Mathematical Portfolio Theory

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Module 02: Basics of Financial Markets
Lecture 03: Binomial and geometric Brownian motion (gBm) asset pricing models

Hello viewers! Welcome to this next lecture on the MOOC course on Mathematical Portfolio Theory. I would recall that in the last two classes we have been focusing on mostly talking about financial markets, and we talked about derivatives. And in the previous class we mostly looked at the two main kinds of assets, one was the risky asset and example of which was bond, and the other was a risk free asset and which was a stock. So, what we are going to do now is basically that we are now going to consider some asset pricing model. And for that purpose we are talking about the risky assets, and in particular we will look at two models namely the Binomial model and the geometry Brownian motion model which are respectively the discrete and continuous models for asset pricing that are most commonly in use. So, we begin this lecture by first talking about the binomial asset pricing model, for which we will make use of the binomial distribution. And then we will talk about the geometric Brownian motion, which will be driven by the normal distribution which justifies the introduction of these two distributions in one of our earlier classes.

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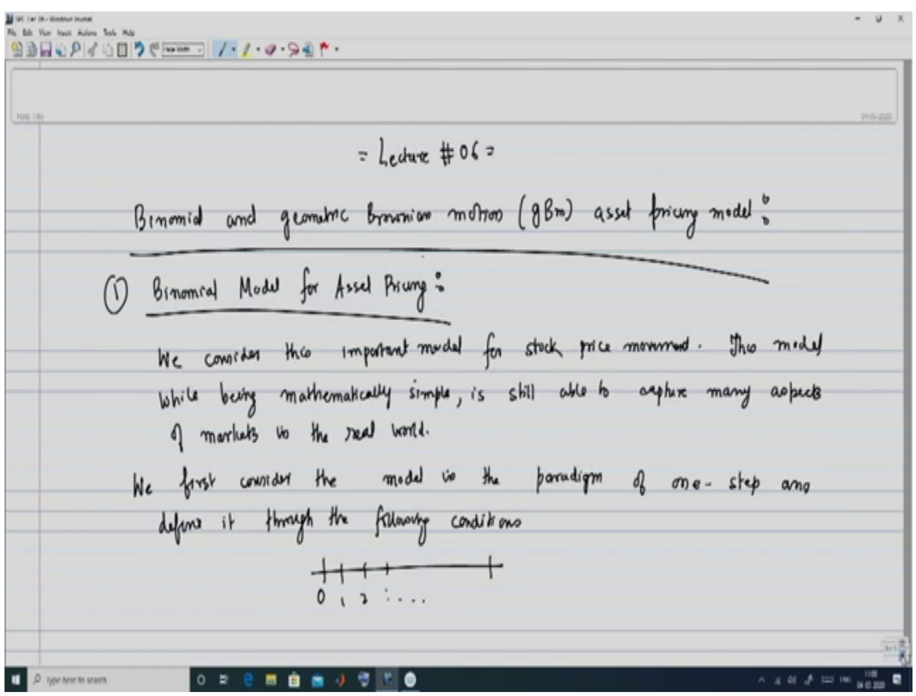

So, we start this lecture, the topic of this lecture is Binomial and geometric Brownian motion or gBm asset pricing model. So, let us first begin with the binomial model. So, we consider this important model for modeling the stock price movement. Now, note that this model while being mathematically tractable and simple is still able to capture many aspects of markets in the real world. So, we first consider the model in the paradigm of one step and define it through the following conditions. So, before I start with the condition let me just say that we are basically we will look at some time point 0, 1, 2 and so on. So, these are just the index for the different time points.

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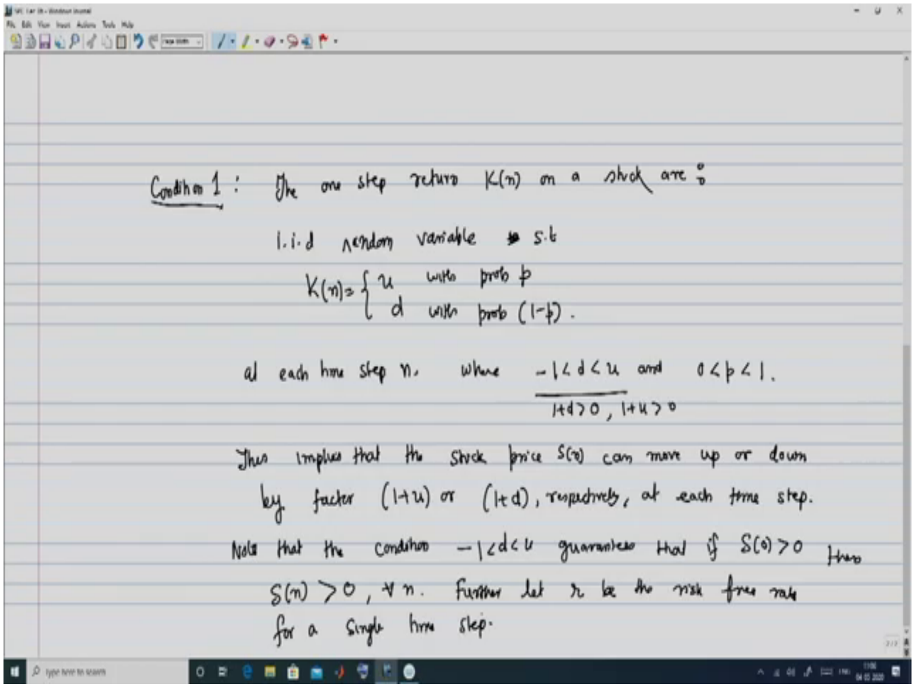

So, we start looking at the first condition. So, the one step return K_n on a stock are. So, what do you mean by return? So, just to illustrate it through a simple example suppose that you invest in amount of 100 today and then at the end of 1 year it becomes 110. So, this means that you have made an additional amount of 10 in 1 year upon your original investment of 100. So, basically this will be given by 110 minus 100 divided by 100. So, K_n here basically will mean that it is going to be the difference in the stock price between two subsequent, two consecutive stock prices at two different times, and divided by the price at the previous time. So, I suppose we take some time point n and $n+1$, then the return in this particular time interval will be given by the stock price at time $n+1$ minus the stock price at time n divided by the stock price at time n. So, then this particular return is going to be a random variable. So, this stock these are i i d random variable. So, this return K_n these are all random variables such that K_n is going to be equal to u with probability p and d with probability 1 minus p. So, this means that if the stock price to today is say S of n this means that tomorrow the stock price can be S of n multiplied by $1+u$ with probability p, or it could become $S(n)(1+d)$ with probability $1-p$. So, this essentially what it does is that between any two time intervals or any two time points the change in the stock price can only happen in two possible ways. One when it is changes by a factor of $1+u$ and 1 when it changes by a factor of $1+d$ and since this is a random variable. So, we take the corresponding probabilities to be p and $1-p$ respectively. So, now, this is this will happen at each time step n, where I need to put this condition. So, one of the condition would be a $1<d<u$, all right and the second is $0<p<1$. So, we take the condition $-1<d<u$ the reason being that that I want first of all that $1+d>0$. So, that the stock price and also I then I need; obviously, $1+u>0$; so, that the stock price at the next time point when it is obtained by either multiplying by $1+u$ and $1+d$ will still end up giving you the price of the stock to be positive. And I need $0<p<1$ and the reason I cannot allow $p=0$ or 1 because, if either $p=0$ or $p=1$; that means, the returned $K(n)$ is no longer going to be a random variable and then it will become a certain return and then it will start acting like a bond. So, next this implies that the stock price. So, just to sum up whatever I have said this implies that the stock price S_n can move up or down by factor $1+u$ or $1+d$ respectively at each time step. So, here typically u is what is known as the up factor and d is what is known as the down factor. So, typically we say that the stock price from the current time of S_n, in that in the next time point it can either go up to $S(n)(1+u)$ with probability p or it can come down to $S(n)(1+d)$ with probability of $1-p$. So, then note that the condition so, as I had mentioned before this condition of $-1 < d < u$ guarantees that, if you start off with a positive stock price; then $S(n)$ will also be positive for all n. Now, further we let r be the risk free rate for a single time step. So, r being the risk free rate basically means that if you invest an amount of $S(0)$ in a bond today. And r is the interest rate that is prevailing for a single time period, then at the end of that time period you will receive an amount of $S(0)(1+r)$.

