Mathematical Portfolio Theory
Module 07: Risk Management
Lecture 33: Average Value-at-Risk and its properties

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Hello viewers, welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You would recall that in the previous two lectures we were talking about Risk Management and we essentially looked at what is going to be the definition of upper and lower quantiles and then we moved on to the definition of value at risk and we looked at some of the properties of value at risk. So, in this concluding lecture on this part of the course we will talk about a couple of deficits that is observed in case of value at risk. And, then we will introduce the notion of coherent measure of risk with an emphasis on a particular coherent measure of risk namely a AVaR.

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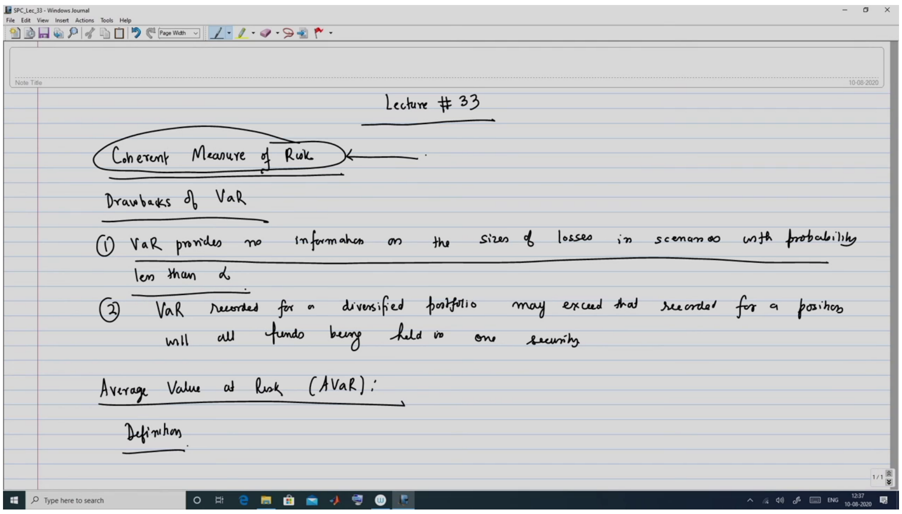

So, accordingly we start this lecture today, with a brief introduction to coherent measure of risk. So, this is motivated by couple of observation on drawbacks of value at risks. The first one is that the value at risk or VaR, this provides no information on the sizes of losses in scenarios with probability less than alpha and secondly, the VaR recorded for a diversified portfolio and you have seen in the example of this may exceed that recorded for a position with all funds being held in one security.

So, we will just have a brief look at the coherent measure of risk at a later stage, but what I want to start off with is a particular case of coherent measure of risk and that is what is known as the average value at risk or AVaR.

So, naturally we are going to start off with the definition of AVaR and I said that you know VaR provides no information on the size of losses in scenarios with probability less than alpha. So, what AVaR does is actually it takes into account the losses which have a probability of less than alpha.

