Mathematical Portfolio Theory
Module 07: Risk Management
Lecture 31: Quantiles and their properties

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Hello, viewers welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. From today's class we will begin a new topic and that is on risk management in case of portfolio theory. And so far you will recall that in the Markowitz framework, we have talked about the variance or equivalently the standard deviation as the measure of risk.

And in this part of the course we will introduce a couple of more risk measure namely value at risk and the conditional value at risk. So, accordingly we begin this lecture today we will be talking about mostly about quantile and then that will be used in to define what is going to be the value at risk and we will look at some of it is properties.

And, identify some of the shortcomings of value at risk and then that would motivate us moving to the next topic, namely, the conditional value at risk. So, let us begin today's lecture by starting to talk about what are quantile.

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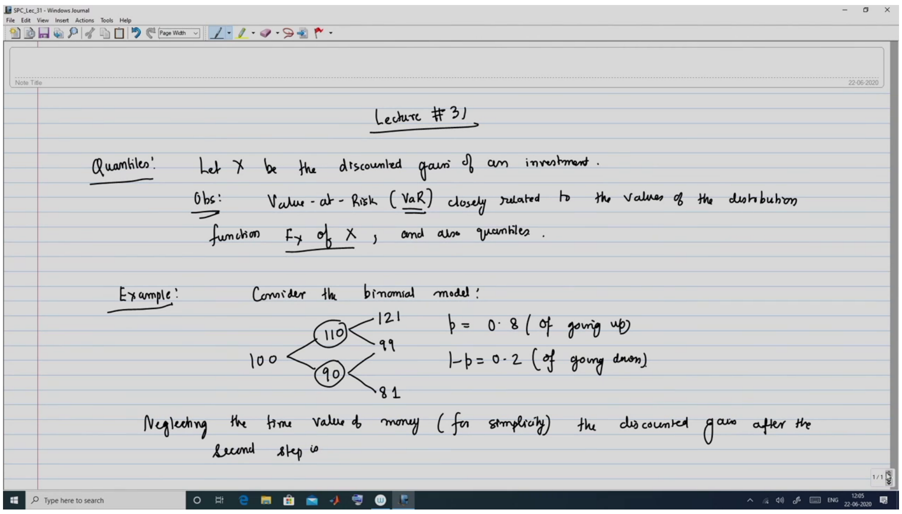

So, first of all we start off with the definition of what are Quantiles. So, let X be the discounted gain of an investment. So, we consider an investment and we consider its discounted gain and we begin with the following observation that the Value at Risk which will define later on or abbreviated as VaR. So, in this case the R is capital unlike the small r in case of variance.

So, this is closely related to the values of the distribution function. So, recall that we had defined what is the distribution function in the first week of the course and this distribution function was given by capital F subscript x of the random variable X and also what are known as quantiles. So, quantiles will be defined in terms of F_x.

So, let us begin with an example on distribution function. So, what you do is now we go back to the binomial model and consider the following binomial model. Suppose that the initial stock price is 100 and then it can go either up to 110 or come down to 90 at time 1 and from 110 it can go up to 121 or come down to 99 and from 90 it can either go up to 99 or come down to 81. So that means, the up factor is a 10 percent and the down factor is also 10 percent.

Now, the probability of going up is taken to be 0.8. So, this is of probability of going up and consequently 1 minus p 0.2 and that is the probability of going down. So, neglecting or ignoring the time value of money this is just for the purpose of simplicity and for illustrative purposes, the discounted gain after the second step is given by $X = S(2) - S(0)$.

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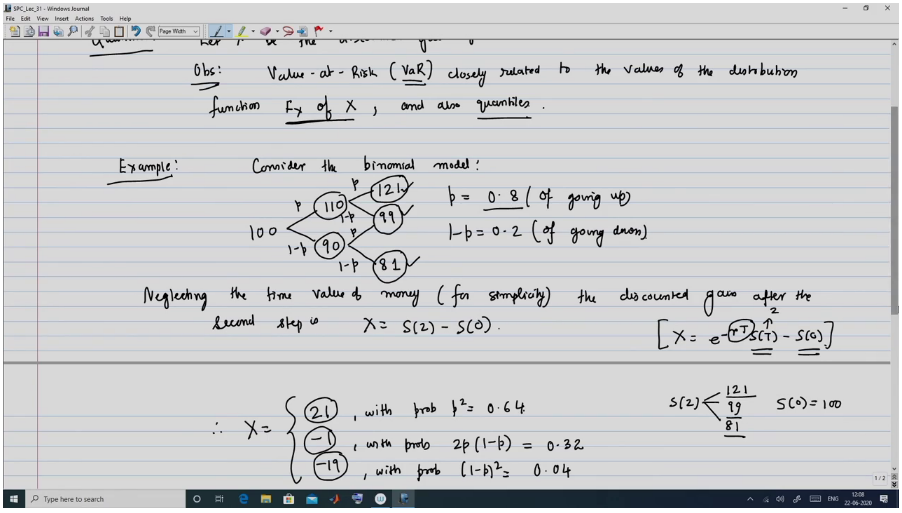

So, normally your discounted gain would be something like the discounted value of

$$X = S(T)e^{-rT} - S(0)$$

But here I am saying that you know that $r = 0\%$. So, consequently I am taking my

$$X = S(t) - S(0)$$

where my t is going to be equal to 2. So, now you observe carefully that this value of S of 2 has 3 candidates, what are those candidates? This is either 121 or 99 or 81 and S 0 of course is deterministic and that is equal to 100. So therefore, X can take three values what are those 3 values? It is going to be 121 minus 100 that is 21, 120 99 minus 100 that is minus 1 and 81 minus 100 that is minus 19.