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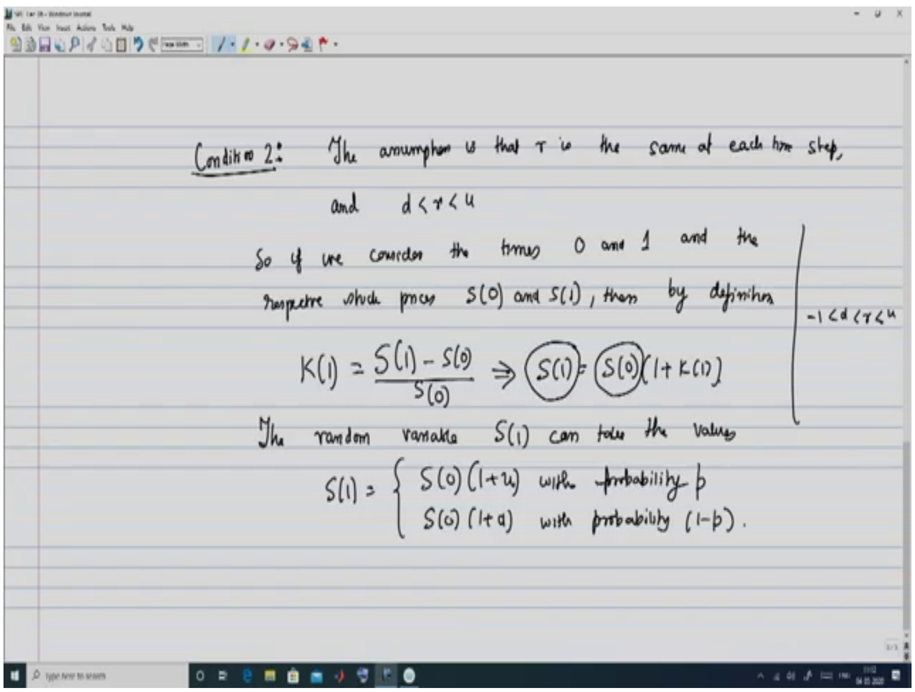

We now come to the second condition and this condition will be in terms of this r that we have just introduced. So, the second condition is involves the assumption is that r is the same at each time step so; that means, r remains constant as long as you are considering the model of the asset and $d < r < u$. So, if we consider the times say 0 and 1 and the respective stock prices $S(0)$ and $S(1)$, then by definition. What will we get? We will get so, by definition $K(1) = \frac{S(1)-S(0)}{S(0)}$ and this implies is that $S(1) = S(0)(1+K(1))$. So, the random variable now remember that $K(n)$ and in particular $K(1)$ can take two values. So, the random variable $S(1)$, then can take the values $S(1) = S(0)(1+u)$ and $S(0)(1+d)$ with probability p and with probability $1-p$. So, here I would like to just make one more observation, that we had this condition that $-1 < d < u$ and then I put in r in between. So, the reason why this is very crucial; so, I have already explained why we must have > -1, because we need $1+d > 0$ resulting in the stock price at our subsequent points being positive if we start off with a positive stock price. And by the same logic I need $1+d = 1+u > 0$. So, that the stock price remains positive along subsequent points. Now, the third point that we have just introduced in the second condition, where basically r lies between d and u, the reason for this is that suppose that $r < d < u$. So, this means that the return in that so, the stock price in any single time interval can either go up by a factor of $1+u$ or come down by a factor of $1+d$. Now, if my$r$ is less than both d and you this means that if I invest in the stock in the worst case scenario, I will end up with $S(0)(1+d)$, but as in comparison to that if I invest in a risk free asset that is a bond, then ill end up with $S(0)(1+r)$. So, they should mean that $S(0)(1+r) < S(0)(1+d) < S(0)(1+u)$. So, this means that in the worst case scenario the stock price will still be higher, at the end of the time period as compared to an investment in a bond which cannot happen, because the stock is a risky asset. And also we need my $r < u$, because if $r > u$, this means that your return on a risk free investment is always going to be more than the best case scenario that is $S(0)(1+u)$ in case of a risky asset, which leaves no incentive for anyone to invest in the risky asset. Because, an investment in a risk free asset at a return of r obviously, is always going to be higher. So, with these two cases ruled out you basically then need that your r must lie between d and u in both the cases, ok. So, now what you do is that I have just looked at the kin scenario, when I look at some time point index by 0 and 1. And now, let us look at some n time process suppose that we are looking at a stock price say over at prime interval. And suppose you are looking at time 0.0123 this could be days first day second day and so on. So, suppose we consider a general scenario and try to see what is going to be if you start off with a stock price of $S(0)$ which is deterministic. So, in this case your $S(0)$ is known it is deterministic its only $S(1)$ that is random because setting at time $t = 0$, you do not know for certainty with what $S(1)$ is going to be. So, likewise instead of just considering the time point 0 and 1 we consider some generic time point n. And look at what the stock price is likely to be or what are the possible stock prices that we can have at the time and that is what are the possible candidates for $S(n)$ if we start off with the initial stock price namely $S(0)$.

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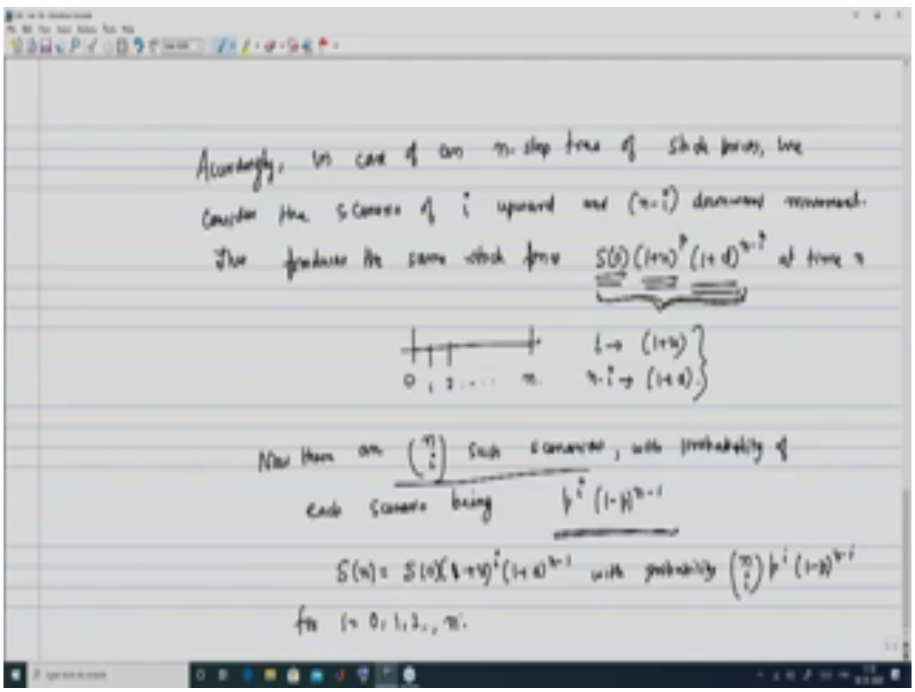

