Discrete-Time Markov Chains and Poisson Processes

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Lecture 14

Module: Classification of States

Number of visits

Welcome to the 14^{th} lecture of the course Discrete-Time Markov chains and Poisson processes. Recall that in the last lecture, we have talked about what is called recurrent state and what is called transient state and to define those recurrent state or transient state we need what is called first passage time. It defines first passage time which basically means that after the time 0, times 0 is excluded when the chain will visit for the first time the state i. And then if the probability of visiting the state i in finite time for the first time given the chain is starting at the state i is 1 we call such states as a recurrent state if it is strictly less than 1 we call such states as this transient state. And then we talked about the fact that or in case of the recurrent state that basically means that if i is recurrent then starting from i, I will come back to the state i again and again again and again. And it was a basically if and only if condition that i is recurrent if and only if the probability starting from i, $X_n = i$ for infinitely many n is 1. So, I will visit the state i again and again starting from i if i is recurrent, on the other hand, if it happens that starting from i, I am keep on visiting the state i after sometime then i has to be recurrent. Those things we have seen then we talked about what is called k^{th} passage time which is nothing but after time 0 when the time at which the chain is visited the state i for k^{th} time. And then we have defined what is called the length of the excursion which is nothing but the time difference between two consecutive visits to the state i and we see what is the distribution of S_i, what is the distribution of $S_i^{(k)}$ that we have seen. With that, we start with today's lecture and today's lecture, we are going to talk about what is called number of visits.

Number of visits is just what is basically mean that how many times the chain is visited some state. Definition goes like that for a state i, we define the number of visits to state i by this particular quantity just recall that we define $\delta_i(x)$ to be 1 $x = i$ and it is 0 otherwise, that means that number of times $X_n = i$ occur. So, I am taking the collection of all possible values of $\{n \geq 0 : X_n = i\}$ and I am taking the cardinality of that set that is basically this one. That basically means number of times the chain entered into the state i that is basically nothing but our V_i and V_i is called number of visits to the state i.