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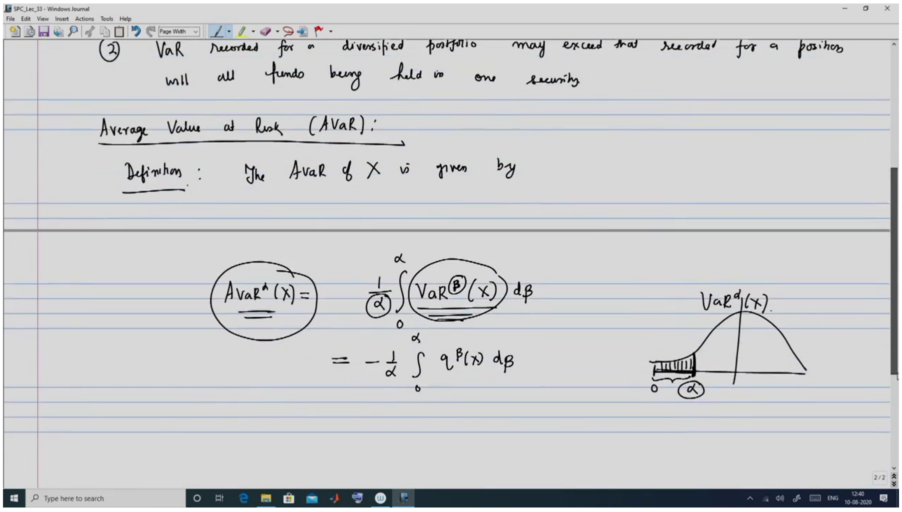
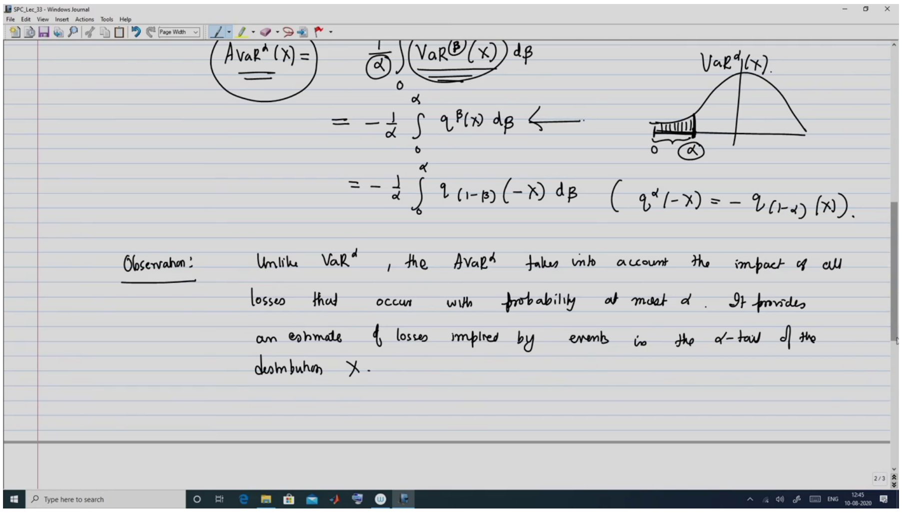
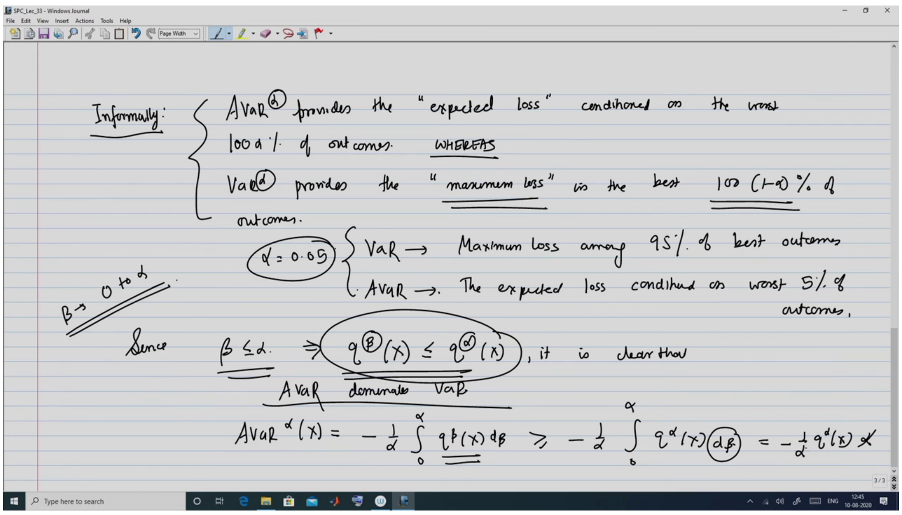


So, then accordingly the AVaR of X where X is some random variable is given by the following:

$$AVaR_{\alpha}(X) = \frac{1}{\alpha}\int_0^{\alpha} VaR^{\beta}(X)\, d\beta = -\frac{1}{\alpha}\int_0^{\alpha} q^{\beta}(X)\, d\beta$$





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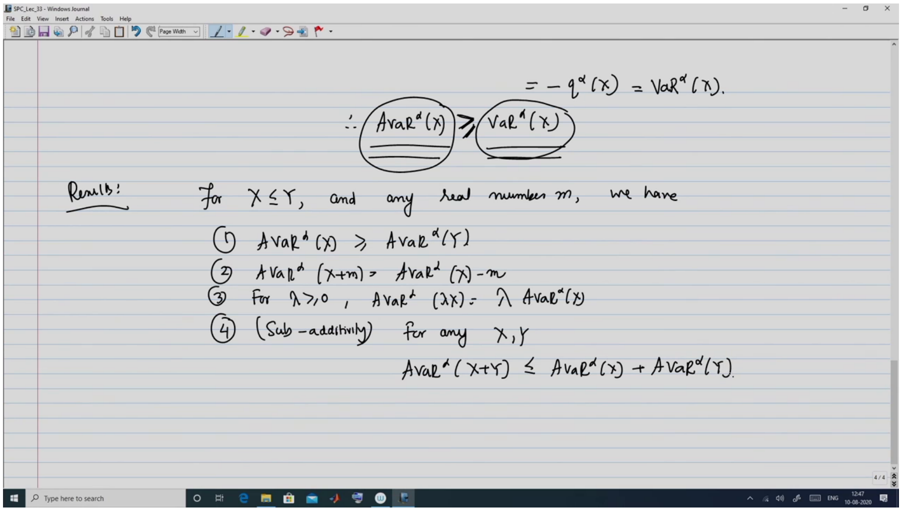

Properties of AVaR:
For $X \leq Y$,

- $AVaR^{\alpha}(X) \geq AVaR^{\alpha}(Y)$
- $AVaR^{\alpha}(X+m) = AVaR^{\alpha}(X) - m$
- For $\lambda \geq 0$, $AVaR^{\alpha}(\lambda X) = \lambda AVaR^{\alpha}(X)$
- For any X, Y: $AVaR^{\alpha}(X+Y) \leq AVaR^{\alpha}(X) + AVaR^{\alpha}(Y)$