So, accordingly in case of an n step tree of stock prices, we consider the scenario of i upward there is i number of $1+u$ and $n-i$ downward movement. So, this produces the same stock price $S(0)(1+u)^i(1+d)^{n-i}$ at time n. So, this means that say you have this time interval $0,\ 1,\ 2,\cdots,\ n$, and I say that as we move from 0 to n while reaching there you will basically have i number of cases of $1+u$, and you will have consequently $n-i$ number of cases of $1+d$ movements. And so, consequently the stock price is then be going to be as $S(0)(1+u)^i(1+d^{n-i})$, where your I can be $0, 1, 2, \cdots, n$. So, this means that you could have if $i = 0$; that means, that right from the beginning to the end. There is no going to be no upward movement and is going to be all downward movement. If $i = 1$; that means, in this from 0 to n you will have only one upward movement at some test single time step and for the remaining time steps you will just have all downward movements and so on. So, generically if we have if you are looking at n steps and if start off with $S(0)$ and you want to see what is the stock price is going to be at the end of n steps, where between 0 to n you have a i number of up movements and $n-i$ number of down movements. This is what you are going to end up with $S(0)(1+u)^i(1+d)^{n-i}$. So, accordingly now there are n choose one such scenarios. So, this means that between in the n time intervals that are between 0 to n, you can have i number of ups in many different ways. It could be that at the first i movements from step 0 to i you have all up movements it could be that the last i step it could be that there are alternative type step. So, there are different and a sequence in which you can have i number of upward movements, and that number of such possible combinations is going to out of this n is going to be $\binom{n}{i}$. And since there are $\binom{n}{i}$ such scenarios of i upward movement and $n-i$ downward movement. So, this means that so this means that it will have the probability of each scenario being. So, remember that the movement from one time step to another time step, where there is an upward or downward movement they are all independent of each other. So, this means that the probability of i upward movement is going to be given by p^i, and the consequent $n-i$ downward movement will be given by $(1-p)^{n-i}$. So that means, that there are n such scenarios with the probability of this scenario being $p^i(1-p)^{n-i}$. So, now we have both the identifiers as far as the stock price $S(n)$ is concerned, we know what is going to be the stock price and we also know what is going to the corresponding probability. So, this means that I can write $S(n) = S(0)(1+u)^i(1+d)^{n-i}$ with probability $p^i(1-p)^{n-i}$ and this can happen in $\binom{n}{i}$ number of ways. And this can hold for $i = 0, 1, 2, \cdots, n$.

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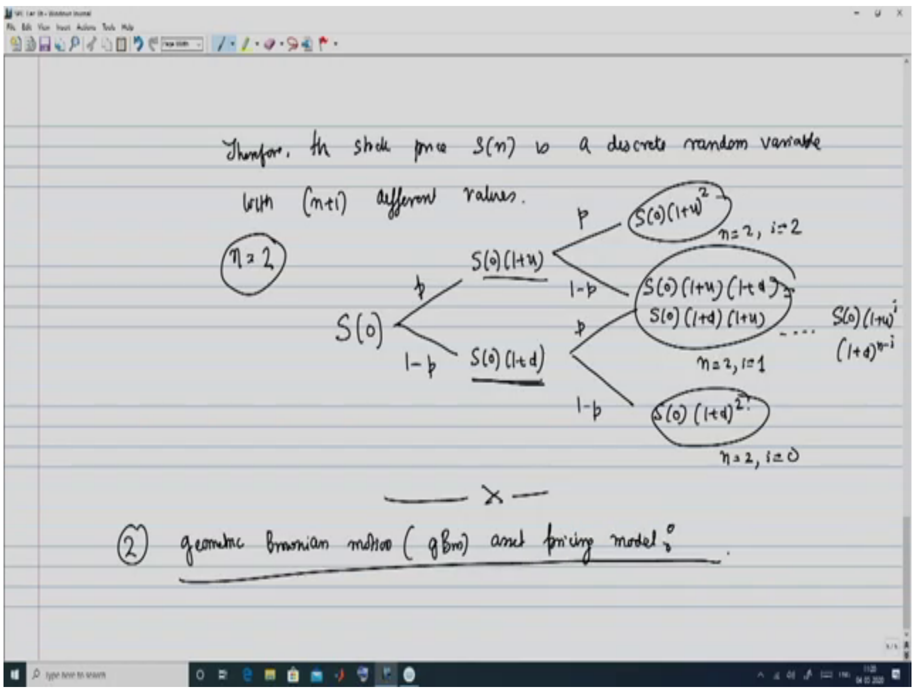

So, therefore, the stock price $S(n)$ is; obviously, a discrete random variable with $n+1$ different values. So, I just conclude the discussion with a graphical representation for $n = 2$. So, suppose that you start off with $S(0)$. So, according to the binomial model, we can go up to $S(0)(1+u)$ with probability p or go down to $S(0)(1+d)$, with probability $1-p$. Now, again from here I can go up to $S(0)(1+u)(1+u) = S(0)(1+u)^2$. And this probability is going to be p or I can go down to $S(0)(1+u)$ which is the existing price multiplied by $1+t$. Similarly, here when I start off with the existing price $S(0)(1+d)$, I can go up to $S(0)(1+d)(1+u)$ and note that these are both equal. And so, the probability here is going to be $1-p$, this is p and here with probability $1-p$, I can go down from $S(0)(1+d)^2$. So, that is how we generically end up with our formula for $S(0)(1+u)^i(1+d)^{n-i}$. So, here $n = 2$ so, for $i = 0$ we have this so, this is when $n = 2$ and $i = 0$, this is the scenario when $n = 2$ and $i = 1$. And this is the scenario when $n = 2$ and $i = 2$, ok. So, this concludes the discussion on the binomial model. So, next we look at the other model which is the geometric Brownian motion gBm asset pricing model, ok.

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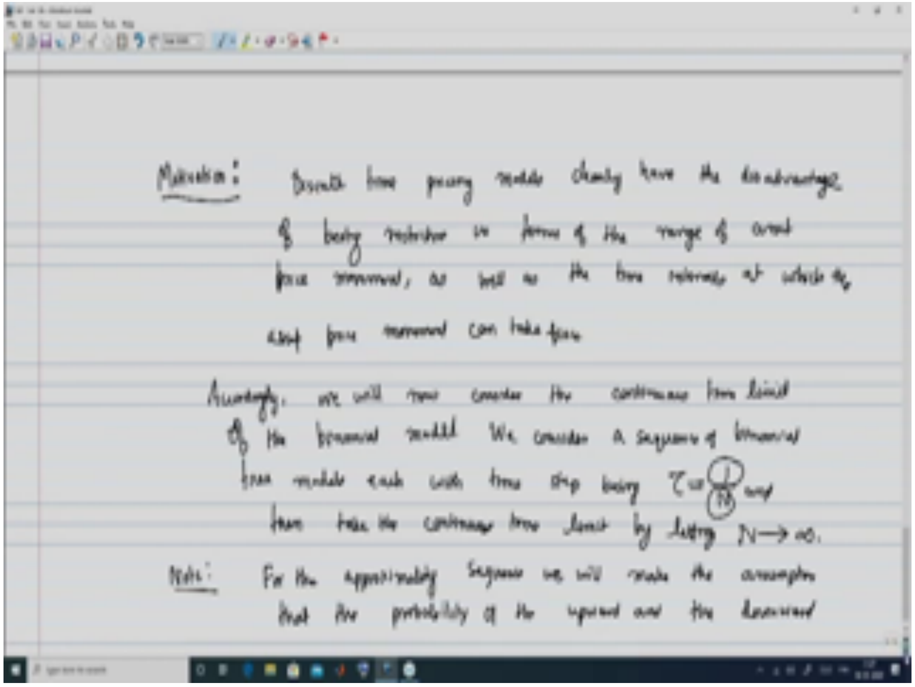
So, let me first of all start with a motivation for this, in from the point of view of it being extended from the binomial model. So, the primary motivation is that the discrete time pricing models clearly have the disadvantage of being restrictive in terms of the range of asset price movement. As well as the time intervals, at which the asset price movement can take place. So, let me explain this two points in a slight amount of detail. So, the first disadvantage I said that it is restrictive in terms of range of asset price



