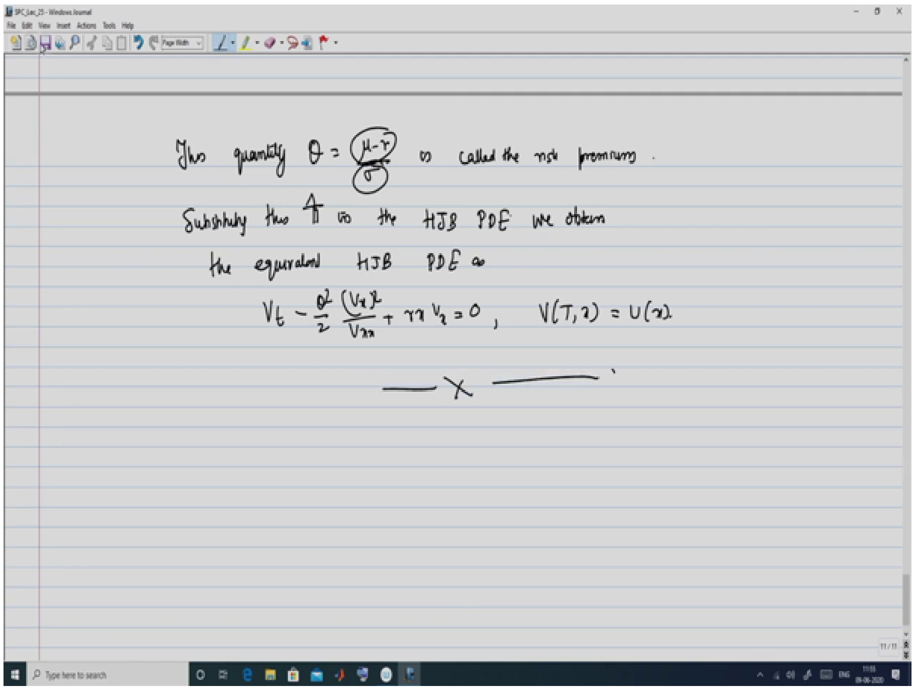
$$\pi\sigma^2 V_{xx} + (\mu - r)V_x = 0.$$

So, solving this for pi we get the optimal portfolio as

$$\hat{\pi}(t,x) = -\frac{\mu - r}{\sigma^2}\frac{V_x(t,x)}{V_{xx}(t,x)} = -\frac{\theta}{r}\frac{V_x(t,x)}{V_{xx}(t,x)}, \text{ where } \theta = \frac{\mu - r}{\sigma}.$$

Now this is something that is familiar.

(Refer Slide Time: 44:22)

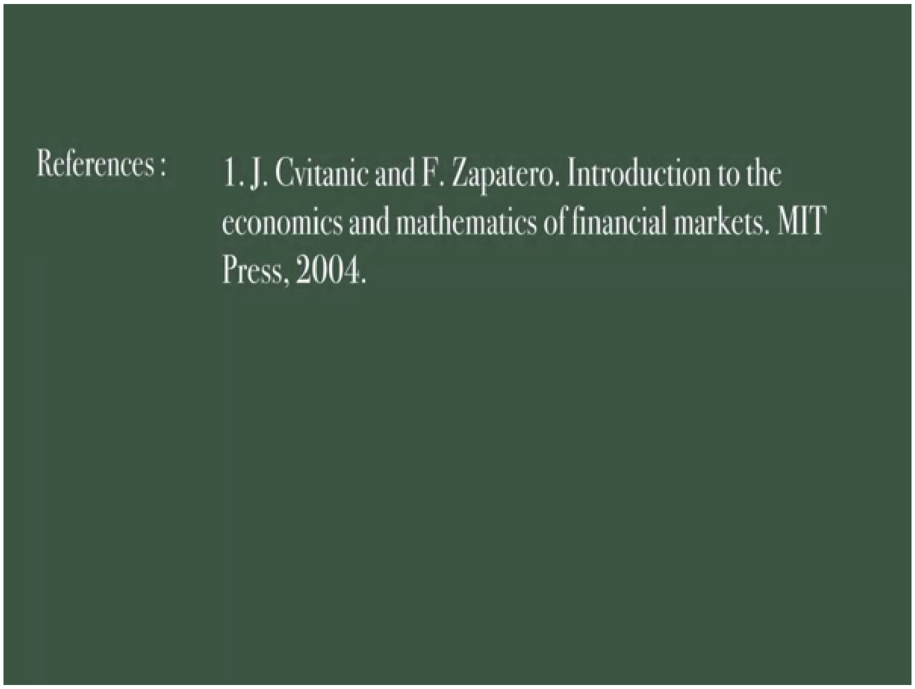


So, this quantity $\theta = \frac{\mu - r}{\sigma}$ is called the risk premium. Remember that we talked about risk premium this is the excess return over risk. Now this pi hat that you obtained here so substituting this pi hat in the HJB PDE; that means, we will substitute this value in this HJB PDE. We obtain the equivalent HJB PDE as

$$V_t - \frac{\theta^2}{2}\frac{(V_x)^2}{V_{xx}} + rxV_x = 0, \;\; V(T,x) = U(x).$$

So, this brings us to the end of this lecture. So, just to do a recap of what we have done in this lecture. So, in this lecture we moved on from the discrete time case to the continuous time case. And we considered the problem of portfolio optimization in the continuous time case. And for that we considered what is known as the Black Scholes Merton framework where we look at the investment in the market comprising of investment in bonds and investment in stocks. And the investment in stocks was modeled through the geometric Brownian motion. And using the models for the stock movement using the GBM and the continuous time model for the price of the bond we looked at what is going to be the wealth process. Then we proceeded to the problem of defining what is the value function and then the ultimate goal was to obtain what was the Hamilton Jacobi Bellman equation. So, accordingly what we did was we first of all gave a proposition and that proposition has a direct ramification on the derivation of the Hamilton Jacobi Bellman equation whose solution will give you the value function in the continuous time setup. And also the in the process of deriving the Hamilton Jacobi Bellman equation we are able to obtain what is going to be our optimal portfolio. So, in the next class we will continue our discussion on this topic and we will look at a few examples. First of all we look at the Hamilton Jacobi Bellman equation that we have derived or rather we have presented here. And then we will look at how to include consumption in the process to look at what is not only going to be the optimal portfolio, but as well as what is going to be the optimal consumption in the continuous time setup.

(Refer Slide Time: 47:25)



Thank you for watching.