Now, this value of 121 is a result of the stock price reaching 121 at the time t equal to 2. So, this will happen as a result of 2 upward movements with probability p followed by probability p. So, this X value of 21 can happen with probability of $p^2 = 0.64$, remember that $p = 0.8$.

Now, this value of minus 1 is a result of u reaching amount of 99 at time t equal to 2 and this can happen either as a result of an upward followed by downward movement with probability of $p(1-p)$ or as a result of downward movement followed by an upward movement with probability of $p(1-p)$.

So, the probability of reaching 99 this is going to be equal to 2 into p into 1 minus p and remember p is 0.8 and 1-p= 0.2, so this become 0.32. And finally you arrive at the value minus 19 in the situation where S of 2 is equal to 81. So, in this case what is going to be the probability? So, its a its a result of 2 consecutive downward movements. So, the consecutive or the concurrent probability is going to be 1 minus p square equal to 0.04 ok. Now, what you are interested in is in the context of discussing value at risk we first need to talk about quantize and before we start quantize we need to talk about what is the distribution function.

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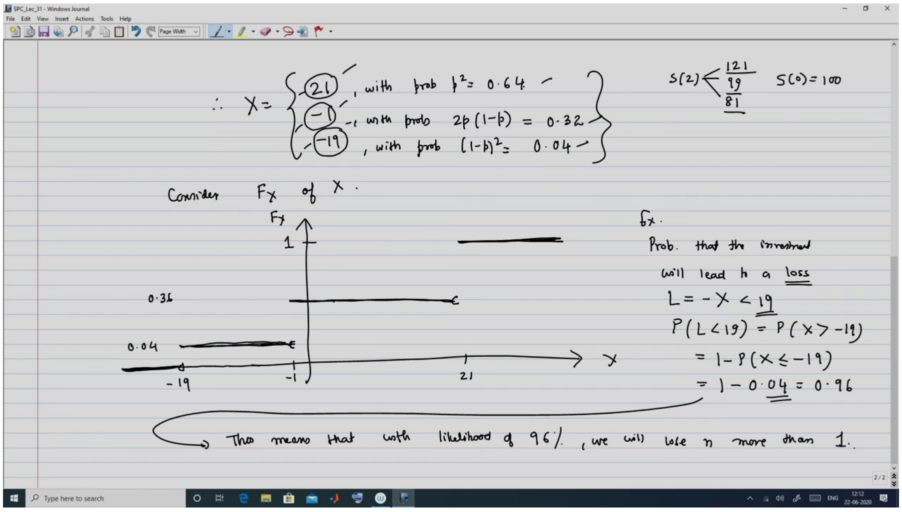

So, accordingly we now consider the distribution function F_X of X. So, how is the distribution function going to look like graphically? So, this is the distribution function and I want to look at it in from a graphical point of view. So, on the X axis I have X and on the Y axis I have $F(x)$ and remember the maximum value of $F_x = 1$ and the values of X that are allowed are 21 minus 1 and minus 19. So, minus 19 will be somewhere here minus 1 will be here and 21 is going to be here.

So, the probability of a value being less than minus 19 is 0 the F_x is going to be 0 here. Now, at 19 the probability is 0.04, so this is 0.04 this height and this is the probability from minus 19 to minus 1. Now, at minus 1 the cumulative probability is going to be 0.04 plus 0.32, so that is going to be 0.36.

So, it is going to be 0.36 and this is the probability at 1 and this is the probability that will last till 21 and when you reach 21, then the total probability is going to be 1. So, the graph will look something like this.

So, this means that from 0 to less from up to less than minus 19 the cumulative probability is 0, from minus 19 but less than minus 1 the cumulative probability is 0.04. From minus 1 but less than 21 the cumulative probability is 0.36 and from 21 onwards the cumulative probability is going to be equal to 1.

So for example, the probability that the investment will, so this is for example, the probability that the investment will lead to a loss of L and what is loss? Loss is negative of gain, so that is minus $X \leq 19$. So, I want to find out what is the probability that the loss will be less than 19. So, this is nothing but the $P(L \leq 19)$ is the same as probability of X being greater than minus 19.

And this is nothing but 1 minus probability, so using the complement property of probability this is 1 minus probability X less than or equal to minus 19. What is the probability that X is less than or equal to minus 19? This is nothing but 1 minus probability that X equal to minus 19 which is 0.04 and so 1 minus 0.04 is 0.36.

So, this means that 95 percent chances are that the loss will not be more than 19 and there is only a 4 percent chance that the loss is going to be more than 19. So, I can say that so from here this means that with likelihood of 96 percent we will lose no more than an amount of 1 alright.

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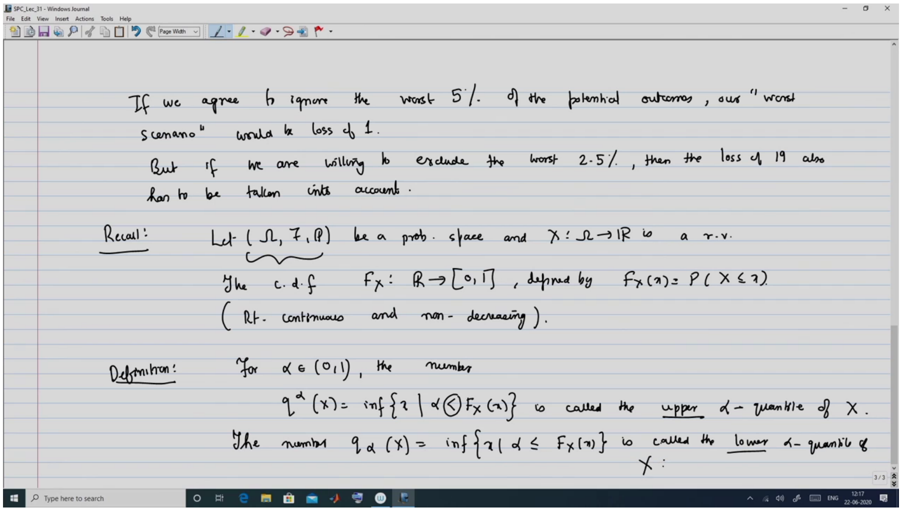

So for example, if we agree to ignore the worst 5 percent of the potential outcomes our worst scenario would be loss of 1. But if we are willing to exclude the worst 2.5 percent, then the loss of 19 also has to be taken into account.