movement. So, it is easily sort of observable from the previous discussion regarding that, if you start off with an amount of $S(0)$ at time 0. And then you want to look at the asset price $S(n)$, and we saw that this is going to be $S(0)(1+u)^i(1+d)^{n-i}$. So, this means that after if you are looking at a n point n time interval discrete model, then you end up with only $n+1$ possible values of the stock price which are the random variables at time n. So, this means that you only have a limited number of possible stock price, which is not consistent with what actually happens in the real world. And secondly, you are talking about the time points at which the stock prices can change. So, we are talking about some time point $0, 1, 2$ perhaps on a daily basis, but given the way stocks are traded now, this is more of a continuous process. And what this binomial model does is that it actually restricts the time point at which the stock prices can change. So, in order to address these two key shortcomings of the binomial model, a natural way is to move on to the continuous time model. So, this can take place ok. So, accordingly we will now consider the continuous time limit of the binomial model. So, what do you do is that we consider a sequence of binomial tree models, each with time step being tau is equal to $\frac{1}{N}$ and then take the continuous time limit by letting $N \to \infty$. So, what this means is that we basically look at some time length of 1 and we divide it into n number of subintervals. So, the length of each of those sub intervals is going to be one over N. So, suppose we take our time interval of 1 year and you want to look at the asset price movement over 1 year, and you take $N = 2$. So that means, τ is going to be equal to half a year which is 6 months, then we take say $N = 12$; that means, τ is going to be 1 month if we take $N = 365$, then $\tau = 1$ day. So, what do you do is that we basically look at various different scenarios of such binomial models with different values of N. And as we increase N we observe that the time interval between any 2 consecutive asset price movement, which is given by τ that becomes smaller and smaller. And as your $N \to \infty$, your $\tau \to 0$ so; that means, that the asset price movement model, now is effectively a continuous time model, because the time interval between any two prices movement it tends to 0. So, what you are going to do is that we will look at a binomial model for asset pricing with N. And, then you make n smaller and smaller and let N tends to infinity, and that would give us a continuous approximation of the binomial model which in turn is going to result in the geometry Brownian motion for asset pricing, ok. So, now what you do is that we make a note that we will have to make a simplifying assumptions. So, that the proof is accessible.

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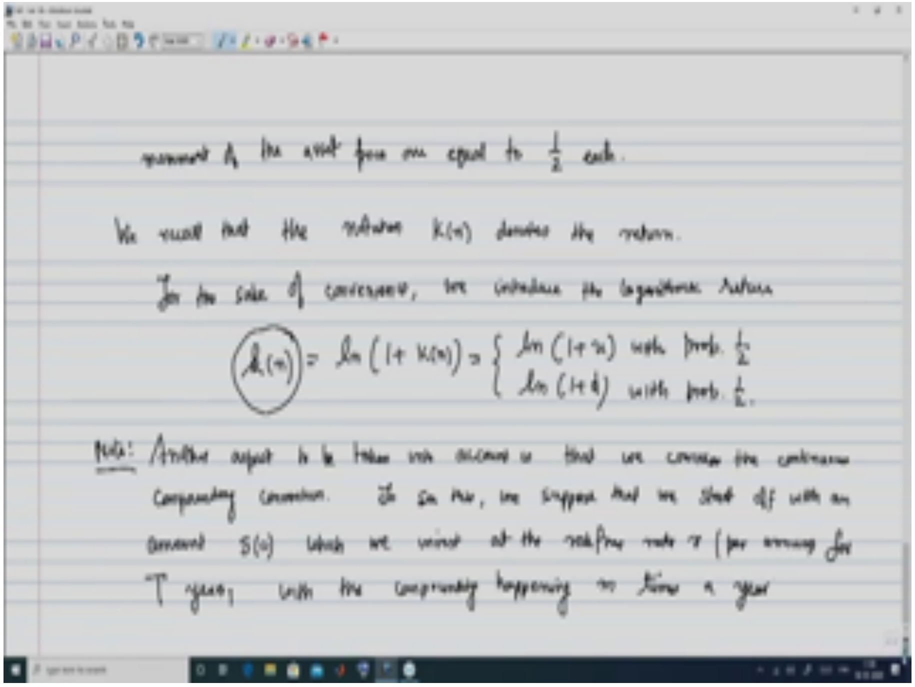
So, for the approximating sequence we will make the assumption that the probability of the upward and the downward movement of the asset price are equal to half each. That means, we take p is equal to half



and $1-p$ equal to half, ok. So, now we recall that so, recall the notation $k(n)$ which denoted the return at the nth time step. Now, for the sake of convenience and we will later on see why this needs to be done, we introduce what is known as the logarithmic return. So, logarithm return is nothing, but the natural log remember a in finance log always miss natural log that is busy. So, $\ln(1+k(n))$, I will define this to be the logarithmic return which is $k(n)$. And this will be nothing, but this can take the value $\ln(1+u)$ and $\ln(1+d)$, and remember here we took the probabilities to be identical. So, $k(n)$ can have probability half and half in the two scenarios. So, for now from now on will primarily be using the logarithmic return. Now, another aspect so, I will make another observation here. So, the another aspect to be taken into account is that we consider the continuous compounding convention. So, to see this in more detail we suppose that we start off with an amount $S(0)$, which we invest at the risk free rate or the rate of the bonds are per annum for say T number of years and with the compounding happening m times a year.

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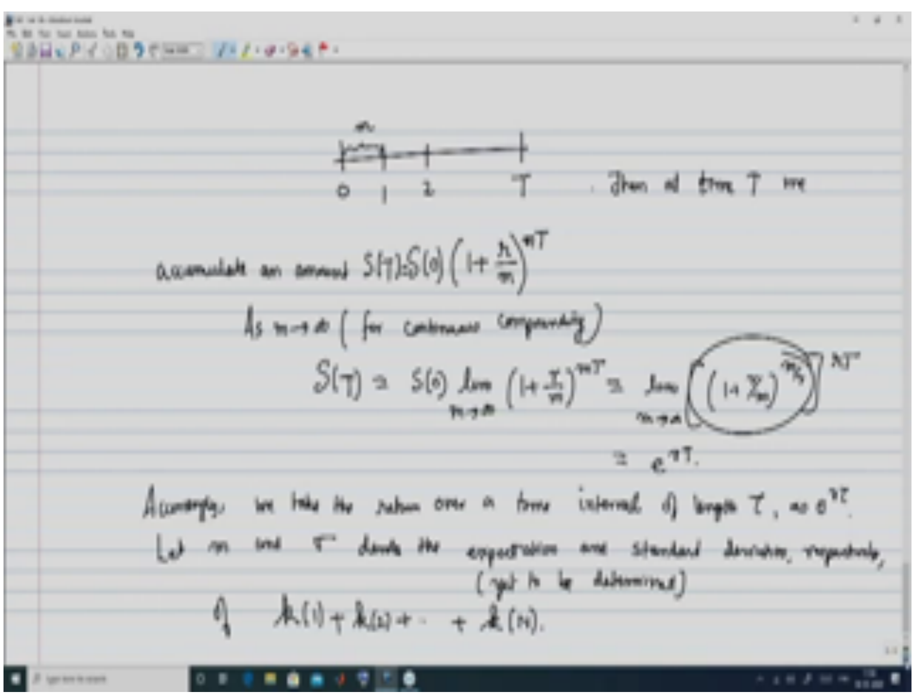