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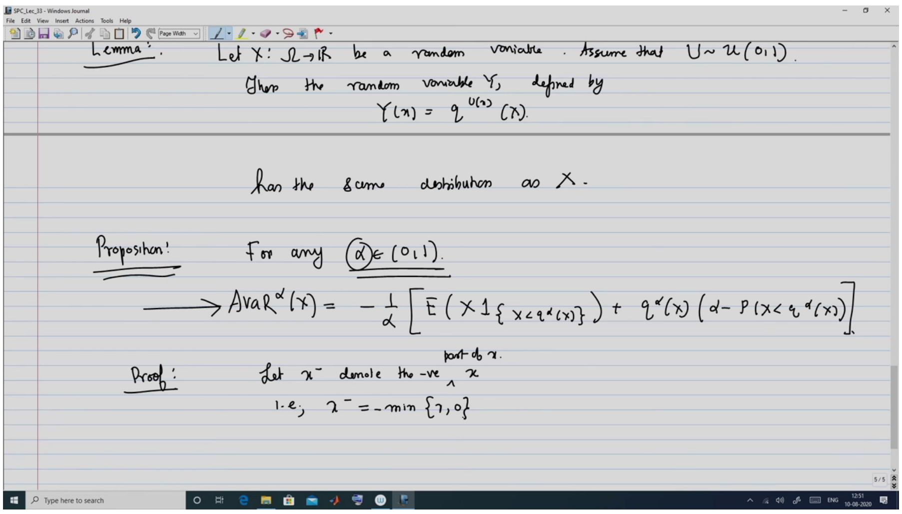

At this point I am going to just note a lemma Let $X : \Omega \to \mathbb{R}$ be a random variable. Assume that $U\ u(0,1)$, then

$$Y(x) = q^{U(x)}(X)$$

Let us now state a proposition to give a more concrete form to what is going to be how it can one make an estimate of AVaR. For $\alpha \in (0,1)$:

$$AVaR^{\alpha}(X) = -\frac{1}{\alpha}\left[E(X 1_{X \leq q^{\alpha}(x)}) + q^{\alpha}(X)(\alpha - P(X < q^{\alpha}(X)))\right]$$

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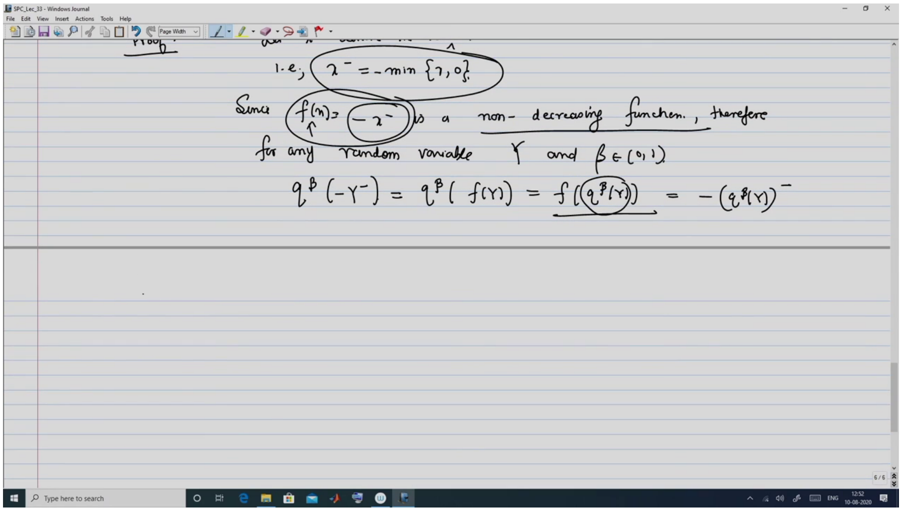

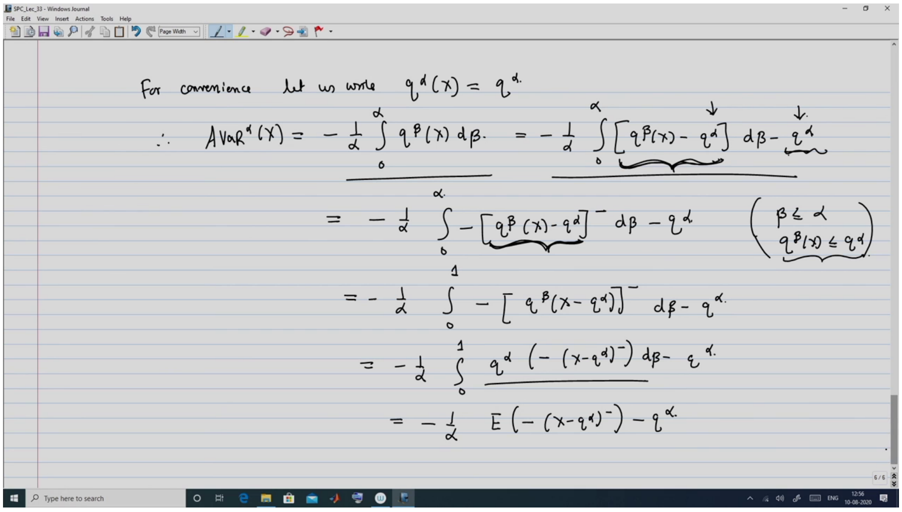

So, for convenience let us write that this q alpha of X remember this is a fixed quantity. So, the q alpha of X is equal to q alpha. So, this is for just for the ease of notation ok. So, now, that I have this result set up now what I can do is that I can start off with getting the result that I have stated here in order to have an expression for AVaR of alpha.