So, let us just recall the following that if your say $(\Omega, \mathcal{F}, P)$ be a probability space, so here Ω is the sample space, $\mathcal{F}$ is the sigma algebra and P is the probability measure. So, this order triplet is the is a probability space and $X : \Omega \to R$ is a random variable. You recall that the cumulative distribution function (cdf) is nothing but a function F_x on R to $[0,1]$ remember F_x can only be between 0 and 1. And this is defined by $F_X(x) = P(X \leq x)$, and you recall that this is right continuous and non decreasing. So, now that we have brought the definition of a distribution in case of; in case of a random variable and more specifically we talked about the cumulative distribution function.

And we looked at an example in the context of our application in finance namely what is going to be the discounted gain. It is now time for us to move into quantize which as I had mentioned earlier is the prerequisite or the prelude to the definition of value at risk. So now, accordingly we start off with the definition of quantize.

So, the quantize says that the following that for $\alpha \in (0,1)$. So, any value of alpha lying strictly between 0 and 1, the number and I will define a couple of numbers. So, there is a number which I will denote by q subscript alpha of x which is the infimum of x that is the smallest x such that alpha is less than the F_X or the cumulative distribution function of x.

And this is called the upper alpha quantile remember alpha here is pre specified of the random variable x. And motivated by the same this setup, we now say that, so now we have talked about a upper alpha quantile. So, then we can now accordingly define the lower alpha quantile and then this number q subscript alpha is going to be the infimum of x such that alpha.

So, remember here alpha was strictly less than the c d f and here I will take alpha to be less than or is equal to the c d f of x. And this case we call the $q_\alpha(x)$ is called the lower alpha quantile of X.

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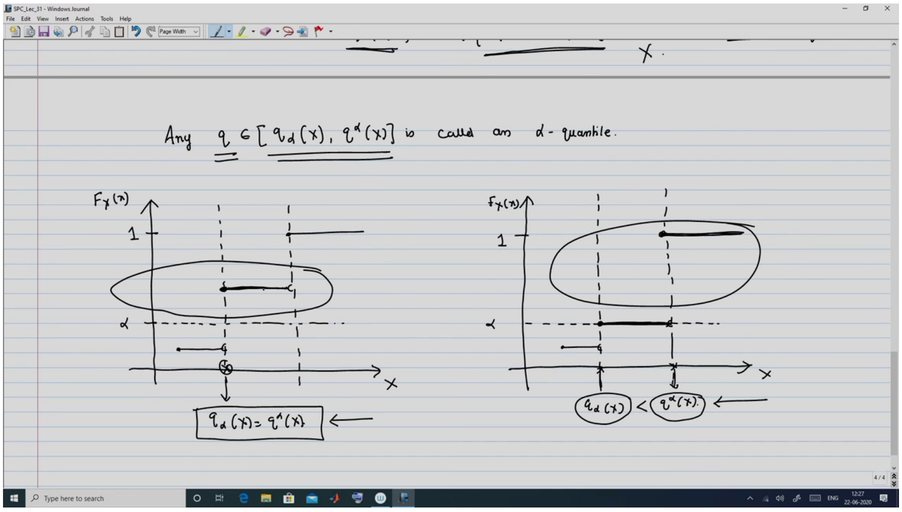

So, to reconcile this upper and lower quantile we make the observation that any q which lies in the closed interval of the lower quantile and the upper quantile this is called an alpha quantile.

So, the next thing that we do is we now look at the interpretation of the quantiles in a graphical manner. So, we take the x on the x axis and the cdf F_X in the y axis and remember that the maximum value for this is going to be 1. So, let us now fix our alpha in both the cases, but you consider two different distributions.

So, this is the first distribution and this is the second distribution. So, in this case you observe that this point that I have here, this point is both the lower as well as the upper quantile. Now, why do I say that both of them are at the same point both the upper and the lower quantile?

So, if you look at the lower quantile definition, the lower quantile definition says that we will take the smaller (Refer Time: 17:58), so that alpha is less than or equal to F x of x. So, the region of alpha which satisfy alpha is less than or equal to F_x of X it is going to be thus this region which is above this line.

So, $\alpha \leq x$ if you look at all the values of x such that alpha is less than or equal to x the smallest of that x value is this and likewise if you look at those values such that alpha is strictly less than F x of x again the smallest value of x is this. So, as a result you have the lower quantile being the same as the upper quantile and so here this q is just going to be a single point, because this interval reduces to a single point.

However, in the second case if you observe carefully, we have a distribution the graph lying in a coincidental manner at alpha. So, if you in this case this particular point here. Let us now look at the second case, so here this particular point this is going to be the lower quantile and this point is going to be the upper quantile and the reason for this is the following that for the lower quantile you need alpha less than or equal to $F_X(x)$.

So, here in this line the alpha is going to be equal to $F_X(x)$, in the previous case you know $\alpha \leq F_X(x)$ involve this line. But, here since alpha is coinciding with $F_X(x)$ we have to consider this line when you are looking at the lower quantile and so accordingly the smallest value of x for this part is going to be the q alpha of x.