So, this means that I have this interval $[0, T]$ and this is 1 year 2 year and so on. And in each year I compute the interest m number of times. So, this means that if I start off with an amount of $S(0)$, then how will you do the compounding. So, for each period we will do the compounding so, if r is the annual interest rate then the per period interest rate is $\frac{r}{m}$. So that means, for the whole year it is going to be $r(1 + \frac{r}{m})^m$. So, this is the amount of money that I am going to get starting off with an amount of S naught. So, slight correction I start off with $S(0)$. So, this is the amount of money that I will get at the end of 1 year. So, at the end of T years I will get the power to be mT. So, then I can write so, then I can make the statement that at time T, we accumulate an amount given by $S(T) = S(0)(1 + \frac{r}{m})^{mT}$. Now, if I want to do continuous compounding this means that you are basically having the interest is being calculated on a continuous basis. So, this means that I will take as m tends to infinity for continuous compounding what do you get, then what is going to be $S(T)$ if you start off with $S(0)$. So, this is going to be $S(0) \lim_{m\to\infty}(1 + \frac{r}{m})^{mT}$. And this can be written as $\lim_{m\to\infty}[(1 + \frac{r}{m})^{\frac{m}{r}}]^{rT}$. And this limit you know is the exponent. So, this is going to be e^{rT}. Now, here I took T to be some fixed number of years, but it is $2e$ in general for any other time point. So that means, that if your r is the interest rate, then at any and you start off with an amount of $S(0)$, then you can see that at time t your $S(T) = S(0)e^{rt}$, ok. So, let us now come back to our main discussion on the gBm. So, accordingly we take the return over a time interval remember, we took the time interval notation to be $\tau = 1/n$. So, we take a time interval of length tau as $e^{r\tau}$, ok. So, let m and sigma denote the expectation. So, I move on to the next step and I will introduce two variables. So, denote the expectation and standard deviation respectively and these are yet to be determined remember the so, here at this point we do not know what m and sigma. So, these are the expectation standard division respectively of what random variable $k(1) + k(2) + \cdots + k(n)$. So, this is basically the sum of the log returns of each individual intervals.

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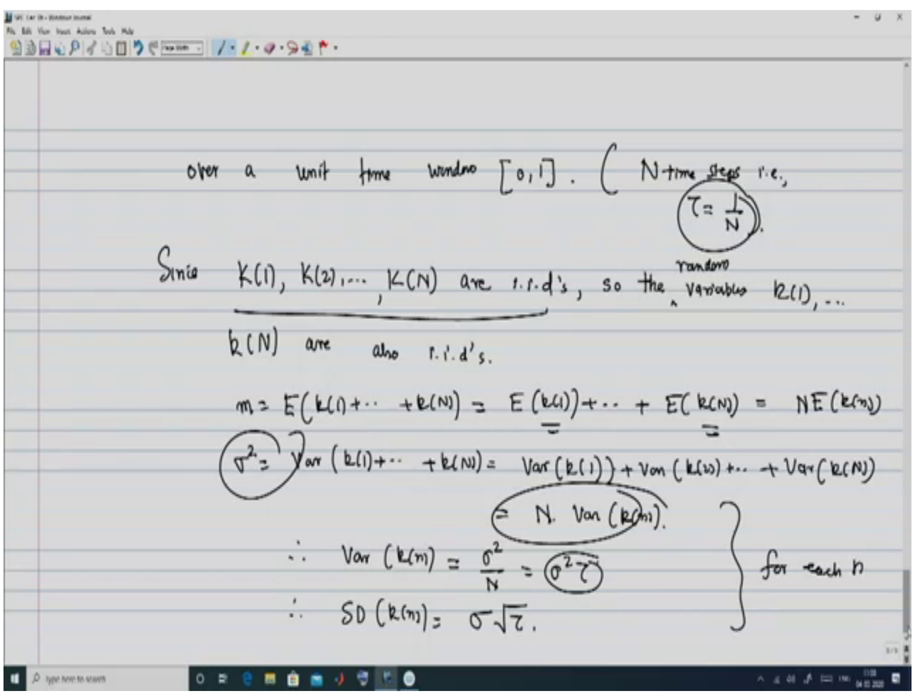

And this is over a unit time window of $[0, 1]$. So, just recall that you had N time steps and so, that is we had τ that is each time interval to be $1/N$. Now, since your original return $k(1), k(2), \cdots, k(N)$, they were remember that they were i.i.d, because they were modeled through the binomial model. And so, they were independent and so they are identically distributed and of course, they are independent because, we assume that the change in the asset price between any two consecutive time steps, they are all independent of each other and they of course, follow the identical binomial distribution. So, from for capital $k(1), \cdots, k(N)$ which is the original variable for the distribution. So, I can since these are i.i.d's so, consequently I can say that so, the variables $k(1)$. So, the random variables $k(1)$, that is the log return through $k(N)$ are also independent and identically distributed ok. Now, let us go back to this I said that m is the expectation of this random variable so; that means, I can write this as m is equal to expected value of $k(1), \cdots, k(N)$. And this by the linearity property I of expectation; so, this is where we use the linearity property of expectation that we discussed earlier. And now, since these are all independent and identically distributed they are basically going to have the identical mean of say $E(k(n))$ of some generic $k(N)$. And there are n number of such variables so, this will be $NE(k(n))$. Similarly sigma square is what? It is basically variance of $k(1), \cdots, k(N)$. And this is going to be nothing, but variance now again these are since these are independent so, the covariance terms will not show up. So, this is going to be simply nothing, but $NVar(k(n))$. So therefore, variance of $k(N)$ from this relation will become equal to σ^2/N and remember $N = 1/\tau$ so, this becomes $\sigma^2\tau$. So, therefore, the standard deviation of $k(n)$ this is going to be nothing but the square root of this term which is $\sigma\sqrt{\tau}$. And please remember that these two these are these will all hold for each n, ok. Now, here we will essentially look at a slightly simplifying assumption.

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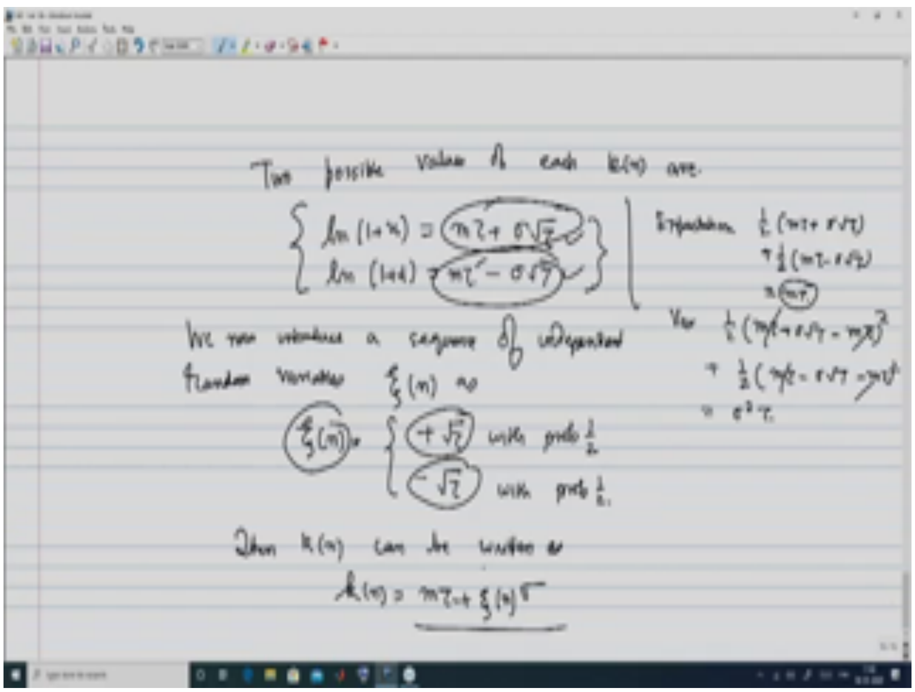