It is shown that:

$$AVaR^{\alpha}(X) = -\frac{1}{\alpha}E(-(X-q\alpha)^{-}) - q^{\alpha} = -\frac{1}{\alpha}\left[E(X1_{X \le q\alpha(x)}) + q^{\alpha}(X)(\alpha - P(X < q\alpha))\right]$$

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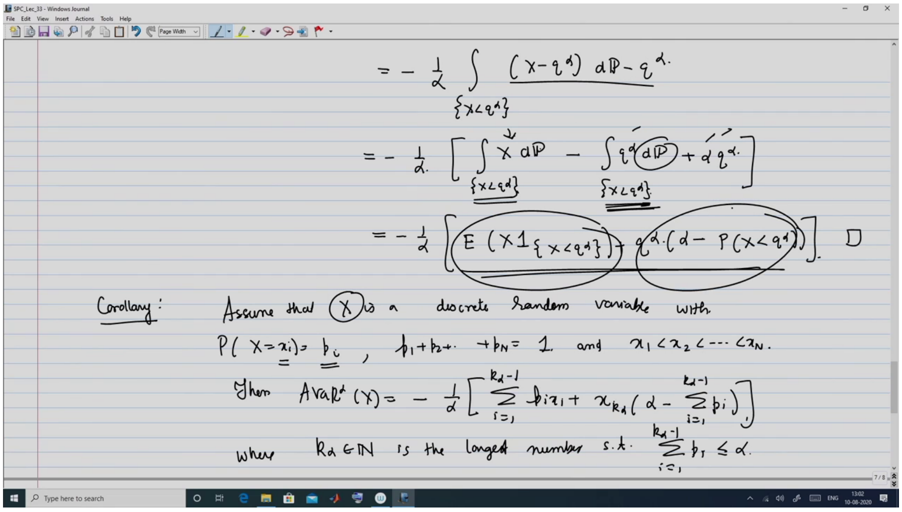

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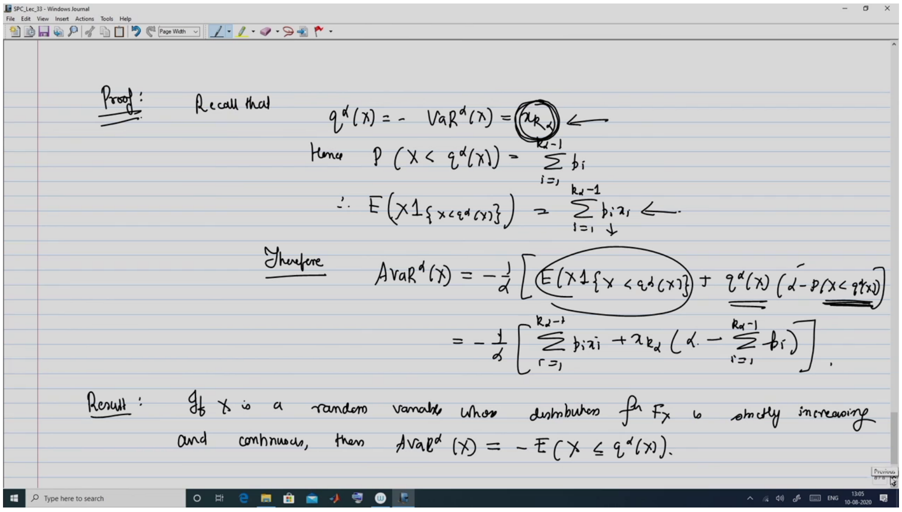

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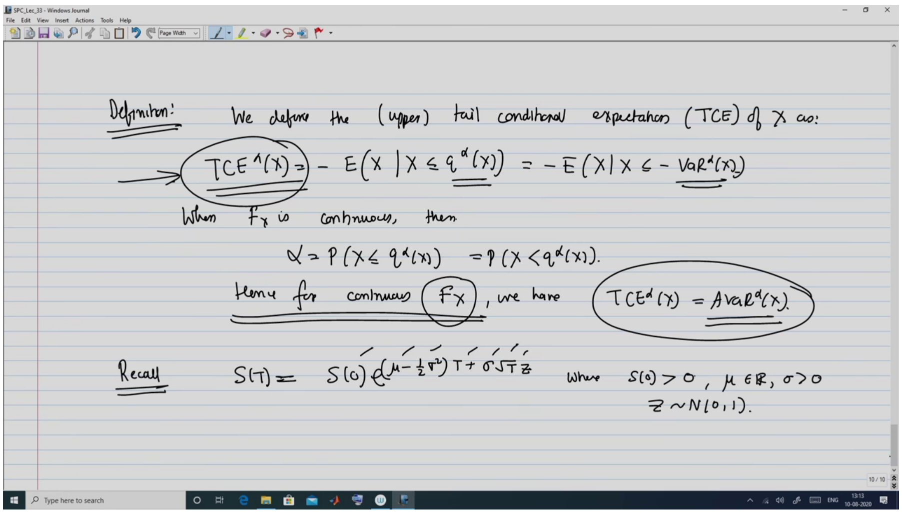