However, the region for the upper quantile the region which I have to consider is $\alpha \leq F_X(x)$. So, accordingly I have to consider that part of the distribution which lies here and the smallest value of x which satisfies the region of alpha strictly less than x is going to be $q^{\alpha}(x)$.

So, here the lower and the upper quantile coincide, but here the lower and the upper quantile are distinct from each other. And the primary reason why this was happened is that because the c d f graph coincides with the value of alpha in the second case, but not in case of the first one.

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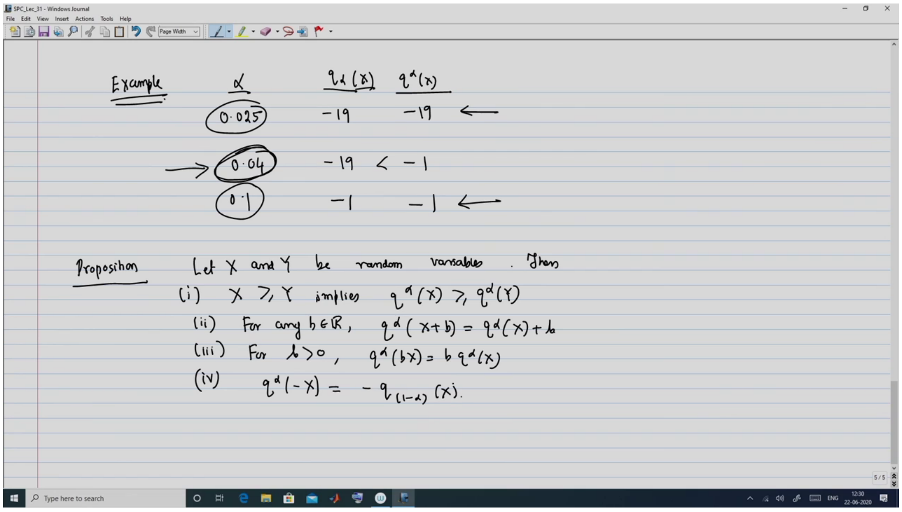

So, if you go back to the example you can verify for yourself, the example that you had done for $F_X(x)$. So, if your alpha I you take three different values that is alpha is 0.025, alpha is 0.04 and alpha equal to 1 a 0.1. Then the lower quantile is going to be minus 19 and minus 1 and the upper quantile is going to be minus 19 minus 1 and minus 1. So, you observe carefully that amongst these 3 values of alpha the only scenario where the cdf matches alpha is going to be 0.04.

So, in the other cases these two quantiles are identical, but in this case because it coincides with the F_X values. So, in this case the lower quantile q alpha of x this is going to be strictly less than the upper quantile q superscript alpha of x ok. So now, let us do a proposition and the proposition states the following that if you let x and y be random variables, then I state 4 results.

So, the first one is if your x is greater than or equal to y, this implies that q alpha of x that is the upper quantile of x is greater than or equal to upper quantile of y. The second result is that for any real number b the upper quantile of x plus b is going to be upper quantile of x plus b.

The third result is that for a positive b the upper quantile of b x is b into upper quantile of x and finally and this is a very crucial result that the upper quantile of minus x of alpha, the upper quantile alpha of minus x this is going to be minus q of lower quantile with respect to 1 minus alpha.

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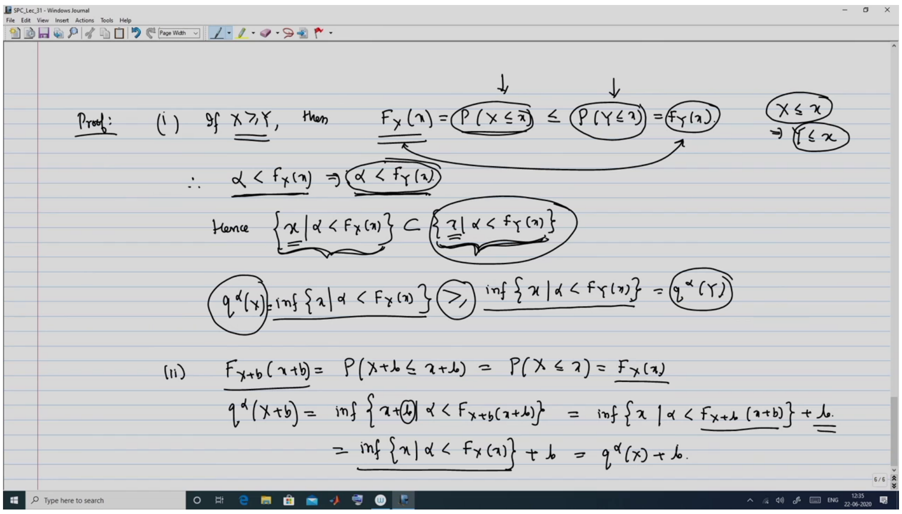

So, let us start doing the proof of this. So, the first proof is that, so what is the assumption in the first result that your x greater than or equal to y implies that this quantile relation holds. So, if your x greater than or equal to y, then in order to prove this result I will need to make use of the c d f. So, F x of x by definition this is probability that x is less than or equal to x.

And this probability is going to be less than or equal to probability of y less than or equal to x which is F y of x. And the reason is the following that if you have your X less than or equal to x, then automatically your y. So, if you have x less than or equal to x this will automatically imply that y less than or equal to x.

So that means, the probability of this happening which is given here must be greater than or equal to the probability of being x being less than or equal to x which is here and this is a probability of x less than or equal to x is by definition F x of x and probability of y less than or equal to x is by definition F y of x.

So, from this relation involving this and this term what we can say is that, so therefore so for upper quantile we need alpha strictly less than F x of x. So, if your alpha is strictly less than F x of x, then obviously it will mean that since your F x of x is less than or equal to y. So it obviously implies that alpha is less than F y of x. So, this means that if your alpha is less than F x of x it means that alpha is less than F i of x.