Now, what we are going to see is that what is the distribution of V_i, the distribution of V_i is given here, which is basically that for $k = 0, 1, 2, \cdots$. The probability that $(V_i > k)$ is given by f_i^k and this is a conditional probability. So, this probability as we know it is nothing but probability that $(V_i > k)$, that starting from $X_0 = i$ that is basically f_i^k and what is f_i?. f_i is nothing but probability that starting from i, what is the probability of $X_i < \infty$. Recall that, when we define that recurrence and transients in terms of f_i, if $f_i = 1$ we call i to be recurrent, if f_i is strictly less than 1 we call i to be transient. This is basically the distribution of V_i and notice that if I know this quantity for all values of k, I know all kind of probability for V_i. For example, I can easily calculate this probability also this probability is nothing but as follows. Basically if this point is k, then basically what I have to do is that I have to take probability in this side minus probability at $k+1$. That means, this is greater than $k-1$ minus probability of $Vi > k$, i.e., $P_i(V_i = k) = P_i(V_i > k-1) - P_i(V_i > k)$. And now, the first quantity from here is basically nothing but $f_i^{k-1} - f_i^k$. So, in this way I can able to find all the probabilities for $V_i = k$ and if I get that I can able to find what is the probability that $V_i = 0$ plus probability V_i equals to maybe 100. This now, I can easily find out the first quantity will be $V_i = 0$ that basically 0. Because that is basically k start from 1 here. Because I take $k = 0$ there is a problem. So, I can write this probability as probability of V_i is I make it 1 because that 0 is basically does not make much sense here. So, I make it 1 and then it is basically nothing but $P_i(V_i > 0) - P_i(V_i > 1)$, this is basically the first part plus I have the second part that $P_i(V_i > 99) - P_i(V_i > 100)$. So, this way actually I can find all the probabilities just 1 thing I by mistake, I have written 0 here, but 0 will not be there because that $V_i = 0$ does not make any sense in this case, because the thing is that actually, that probability you can take as 0 because I am starting from i and $n = 0$ is included here. So, V_i has to be greater than 0 with probability 1. So, V_i equals to 0 there is no other option for me, that is there, there is no other option for me. So, this way that my main point here is that if I know this probability, I can actually all kinds of conditional probabilities regarding V_i when $x_0 = i$ regarding V_i, I can calculate all the conditional probabilities if I know this particular thing. Now we will go for proof, but to prove this one we are going to use what is called mathematical induction. I know many of you know what is mathematical induction, but just recall that or if it is new to anybody, that is for them to introduce to what is mathematical induction. Let us first discuss what is mathematical induction is, and then we will proceed to what is the proof of this particular theorem. The mathematical induction can be stated as follows. Suppose, I want to prove the statement P_n holds for all n equals to, maybe $n = 0, 1, 2, \cdots$, or sometimes 0 may not be there or 1, 2 may not, or 0, 1 may not be there those kinds of scenario occurs, but I am taking the most general case. For example, I may think of the statement $P_n : 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. The problem is that I am trying to prove that $P(n)$ is a statement which holds for all possible values of n. Like P_n could be that statement, I may try to solve that the sum of first n natural numbers is $\frac{n(n+1)}{2}$. So, the way of mathematical induction goes like that we generally show the step 1 which is called a basis is that to show that P_0 is true that basically means, in our example, P_0 is basically nothing but 0 equals $\frac{0(0+1)}{2}$. So, I take $n = 0$ here and I need to show this and this is trivially true, in this case. So, the first basis part is that I have to show that if our $n = 0$ the statement is true. So, the first point that statement is true whatever the first one if 0 is not here, if n start from here, then I have to show that this statement is true for $n = 1$. So, basis is that at the first point that the initial point the statement is true, that is that basic, then we go for the induction step. And which is basically to show as in the induction step, the assumption is that $P(k)$ is true. That basically means that $n = k$. We assume that for $n = k$, the statement is true, we need to show that $P(k + 1)$ is also true. So, we assume that if $n = k$ the statement is true, we want to show that for $n = k+1$ the statement is also true. For our example that basically means that $P(k)$, which is basically nothing but $0 + 1 + \cdots + k = \frac{k(k+1)}{2}$, we assume that this is the assumption. Now, based on this assumption, we want to show that $P(k+1)$ which basically means that $0+1+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$. And how we can show?. First part that note that this part is basically from our assumption in that induction step this part says that, it has to be $\frac{k(k+1)}{2}$ and then I have $k+1$, I add these 2 and finally, if you add this one to the algebra, you can easily show that this turns out to be this one and that is that completes the proof. So, the idea here is that first we will show that $P(0)$ is true, then we assume that $P(k)$ is true and based on that assumption, we will show that $P(k+1)$ is true. And then I am done because 0 is true that I can take k equals to 0 and it says that $P(k + 1)$ is true. So, $P(1)$ is true. Now, if $P(1)$ is true, then using the induction step I can tell $P(2)$ is also true then because $P(2)$ is true, I can also now show that $P(3)$ is also true and so on so forth. So, that is basically the idea of induction and to prove this lemma we will basically going to use the induction so in our case, basically induction means that we will show that for k equals to 0 the statement is true then we assume that this statement is true for k and we will show that for $k + 1$ also this statement is true. So, in the Layman's word that mathematical induction can be put in this way, suppose I am trying to climb a ladder and the way we climb is basically step by step we go off. So, mathematical induction basically tells that if I am in the bottom rung, then basically the basis part is that I am in the bottom rung that is true I can stand on the bottom rung that is true and then induction step says that if I am in the k^{th} rung then I can move to the $(k + 1)^{\text{th}}$ rung. If these two is true that I can be on the bottom rung and then if I am in any rung, then I can move to the next rung, then using the mathematical induction this says that I can climb as high as I want that is basically the mathematical induction in very Layman's word. With this understanding, let us now proceed to the proof of this lemma.