So, we next come to a definition which is going to lead us to the ever in the context of the black shows framework. So, we define the upper tail conditional expectation which will abbreviate as TCE of

X:

$$TCE^{\alpha}(X) = -E(X|X \leq q^{\alpha}(X)) = -E(X|X \leq -VaR^{\alpha}(X))$$

And when F_x is continuous then we have alpha

$$\alpha = P(X \leq q^{\alpha}(X))$$

, then for continuous F_X,we have

$$TCE^{\alpha}(X) = AVaR^{\alpha}(X).$$

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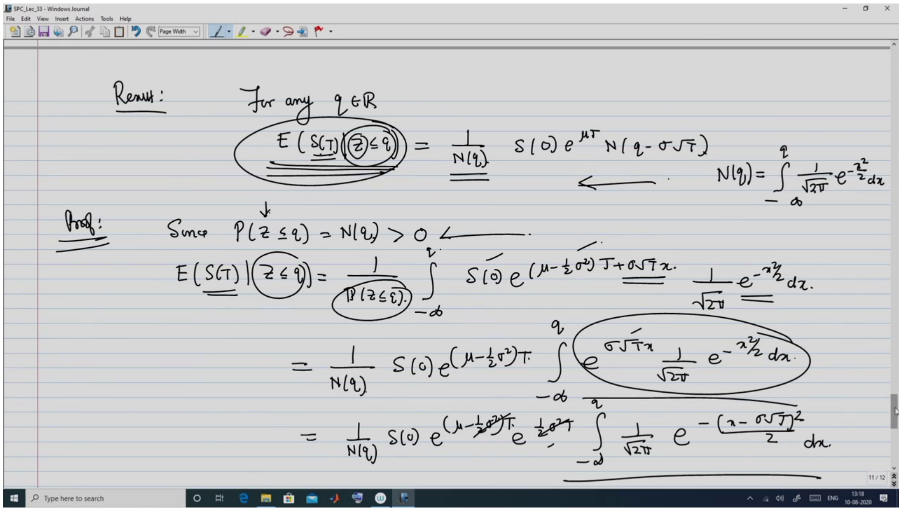