So that means, that all the values of x which satisfy alpha less than F x of x must also satisfy alpha less than F y of x and maybe there are other values of x which satisfy this condition. So, hence what do you have? All those values of x such that alpha is less than F x of x the set of all such x that must be a subset of all those x such that alpha less than F y of x.

Now, since all the values of x on the for $F_X(x)$ is a subset of all the values of x such that for F y of x. So therefore, what will we have? You will have the smallest since this contains more number of x; so that means, that the infimum of all those x such that alpha is less than F x of x. The smallest x satisfying this condition, obviously must be greater than or equal to the smallest x which satisfies this condition that is infimum of x; such that alpha less than $F_Y(x)$ right.

Simply because this contains more values of x, so it might actually have a smaller value of x that is not present in this set. Now, what is infimum of x such that alpha strictly less than $F_X(x)$? This by definition is q alpha of x and infimum of x such that alpha strictly less than F i of x this by definition is the upper quantile of y.

So, what do you get? So, you basically get q alpha of x the upper quantile of x is greater than or equal to the upper quantile of y which is this result ok. Let us come to the second result now. So, for the second result we consider that F of x plus b of I look at the cumulative distribution of x plus b, this by definition is probability that the random variable x plus b is less than or equal to little x plus b. And if we cancel the b on both the sides, so this will become probability x less than or equal to x and this is F x of x.

So now, we are in a position to prove that q alpha of x plus b the upper quantile of x plus b. What is this? This is by definition infimum of all those x plus b such that alpha is strictly less than F of x plus b of x plus b. And I can since b is a constant I can take this outside, so this is going to be infimum of all those x such that alpha less than F of x plus b of x plus b plus the b which goes outside. And this from the this result that you have obtained here, I can replace F of x plus b to obtain infimum of x such that alpha is less than F x of x plus b. And this infimum by definition is the upper quantile of x plus b.

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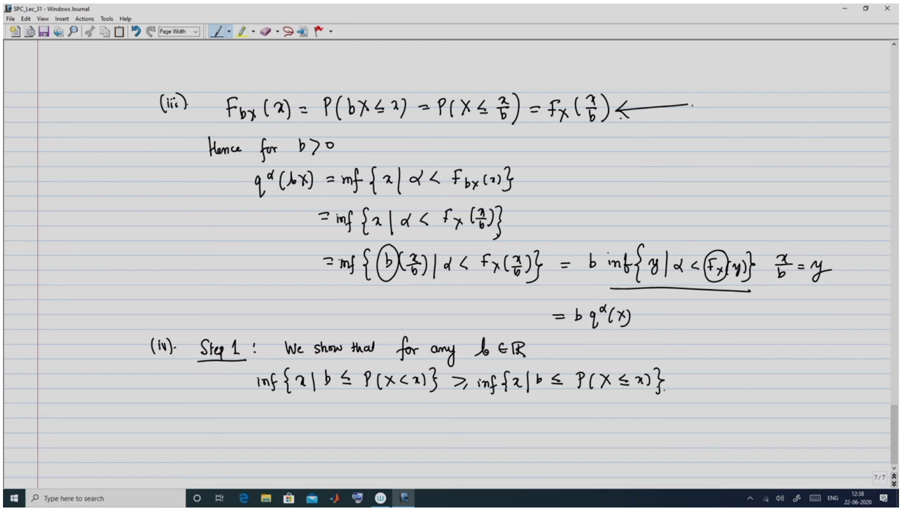

So, let us now come to the third result which is the upper quantile of bX, where b is positive. So, for this again we consider the distribution function or c d f of $F_{bX}(x)$. So, this by definition is $P(bX \leq x)$ and it is finally $F_X(x/b)$

Now, we can show that

$$q^{\alpha}(bX) = bq^{\alpha}(X)$$

So, this proof goes in a couple of steps. So, the 1st step would be what we do is that we show that for any real number b infimum of x such that b is less than or equal to probability of x strictly less than x. This must be greater than or equal to infimum of x for such that b is less than or equal to probability of x are no longer strictly less than X, but less than or equal to little x ok.

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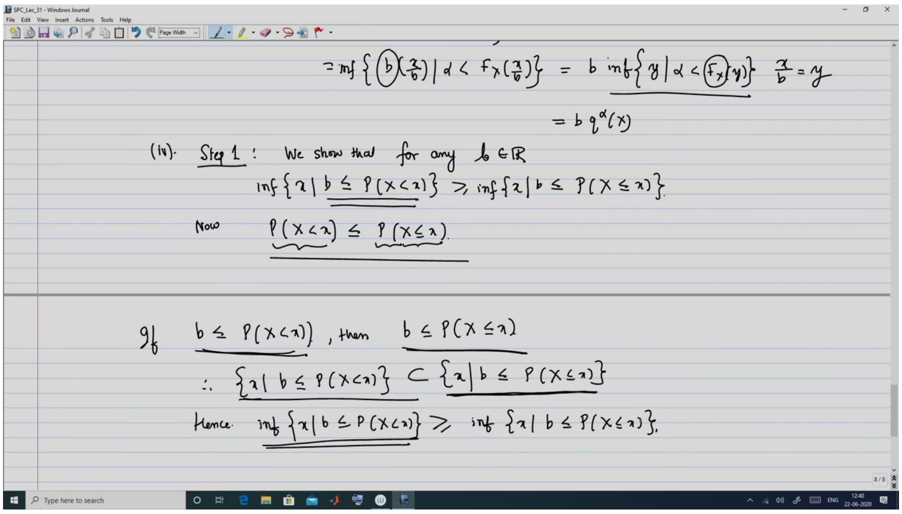

So, now $P(X \leq x)$ is obviously less than or equal to probability of x less than or equal to x. Because here we obviously I have also included the event that X is equal to x. So obviously, this probability is going to be greater than or equal to the probability of capital X being strictly less than x. Now if b is less than or equal to probability of X strictly less than x right.