The proof of this lemma goes like that as I already pointed out, we will basically use the mathematical induction to prove this lemma. Now, the first observation in this lemma is this one that see that $V_i > k$ what is V_i?. V_i is number of time the chain visits the state i. So, this part is saying that the Markov chain visits the state i at least $k+1$ times. And so, that obviously true now is that k^{th} passage time has to be finite. If the Markov chain visits the state i for at least k times then k^{th} passage time has to be finite. That basically if I for example, if I take k equals to 1 and what I am saying here is that the Markov chain has visited the state i at least twice that basically mean that this is $T_i^{(0)}$ then somewhere I have $T_i^{(1)}$ and somewhere I have $T_i^{(2)}$. And it has visited at least twice that means basically it may be this or it may be this or if I start from something else, then these two has been visited. Finally is that that if I take k equals to 1 then this time has to be finite there is no question about that, this has visited at least twice. So, this time has to be finite there is no problem I mean that this has to be true. And on the other hand, if this is true that this one is finite, then the Markov chain has to visit the state i at least twice that means that V_i has to be greater than 1 that is there. Of course, there one assumption is there because always I am talking that I am starting from i. So, this observation is true under the condition that $X_0 = i$ these two things are same. So, this is i of course, this is i. So, V_i is greater than k that means, if $k = 2$, then V_i is at least 2. So, it has visited here it has visited here. So, $T_i^{(1)}$ has to be finite, which is basically this part on the other hand $V_i^{(1)}$ is finite that means i is here and then again it has visited the state i here, so, V_i has to be greater than 1. So, under the condition that $X_0 = i$ that V_i greater than k and $T_i^{(k)} < \infty$, these two are same events. So, if I try to find out the probability of i here, and probability of i here they have to be same, because the two events are equal. So, their probabilities will be equal, but under this condition because these two events are equal under this condition only. Now, let us go for now induction. First point is the basis point this is the basis that statement is true for k equals to 0. Why statement is true for k equals to 0?. The reason is as follows that for k equals to 0 what happens here is that $P_i(V_i > 0)$. Now, as I mentioned that what is V_i just let us write the definition of V_i here, which is $\sum_{n=0}^{\infty} \delta_i(X_n)$. So, because $n = 0$ is included here and these basically means that $V_i > 0$ given $X_0 = i$. So, V_i is at least 1. So, V_i is always greater than 0. So, clearly this is equals to 1. On the other hand, if you take f_i^0 because f_i is the probability is always lies between 0 and 1. So, this quantity is also 1 and now, I am done and our basis step is true that for k equals to 0 this statement is correct. Now, we have to assume that for this statement is true for k. That means that we assume now is that it is true that induction hypothesis is that for k equals to k it is true.

That means $P_i(V_i > k) = f_i^k$ we want to show that $P_i(V_i > k+1) = f_i^{k+1}$ if based on this assumption if we can show this one then we are done using the mathematical induction. I can now claim that this particular statement is true for all values of $k = 0, 1, \cdots$. Let us try to prove that we assume that at k equals to k this is true, I want to show that for $k+1$ this statement is true, which is basically given here. Let us go through it how we can prove that. So, starting from i, $V_i > k+1$ is same as starting from i, $T_i^{(k+1)} < \infty$ these two events are same. So, their probability has to be same these two events are same because of these, but this is basically conditioning on $X_0 = i$ and that condition is kept here. So, these two probabilities are same. Now, this probabilities what? it is basically that I have $T_i^{(0)}$ here then $T_i^{(1)}$ here and so on so, forth I have somewhere $T_i^{(k)}$ and then I have $T_i^{(k+1)}$. So, it says that $T_i^{(k+1)}$ is finite that clearly means that $T_i^{(0)}$, $T_i^{(1)}$, $T_i^{(k)}$ all of them has to be finite there is no question about that this is finite means all of them has to be finite. So, I can write this one as that this one is finite plus this interval this time difference is finite and this time difference is nothing but f_i^{k+1}. So, this one is finite, and then this time difference is also finite, which is nothing but S_i^{k+1}. So, I can write the probability given $X_0 = i$ that $T_i^{(k+1)}$ is finite is same as conditional probability given that $X_0 = i$, $T_i^{(k)}$ is finite and S_i^{k+1} also finite. And these two events are equivalent event in the sense that if this one is true, then this one is also true. On the other hand when this one is true, then this one is also true conditioning on $X_0 = i$. Now, we know that $P(A \cap B)$ can be written as $P(A|B)P(B)$. So, in this case take this one as B and this one as A, this is $A \cap B$ which can be written as $P(A|B)P(B)$. Only thing is that here the condition that $X_0 = i$ is there that I have to keep everywhere, and we have this now, the next second part because of this assumption, the second part is basically equals to f_i^k because again that second part I can write as $P_i(V_i > k)$, because these two events are same under the condition that $X_0 = i$. The next this part is basically same as f_i^k. I have written this one here this probability is same as f_i^k. Now, I have to find out the first one just recall the distribution of this $S_i^{(k)}$ we have pointed out that given that $T_i^{(k)}$ is finite, the distribution of $S_i^{(k+1)}$ is same as the distribution of T_i starting from i. We have proved this thing earlier that this lemma we have seen in the last lecture that if the distribution starting from i that $S_i^{(k+1)}$ is equals to n given that $T_i^{(k)}$ is finite is same as distribution of starting from i, I have $T_i^{(1)}$ which is T_i also is equals to n. This distribution of this one is actually given same as this one it is basically using the previous lemma I mean last lemma we have discussed in the last last lecture. Using that we can get this one. Now, it is true for all values of n. So, if I take the sum over n on both the side that basically mean that this side is nothing but it is basically all possible values of n so, it is nothing but $P_i(S_i^{(k+1)} < \infty | T_i^{(k)} < \infty)$. Because if I take all possible values of n here, so $S_i^{(k+1)}$ takes all possible value and so, $S_i^{(k+1)}$ takes value