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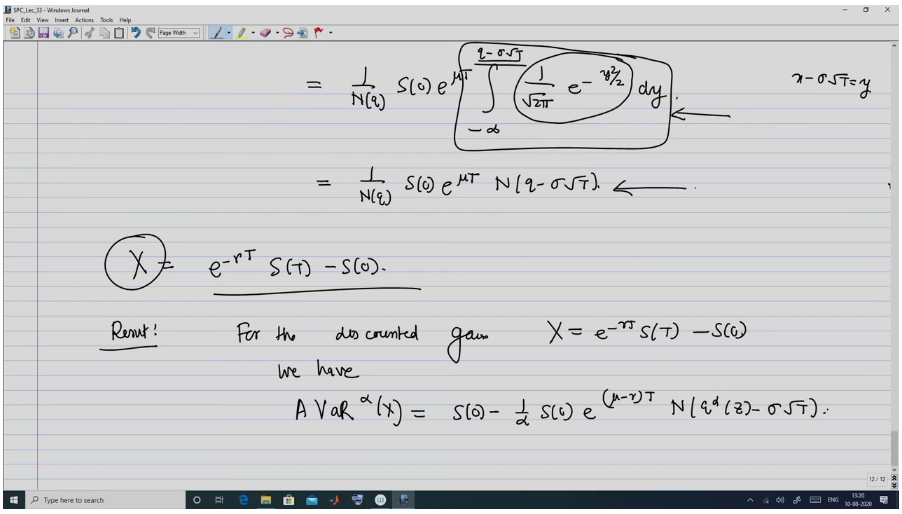
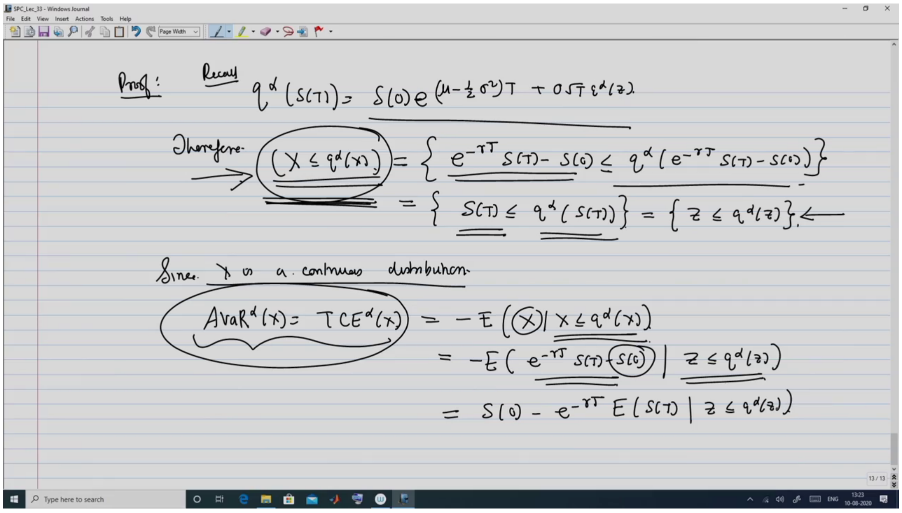
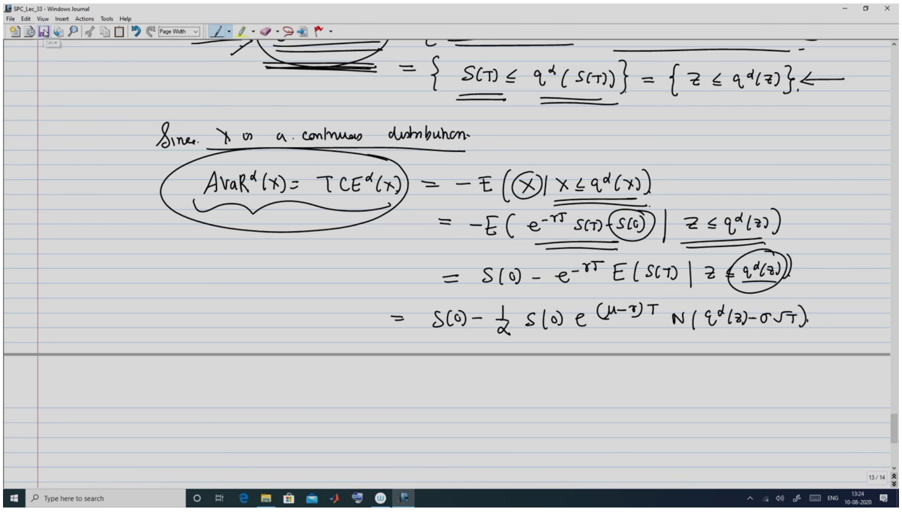


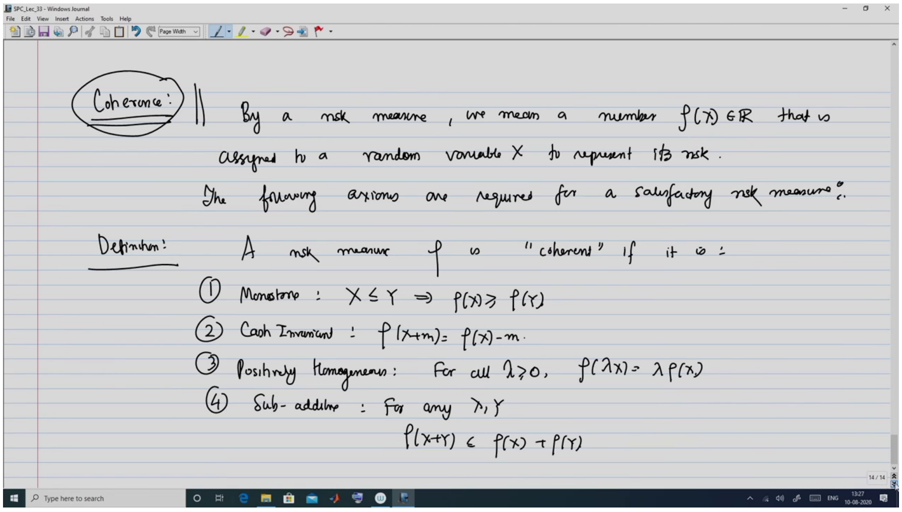
So, now this minus of sigma square to half (Refer Time: 36:53) plus half sigma square will cancel out. So, I will get 1 over N q into S 0 into e raised to mu T and I could defined my x minus sigma square root of T to be equal to y, then this integrand ends up being from minus infinity to q minus sigma square root of T into 1 over square root of 2 pi e raised to minus y square over 2 into some d y and now, what is this?. This is nothing, but the probability density function of the standard normal random variate and this is this integral is the cumulative distribution from minus infinity to q minus sigma square root of T. So, this gives us. So, I will get 1 over N q S 0 e raised to mu T and this term here within my box this becomes N into q minus sigma square root of T. So, hence you end up getting this result. So, you end up getting this result. So, now, what I want to do is I just want to look at. So, remember that our original motivation was to look at X and what does X? X was nothing, but the discounted gain that is e raised to minus r T into S of T minus S 0. So, now I can make use of this result in order to make a final statement that for the discounted gain and what is the discounted gain? This is X is equal to e raised to minus r T into S T minus S 0 what is going to be the AVaR? Remember at the end of the day you are really interested in looking at what is the AVaR of your discounted gain and this is going to be nothing, but S 0 minus 1 over alpha S 0 e raised to mu minus r into capital T N of q alpha of Z minus sigma square root of T. So, how do you go about you know getting this result? The way you go about is the following. (Refer Slide Time: 39:12)