So that means, I am looking at this condition, then from here obviously we get that b is less than or equal to probability of X less than or equal to x. So now, that this implies this so therefore all those values of x such that b is less than or equal to this condition that is probability X strictly less than x, this will be a subset of all those x such that this condition hold that is b is less than or equal to probability of X less than or equal to x.

And hence if I take the infimum, so obviously this right hand side contains more values of x. So, it is likely that the infimum of this might be a value of x that is not contained here. So, accordingly the infimum of x such that b less than or equal to probability X strictly less than x this infimum might be smaller, so this is going to be greater than.

So, this is pretty much on the same line says the proof for part one. So, this infimum is going to be greater than or equal to infimum of x, such that b is less than or equal to probability of X less than or equal to x.

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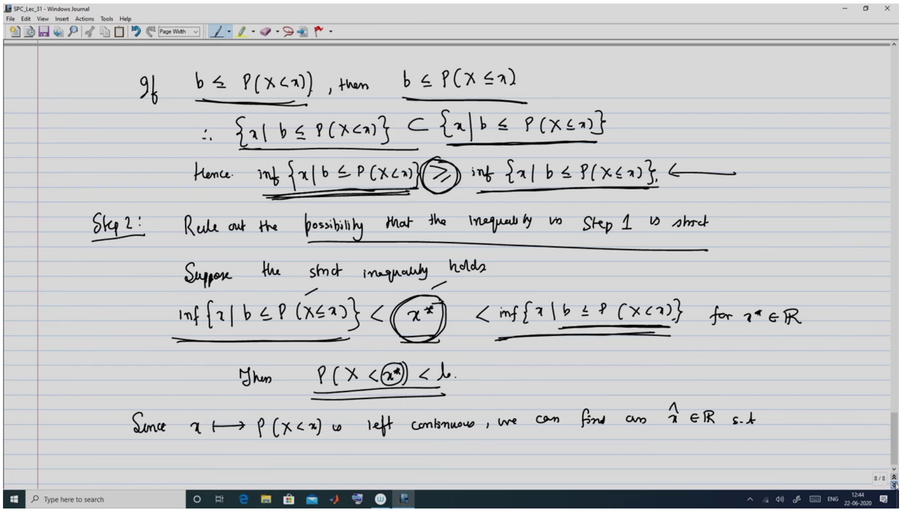

Now, in step 2 we do the following. In step 2 we rule out the possibility that the inequality in step 1 that is this inequality is strict; that means, this is a strict inequality. So, we have to rule out the possibility that this is a strict inequality. Now, let us start off with contradiction and suppose that so you will use the approach of contradiction and begin with the assumption that the strict inequality holds. Suppose that the strict inequality holds.

So, this means that this is strictly greater than the term on the right hand side ok. So, this means that my assumption is that the infimum x such that b less than or equal to probability of X strictly less than x that is this expression on the LHS is strictly greater than this expression on the RHS. So, that is infimum of x such that b less than or equal to probability of X less than or equal to x.

Now, if the strict inequality holds then between these two numbers remember these are just real numbers. So, the between these two real numbers another real number will exist. So, this means that there will be an x star sandwich between them two and this x star would be some real numbers; that mean because between any two real numbers there is a another real number.

Now, when we say this x star we consider that this x star satisfies this property. Then what do we have? So, then we will have probability that X is strictly less than x star is less than b. So, please remember that this x star is less than the smallest value of x which satisfies this condition.

So that means, that since so obviously since this x star is less than the smallest value which satisfies the condition that b is less than or equal to probability of X less than x star. So, this means that this x star will not satisfy the condition that b is less than or equal to probability of X less than x. So that means, that with this means it will satisfy the condition that b is greater than probability of X less than x little x being equal to x star.

Now, since the function x that is this probability of X less than x as a function of little x is left continuous, so we can find and x hat a real number x hat such that it lies between this quantity and this quantity.

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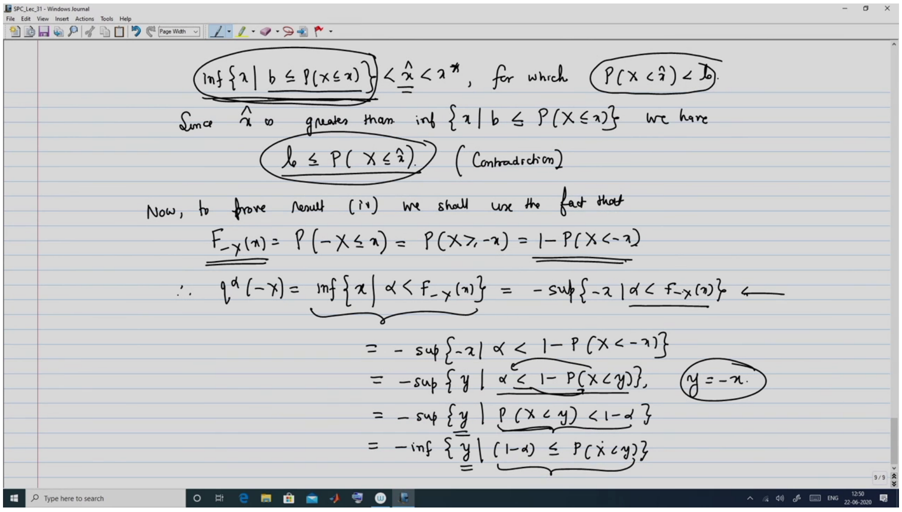

So, this means that infimum of all those x, such that b is less than or equal to probability X less than or equal to x is strictly less than x hat, strictly less than x star for which probability of X less than x hat is less than b.