So, here what you do is that so, we have seen that the expected value of $k(N) = m\tau$ so, here I just forgot to add this here. So, this will imply that so therefore, expected value of $k(N) = m/n = m\tau$. Ok. So, now, here the two possible values of each $k(N)$ are so, I make a total obvious choice. So, I want to figure out what my random variable k n is going to be in terms of this m and σ. So, two possible values of each $k(N)$ are so, this is good to be $\ln(1 + u)$. So, this is going to be $m\tau + \sigma\sqrt{\tau}$. And $\ln(1 + d) = m\tau - \sigma\sqrt{\tau}$. So, you can actually verify that here the expectation, what is this going to be? This is going to be $\frac{1}{2}(m\tau + \sigma\sqrt{\tau}) + \frac{1}{2}(m\tau - \sigma\sqrt{\tau})$ and this is just $m\tau$. So, basically that the expectation is going to be $m\tau$ for each of the $k(n)$. And we want to show that the standard deviation is $\sigma\sqrt{\tau}$. So, that also you can calculate easily so, again just I use the definition. So, for this I will have variance of the random variable $m\tau + \sigma\sqrt{\tau} - E(m\tau^2 + \frac{1}{2})$. Again I will have $m\tau$ the other random variable $m\tau - \sigma\sqrt{\tau} - E(m\tau^2)$. So, these cancels out, this cancels out. So, this just simply becomes $\sigma^2\tau$. So, I have taken basically two particular cases that is $m\tau + \sigma\sqrt{\tau}$ and m term minus $\sigma\sqrt{\tau}$, ok. So, now we observe this two and this motivates us to introduce a sequence of independent random variables, call them $\xi(n)$ as what is $\xi(n)$? $\xi(n)$ I will define this to be $+\sqrt{\tau}$ with probability $1/2$. So, this is motivated by the fact that this term and this term the only difference is that of a sign. And this will be $-\sqrt{\tau}$ with probability $1/2$. Now once I have this definition of $\xi(n)$ so, I can actually combine these two possible values. And so, then $k(n)$ can be written as $k(n) = m\tau$

and then I have the σ term. So, the only place where these are distinguished is $+\sqrt{\tau}$ and $-\sqrt{\tau}$ so, I can write this as $m\tau + \xi(n)\sigma$. So, you can easily see that $k(n)$ takes the value of $m\tau + \sqrt{\tau}\sigma$ probability $1/2$ and $m\tau - \sqrt{\tau}\zeta$ and with probability $1/2$. So, this is ζ.

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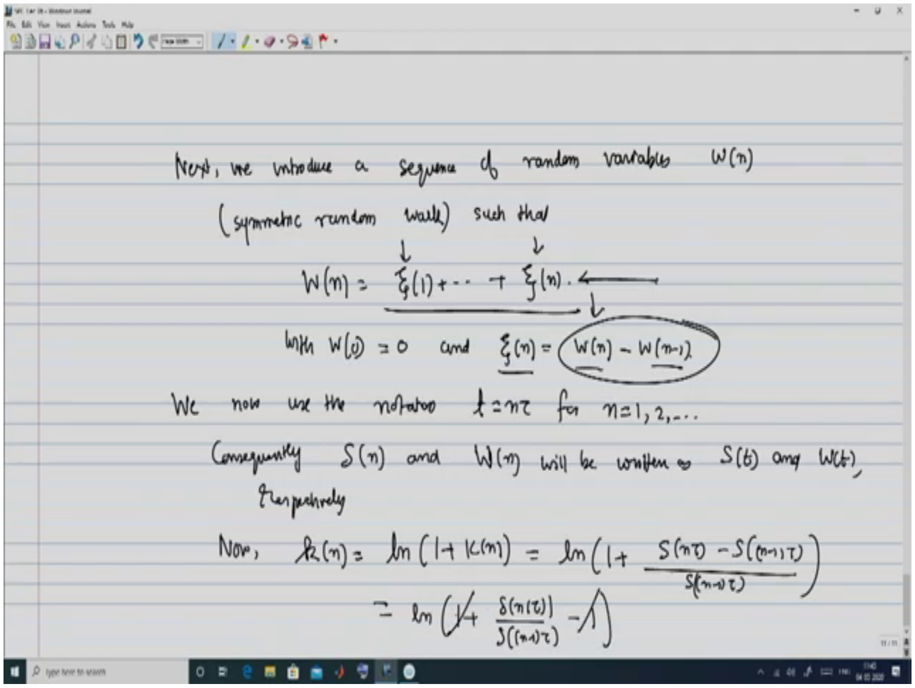

Next we introduce a sequence of random variables say $W(n)$, this is what is known as a symmetric random walk, such that how do I define this? It is going to be nothing, but some of this $\xi(1) + \cdots + \xi(n)$ with $W(0) = 0$. So, this is actually ξ I have been calling it zeta so, please note that this is ξ. So, here $W(0) = 0$ and $\xi(n) = W(n) - W(n-1)$. So, now, I have found an equivalent representation of this random variable $\xi(n)$ and we I am calling this to be my difference between $W(n) - W(n-1)$, ok. So consequently so now, what you do is that we first of all make a slight change of notation. So, we now use the notation $t = n\tau$ for $n = 1, 2,$ so on. And consequently my stock price $S(n)$ and $W(n)$ that we have here and I have defined here this will be written as $S(t)$ and $W(t)$ respectively. So, I will use w here. So, please note that this is the small w. Now, what is $k(n)$ let us go back to k and

$$k(n) = \ln(1 + K(n)) = \ln\left(1 + \frac{S(n\tau) - S((n-1)\tau)}{S((n-1)\tau)}\right) = \ln\left(1 + \frac{S(n\tau)}{S((n-1)\tau)} - 1\right).$$

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So, this will give me that

$$\frac{S(n\tau)}{S((n-1)\tau)} = e^{k(n)}.$$

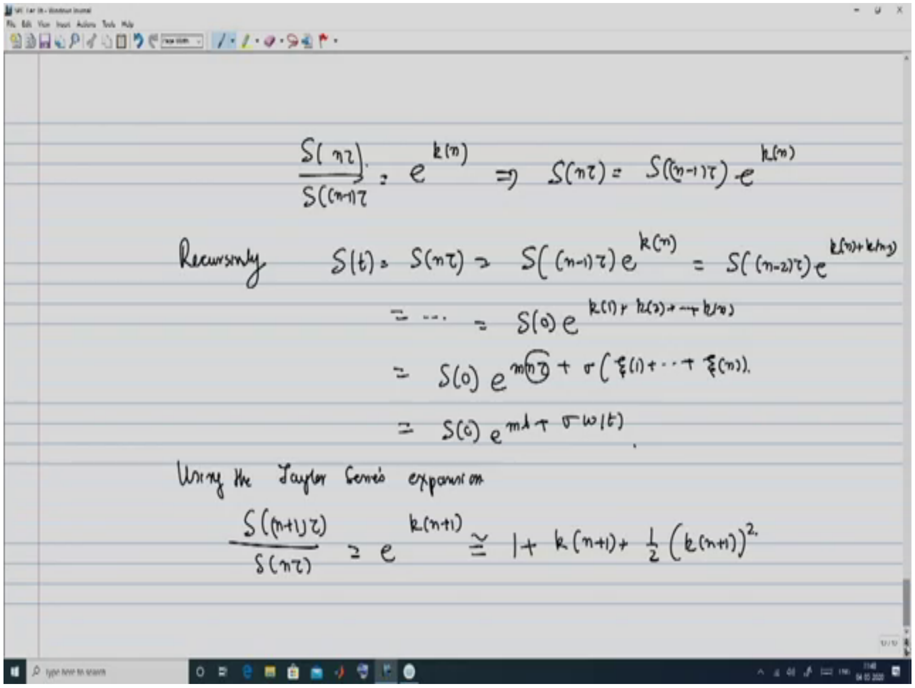
So, which implies that

$$S(n\tau) = S((n-1)\tau)e^{k(n)}.$$

So, accordingly so recursively we will get

$$S(t) = S(n\tau) = S((n-1)\tau)e^{k(n)} = S((n-2)\tau)e^{k(n)+k(n-1)}$$

and I will keep doing this. Until I reach $S(0)e^{k(1)+k(2)+\cdots+k(m)}$. Now, remember that my representation of $k(1)$ through $k(n)$, what did I take I have taken my random variable $k(n)$ to be written in this form. So, I will make use of that so, its $m\tau + \xi(n\sigma)$. So, this is going to be simply $e^{mn\tau}$; that means, there are n