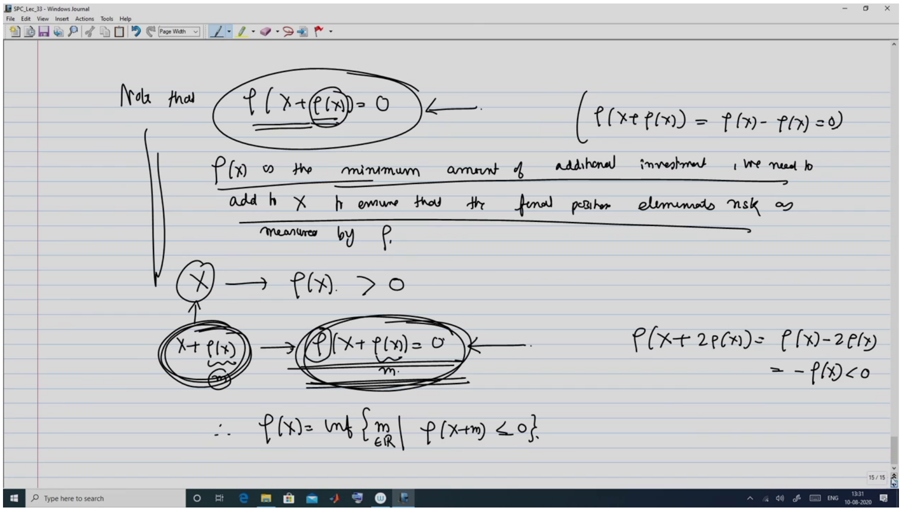
So, this is the proof of the result now remember that. So, recall that q alpha of S T is going to be equal to S 0 into e raised to mu minus half a sigma square into capital T plus sigma square root of T into q alpha of Z. So, therefore, what is X less than or equal to q alpha of X? This now X less than or equal to q alpha of X I can write rewrite this X to be e raised to minus r T into S of T minus S 0 and what is q alpha of X? q alpha of X nothing, but q alpha of e raised to minus r T into S T minus S 0 and now you observe carefully that this actually reduces. So, this relation then reduces to the condition that S of T less than or equal to q alpha of S of T. Now, if we replace the value of q alpha of S of T here and you replace the value of S of T, then this reduces to a much simpler form that Z is less than or equal to q alpha of Z alright. Now why did I abruptly start talking about X being less than or equal to q alpha of X? And the reason is that remember that since X is a continuous distribution remember that X is the discounted gain. So, this is a continuous distribution so; that means, I can take advantage of the fact that if X is a continuous distribution then the AVaR alpha of X is going to be nothing, but TCE alpha of X and the TCE alpha of X involves the definition involves this value. So, in order to take advantage of the fact that these two are TCE alpha and AVaR alpha are identical if X is a continuous distribution I have a priory obtained this relation. So, TCE alpha of X is nothing, but minus the expected value of all those X such that X is less than or equal to q alpha of X now what is this? This is nothing, but. So, what does X? X is nothing, but e raised to minus r T into S T minus S 0 given and remember that this is something that I by definition. An X less than or equal to q alpha of X in the preceding step that is nothing, but Z is less than or equal to q alpha of Z and what is this going to be? S 0 is a constant. So, this is not you know contingent on this condition. So, S 0 will come out and minus e raised to this minus e raised to minus r T will come out. So, all you get is the expected value of S of T given that Z is less than or equal to q alpha of Z alright. (Refer Slide Time: 42:26)



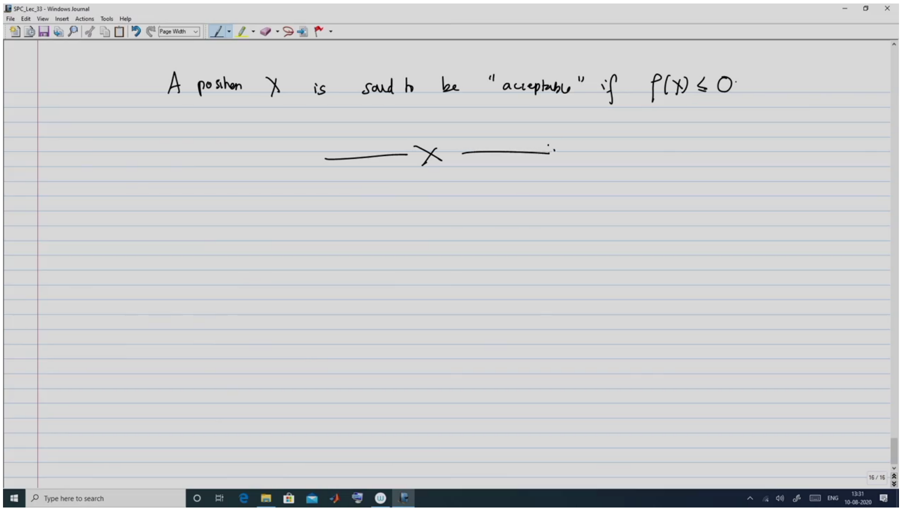
And S of T of Z given Z less than equal to q alpha. So, this turns out to be S 0 minus 1 over alpha into S 0 into e raised to mu minus r of T into N of this q alpha of Z minus sigma square root of T. So, remember that your q alpha is going to be now replaced with q alpha of Z. (Refer Slide Time: 42:57)