Now, since x hat is greater than. So, since this x hat is greater than infimum of the x satisfying this condition. So, it is greater than the infimum of all those x such that b is less than or equal to this condition, that is b is less than or equal to probability X less than or equal to x. We have that b is less than or equal to probability of X less than or equal to x hat right.

So, here you see all that the smallest value of x is such that is this condition is satisfied; so obviously, you will have b being less than or equal to probability of X less than x hat and this is a contradiction. Contradiction to what? So, basically from the previous step that is looking at the region of x star.

So, from the previous step applicable to this part we got that probability X less than x hat is less than b and we have got probability X less than or equal to x hat greater than or equal to b. So therefore, we have arrived at a contradiction ok.

So now, that we have this result that these two are, so the strict inequality does not hold in this relation, we can now make use of this to prove our main result, so to prove result 4. So now to prove result 4 we shall use the fact that F of minus x of x this by definition. What is this going to be? This is going to probability of the random variable minus X being less than or equal to x and this is equivalent to probability of the random variable X being greater than or equal to minus x and this is nothing but 1 minus probability of X being strictly less than minus x.

So therefore, the upper quantile of minus X what is this going to be? This is going to be the infimum by definition is going to be the smallest value of x, such that alpha less than F of minus x into x and what is this? Say infimum; so this is by definition for quantile and this can be written as minus supremum of minus x, such that alpha is less than F of minus x x. So, it is like the infimum of a variable it is going to be minus supremum of the negative of that variable.

Now what can I rewrite this as? Now, let us focus on this term. So, this can be written as minus supremum of minus x such that alpha is less than. So, remember this alpha is less than F minus x of x, see this condition that is same as alpha being less than this condition.

So, I can write this as alpha being less than 1 minus probability of the random variable X being strictly less than minus x. And this can be rewritten as minus supremum of y, such that alpha is less than 1 minus probability of x less than y, where y is equal to minus x.

So, this can now be written as minus supremum of y such that probability of X strictly less than y is less than 1 minus alpha. So, I am just moving this to the left hand side and alpha to the right hand side. So, this can be now rewritten as minus infimum of y such that 1 minus alpha.

So, the this the complement of this is going to be 1 minus alpha less than or equal to probability of X strictly less than y. So, supremum of those y is we satisfies this condition is simply going to be the infimum of y which satisfies the complement of that condition.

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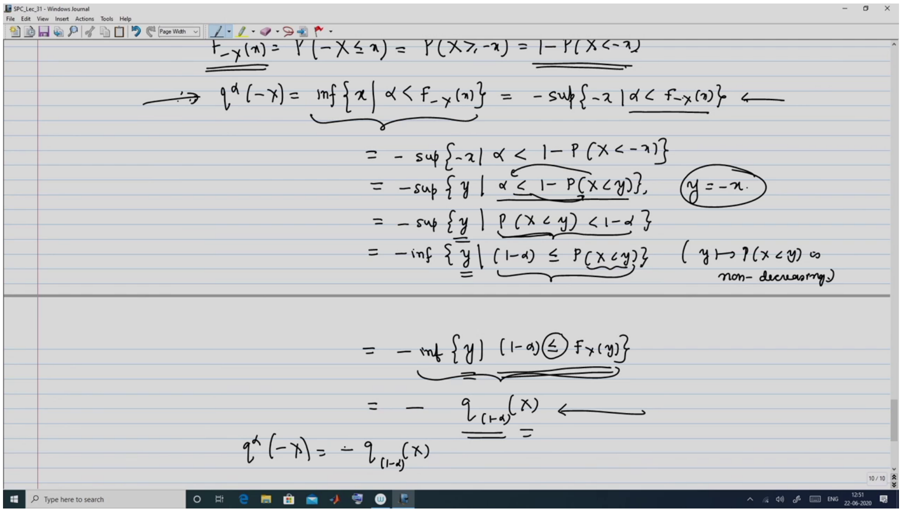

And the reason you can do this is that the function y to a probability of x less than y is non decreasing. And this can now be written as minus infimum of y such that 1 minus alpha less than or equal to P of x strictly less than y. What is this? This is nothing but F x of y. And observe carefully this is y given that 1 minus alpha less than or equal to F x of y. So, since 1 minus alpha is less than or equal to F x of y. So, this is going to be nothing but the quantile of X and since there is a less than or equal to sign this is going to be the lower quantile of 1 minus alpha. So, this entire expression is now this lower quantile y minus alpha of x and of course we have the negative sign. So, hence it proves that. So therefore, from starting from here we have arrived here. So, we get q alpha of minus X there is a upper quantile of minus X is nothing but the lower quantile 1 minus alpha of x with a negative sign. So, this proves the fourth result.