number of $m\tau$'s plus we have a τ. And then for $k(1)$ I have $\xi(1), \cdots, \xi(n)$ and remember. What is this? What is $n\tau$? $n\tau$ is t so, this becomes e^{mt} plus this. So, this is actually a $\sigma\tau$ so, plus σ and remember this is $\xi(1), \cdots, \xi(n)$ and this is defined as $W(n)$ and so, this becomes now $\sigma W(t)$. Ok. So, now, we will make our Taylor series approximation. So, using the Taylor series expansion what do I get? I will get that

$$\frac{S((n+1)\tau)}{S(n\tau)} = e^{k(n+1)} \approx 1 + k(n+1) + \frac{1}{2}(k(n+1))^2.$$

We neglect the higher order terms here.

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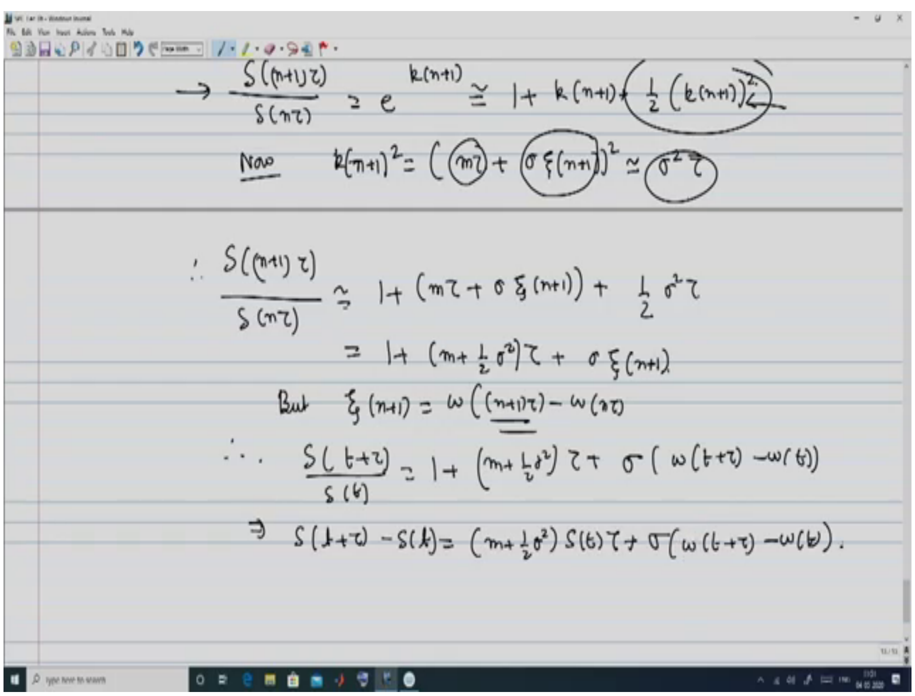

So, this is going to give me. Now, what I need to take care of is a we need to look at this term first. So, now, $k(n+1)^2$, what is $k(n+1)$? $k(n+1) = [m\tau + \sigma\xi(n+1)]^2$. Now, if we expand this say you will have an $m^2\tau^2$ term here. So, that is order of τ^2 and here you will have $\sigma\sqrt{\tau}$, because by definition of $\xi(n+1)$. So, this will give you $\sigma^2\tau$ and the cross term. So, there will be tau here so, the $2m\tau\sigma\sqrt{\tau}$ that will give $\tau^{\frac{3}{2}}$. So, I ignored the as the square term and 3^2 term and retain only the $sigma^2\xi n^2$ term which is τ, because $\xi(n+1)$ is of the order of $\sqrt{\tau}$, ok. So let me now come back to this. So, therefore, I have $\frac{S((n+1)\tau)}{S(n\tau)}$, this is approximately going to be $1+k(n+1)$, which I will now substitute as $m\tau+\sigma\xi(n+1)+\frac{1}{2}$ and this term I will replace with $\sigma^2\tau$. And, this I can rewrite it as $1+(m+\frac{1}{2}\sigma^2)\tau+\sigma\xi(n+1)$. But, recall that by definition $\xi(n+1) = w((n+1)\tau) - w(n\tau)$. So, therefore, from here we get

$$\frac{S(t+\tau)}{S(t)} = 1 + (m + \frac{1}{2}\sigma^2)\tau + \sigma(w(t+\tau) - w(t)).$$

This implies

$$S(t+\tau) - S(t) = (m + \frac{1}{2}\sigma^2)S(t)\tau + \sigma(w(t+\tau) - w(t)).$$

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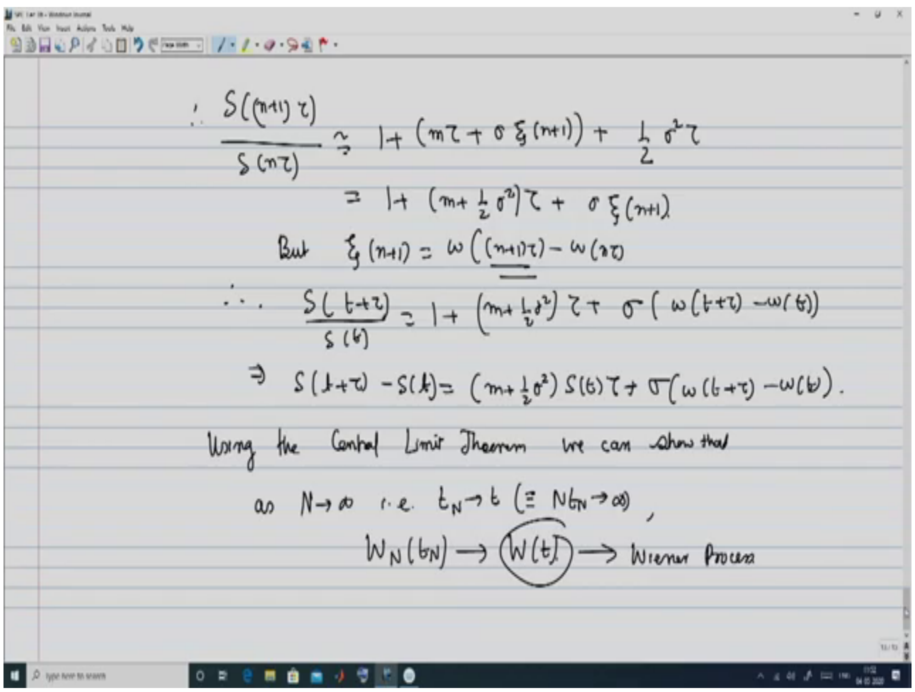

Now, one can make use of the Central Limit Theorem, which we had mentioned in one of the earlier classes, we can show that that as $N \to \infty$, that is $t_N \to t$ or equivalently $Nt_N \to \infty$. The following holds that $W_N(t_N) \to W(t)$ and here this $W(t)$, this is known as the Wiener process. So, what are the properties of Wiener process?