So, I will conclude this topic by revisiting what is coherence? Remember that when I talked about coherence I just focused on AVaR alpha and motivated by what is going to be the defects or shortcomings of VaR alpha. So, in order to define coherence we need to talk about what is the risk measure. So, by a risk measure we mean a number row X which is real number that is assigned to a random variable X to represent its risk. So, the following axioms are required for a satisfactory risk measure. So, this brings us to the definition. So, the definition is as follows that a risk measure rho is said to be coherent if it satisfies the following properties. First of all if it is monotone and by monotone it I mean that if X less than or equal to Y, this implies that rho X is greater than or equal to rho y. Secondly, it is cash invariant that means rho of X plus m is going to be rho of X minus m. Thirdly, it is positively homogeneous and this means that for all lambda greater than or equal to 0 rho of lambda X is equal to lambda of rho X and fourthly it is sub additive that is for any X Y rho of X plus Y is less than or equal to rho X plus rho Y. (Refer Slide Time: 45:44)



So a note that, rho of X plus rho of X is going to be 0. So, here rho of X is a constant quantity so; that means, that rho of X plus so, rho. So, this can be easily observed by the fact that of this property that is rho of X plus rho of X this is going to be nothing, but rho X minus rho X which is equal to 0. Now, the question is why have I suddenly you know made an observation making use of the properties or the axioms that I have stated. And, the reason is that you see rho of X is that quantity which renders rho of X plus rho X to be equal to 0. So; that means, that rho of X is the minimum amount of additional investment we need to add to X to ensure that the final position eliminates risk as measured by rho. So, just to explain it in more detail see you have X and the risk measure is rho of X which is probably greater than 0. Now to this amount of X if you add rho of X which is like your m; what does this make? This means that your new position that is X plus rho of X this becomes rho of X plus rho of X which I have said is equal to 0. So, earlier you possibly had a positive risk, but the additional amount of this cash m this makes the overall risk to be equal to 0 so; that means, that it is the least amount of additional investment we need to ensure that your risk is eliminated that is it is being rendered to be equal to 0. So; that means, that if you add an amount of rho X that this obviously, is the resulting risk as given by rho of this position is going to be equal to 0 and if you add more than this rho of X then obviously, this quantity is going to end up being negative. So, suppose that you add an amount of twice rho X. So, rho of X plus twice rho X what is this going to be? This is going to be rho of X minus twice rho of X this is going to be minus rho of X which is going to be less than 0. So, this entire narrative can then be summarized in particular this statement can then be summarized as that rho of X is nothing, but it is the smallest m right which will make sure that this becomes less than or equal to 0. So; that means, the smallest real number m such that rho of X plus m is less than or equal to 0. And, in the problems of risk management a position X is said to be acceptable as long as the risk is not positive. (Refer Slide Time: 49:08)



So, it is said to be acceptable if rho of X is less than or equal to 0. So, if you observe carefully here if you go back to the definition that we have here a look at the properties that if you observe that all these properties these are satisfied by AVaR, but this sub additivity property is not satisfied by VaR and that is the reason why VaR is not a coherent measure of risk, but AVaR is. So, this brings us to the end of this topic on risk management and just to give you a recap what we have done is that we started by looking at what is the definition of upper and lower quantized in the first lecture. And, then you looked at what is VaR and looked at some of the properties of VaR and in particular look at how are you going to calculate the VaR of a stock in the context of the geometric Brownian motion model. And in today's class we introduce the definition of the average value at risk or AVaR by noting the fact that VaR is not a does not capture what is going to happen in terms of the losses that are that the that have a probability of less than the predefined alpha for which the VaR is being defined. And accordingly we defined AVaR which was nothing, but like the expected loss. And, you looked at several variants of how one can determine AVaR a culminating into a form for determining AVaR first in case of the discrete time setup and also finally, we looked at using the geometry Brownian motion in case of calculating the AVaR for discounted gain of a risky asset. And, we concluded by today's lecture by looking at a general definition or the axioms of coherent measure of risk the most prominent example of which is the AVaR. (Refer Slide Time: 51:17)

Thank you for watching.