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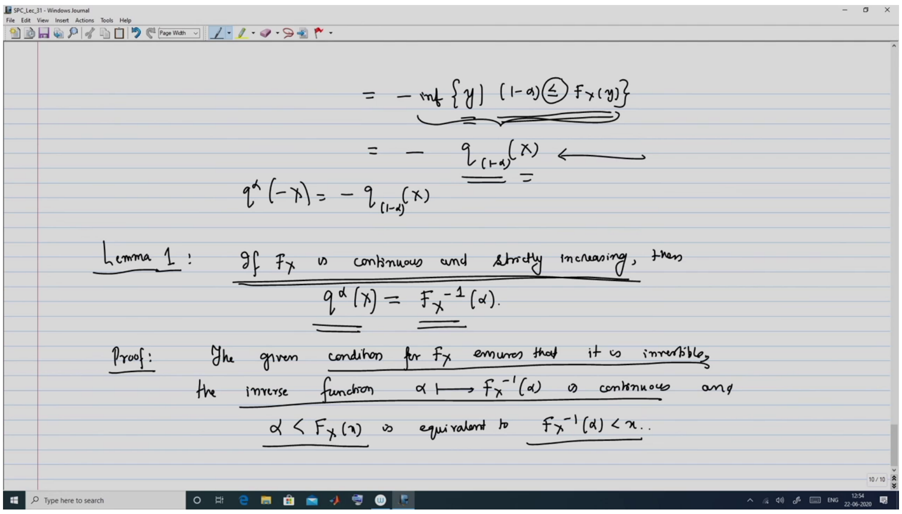

So, we conclude with a couple of lemmas; the first lemma so this is let me call this lemma 1 for today. So, it says that if your F x is continuous and strictly increasing, then the upper quantile of this X is simply going to be F x inverse the inverse of the c d f of alpha. So, how do we get the proof of this? So, what is the given condition? The given condition is that F x is continuous and strictly increasing.

So, I can say that the given condition for F x the c d f this ensures that it is invertible, because I will need to have the inverse of F x. Also it ensures that the inverse function, since it is invertible and the inverse function F x inverse exist. And, this inverse function F x inverse alpha is also continuous and most importantly since we are talking about upper quantile.

So, that the condition that is required for alpha quantile that is alpha strictly less than F x this condition is equivalent to the condition that F x inverse of alpha is strictly less than x. So, based on this condition that F x is continuous and strictly increasing, we have that it is invertible, the inverse function is continuous and most importantly alpha strictly less than F x of x is equivalent to this relation.

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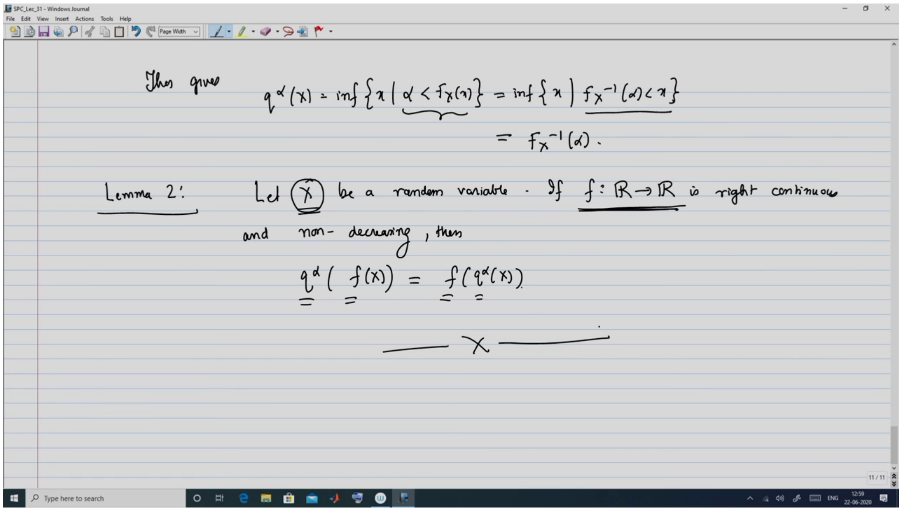

So, what does this give? So, this gives that the upper quantile of x which is by definition infimum of all those x such that alphas strictly less than $F_X(x)$. What is this going to be? It is going to be infimum of those x with an equivalent condition of alpha strictly less than $F_X(x)$ which is this condition. So, it is a smallest x such that $F_X^{-1}(\alpha) \le x$.

And since F_X^{-1} inverse is continuous, so the smallest value of x that this condition is satisfied is nothing but F x inverse of alpha itself ok. So, this is a very crucial result that is lemma 1 and now we state lemma 2 without proof and it says the following, that let X be a random variable. Now, if your f from R to R that is a real valued function on R is right continuous and non-decreasing, then the following condition hold that the upper quantile.

So, remember that if your x is a random variable, then obviously and f is a; f is a real valued function which is right continuous and non-decreasing. Then f of x is also a random variable and this upper quantile of x becomes equal to f of the upper quantile of x. So that means, this quantile and this f interchange their position for this random variable x and this function f.

And remember that for most part we will be looking at x to be the specific random variable of the discounted gain on a asset. So, this brings us to the end of this lecture just to do a brief recap of whatever we have done in this lecture. We started off with recognizing the fact that the only risk measure that we have done so far is the standard deviation or variance that we are seen in case of the Markowitz framework. And, then we decided that we need to move on to something more robust and accordingly we will consider two different risk measures; namely VaR and CVaR which is the value at risk and the conditional value at risk. And in order to motivate the definition of VaR and consequently CVaR we needs the distribution or the cumulative distribution function for a random variable.

So, in this context the random variable that you will consider is going to be the discounted gain on the asset. And then when you talk about the cumulative distribution function for that, we obviously need to then look at how it is going to be useful eventually in the paradigm of VaR.

And, since the definition of VaR involves something for called quantile we essentially looked at the definition of what is the upper quantile and what is the lower quantile and illustrated it through a very simple example on discounted gain. And, then you looked at four important properties of the upper quantile and this was followed by a couple of lemmas.

So, in one lemma essentially that once you are given the distribution of the function it gave you under certain condition how you can find the upper quantile as an inverse of the cumulative distribution function. And, then we did another lemma which is actually even more important, where we talked about this random variable and then the function of this random variable.

And, we saw that how the upper quantile of the function of this random variable can be given as a function of the upper quantile of the random variable itself under certain conditions on the function F. So, in the next class we will continue this discussion and we start of our narrative on value at risk.

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Thank you for watching.