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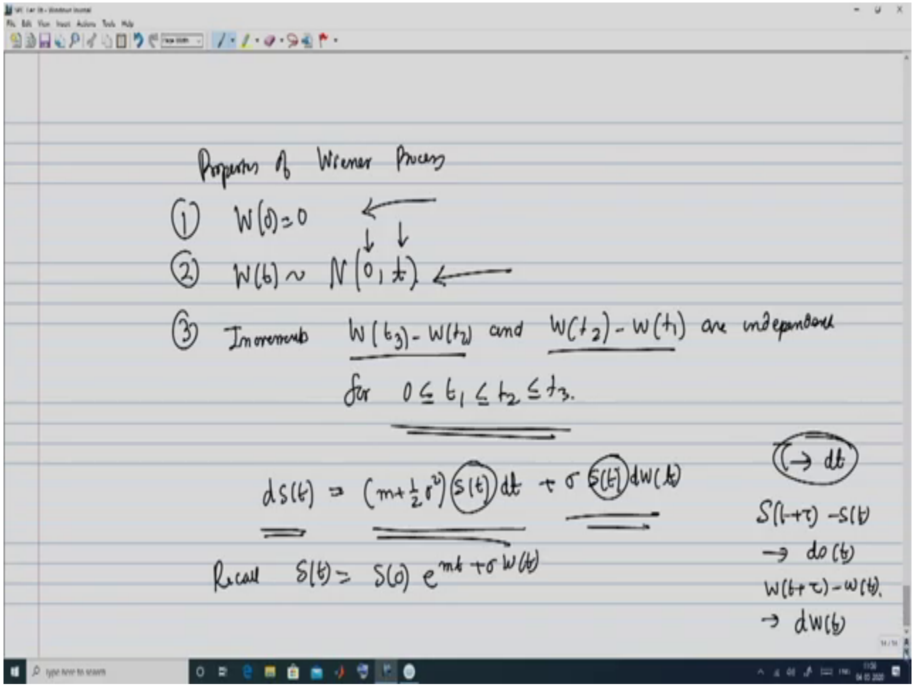
So, the properties of Wiener process the first property is $W(0) = 0$. Secondly, $W(t)$ the it follows a normal distribution with mean 0 and variance of t. And thirdly the increments say $W(t_3) - W(t_2)$ and $W(t_2) - W(t_1)$ are independent for $0 \le t_1 \le t_2 \le t_3$. So, this means that $W(0) = 0$ and $W(t)$ is a random variable which is a normal random variable. So, you see that is why we had to make use of that we define the normal distribution earlier. And the mean of this is 0 and the variance is t, and the last one it says that if we consider two non overlapping intervals (t_1, t_2) and $t_2, t_3)$. Then the corresponding increments of $W(t_2) - W(t_1)$ and $W(t_3) - W(t)$ to these corresponding increments are independent of each other. So, once we have this definition then we can now go back and look at this. So, what I am going to do is I am going to rewrite this. So, I will call this so, I take the small interval and this can be rewritten as $dS(t)$. So, remember that this is the change in the interval. So, instead of $S(t+\tau) - S(t)$ as $N \to \infty$ your $\tau = 0$. So,



accordingly this is going to be $dS(t) = (m + \frac{1}{2}\sigma^2)S(t)dt$. So, I will now replace τ by some small interval dt. So, then $S(t+\tau) - S(t)$, this will become $dS(t)$ and $w(t+\tau) - W(t)$, this becomes $dW(t)$, because of the central limit theorem that I have just mentioned. . So, this becomes $m + (1/2)\sigma^2 S(t)dt$. So, from this term and plus $\sigma dW(t)$. So, plus $\sigma S(t)dW(t)$. And this $S(t)$ which shows up on both sides is because, you have $S(t)$ that was in the denominator. So, we had $\frac{S(t+\tau)-S(\tau)}{S(t)}$, so, that has come to the top part. Now, we recall that $S(t) = S(0)e^{(mt+\sigma W(t))}$. So, what do you do now is so, this is what we had done earlier and this is what we now follows as a limiting case of the binomial distribution.

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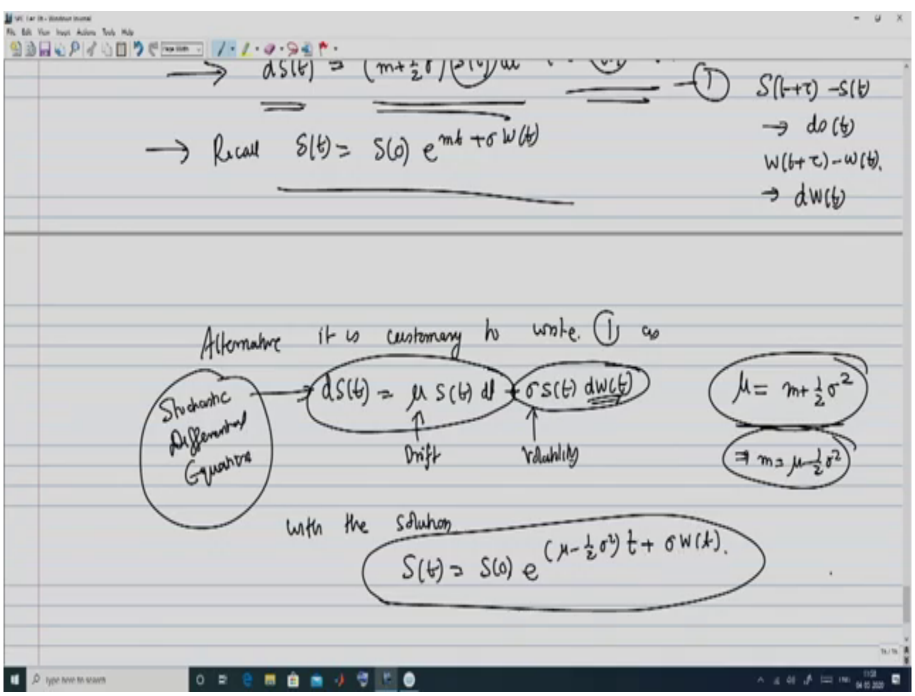

So, alternatively thus it is customary to write. So, let me call this equation 1. So, to write 1 as

$$dS(t) = \mu S(t) + \sigma S(t)dW(t).$$

. So, basically I will replace mu with you I will use μ to replace $m + (1/2)\sigma^2$. So, this is lot of times this is called drift and this is called the volatility of the asset. So, once you choose $\mu = m + (1/2)\sigma^2$. So, then the second relation accordingly becomes so, this has the solution $S(t)$. What is m? m is equal to $\mu - (1/2)\sigma^2$. So, this is

$$e^{(\mu-(1/2)\sigma^2)t+\sigma W(t)}.$$

And this is what is known as a this is an example of what is known as the stochastic differential equation. So, if you had only these two terms this would be an ordinary differential equation, but now that you have added this term which as the Wiener process which is a random variable. So, accordingly this becomes what is known as a stochastic differential equation. And if solution under some conditions is going to be given by

$$S(t) = S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma w(t)}.$$

So, just to do a recap what we are done today is we looked at the binding binomial model. And then we looked at a couple of shortcomings of the binomial model. And, then we moved on to the asset pricing model, in the continuous time driven by Wiener process. And this is also what is known as the geometric Brownian motion and this exercise today of discussing these things also highlights. The background on probability theory that we have used namely we looked at the binomial distribution, we have made use of the normal distribution, we mentioned about the central limit theorem and the properties of mean and variance. So, this brings us to the end of our discussion on markets and the asset pricing models that we discussed today. From the next class we will start our discussion on the main topic for the course, and we will begin with the modern portfolio theory due to Markowitz.

Thank you for watching.