$1, 2, \cdots$. Finally, that means that $S_i^{(k+1)}$ takes all finite value. So, $S_i^{(k+1)} < \infty$. Similarly, if I take the sum in this side that turns out to be $P_i(T_i^{(1)} < \infty)$, because this part is this one and this probability we have taken to be f_i here. That means, the first part of the probability is f_i. So, finally, we get f_i^{k+1}. So, we have shown that under the assumption that the statement is true for k we have shown that the statement is true for $k+1$ also and that completes the proof using the mathematical induction. So, we have proved that starting from i, the probability of $V_i > k$ is same as f_i^k, f_i is nothing but $P_i(T_i < \infty)$ and it is true for all values of $k = 0, 1, 2, \cdots$. This is the distribution of V_i.

Now, let us again revisit the transient just recall that in case of the recurrent we have shown this result. i is recurrent if and only if starting from i, I keep on visiting the chain keep on visiting the state i throughout the expcursion of the chain throughout that process of the chain. Throughout the I mean when chain goes to the future times the i^{th} state come back again and again again and again. That statement we have seen earlier, that basically means that if a state i is recurrent I will see the state again and again if one state is seen for seen again and again after some time, then that state is recurrent one. Very Layman's word or very loosely speaking that is basically the recurrent state. Now, in case of the transient the scenario is just opposite is the statement is exactly same here probability statement only difference is that here it is 0, here it is 1. It says that a state i is transient if and only if $P_i(X_n = i \text{ for infinitely many } n) = 0$ and both of them are necessary and sufficient conditions that means, if I can able to show that that i is transient then this is true similarly in this case also if i is recurrent this is true on the other hand if the statement is true then it has to be recurrent. These two things actually give me the intuitive idea behind transient and recurrent state; recurrent state when it will keep on visiting but for the transient state I mean surely I will not visit the state i for infinite number of times that is the statement means. That means that for transient state after some finite value the state will not be visited. The proof is very simple. Notice that this statement is basically equivalent to write the statement that means that number of visit to state i is infinity. Now, this probability is same as this limit that the reason behind that if you take k goes to infinity that V_i is greater than any finite thing. That means V_i is infinity, that is why this probability is same as the this limit of this probability $P_i(V_i = \infty)$ is same as limit k goes to infinity $P_i(V_i > k)$. Now, because we know that $P_i(V_i > k)$ is same as f_i^k. So, this quantity is same as limit k goes to infinity f_i^k. And in case of the transient state we know that $f_i < 1$ that is the definition because f_i is nothing but probability $P_i(T_i < \infty)$, and we know in case of the transient state by definition this quantity is less than 1 that means this limit is 0. That means that $P_i(X_n = i \text{ for infinitely many } n) = 0$. So, the main takeaway from this corollary and remark is that the intuitive idea behind the rcurrent state and transient state.

Recurrent statement means these will keep on coming after some times but transient state after some finite time that state will not be visited again.

With that, let us proceed and let us now see another set of necessary and sufficient condition for recurrent and transient. This is one set of necessary and sufficient condition for transient. This corollary and this remark another set of necessary and sufficient condition for recurrence and transients are given here. It says that a state is recurrent if and only if, $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$, just recall that we defined $P_{ij}^{(n)} = P(X_n = j|X_0 = i)$ and we define $P_{ij}^{(0)} = 0$ if $i \neq j$ and 1 if $i = j$. This way we have defined. So, it basically says that a state i is recurrent if and only if $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$ and i is transient if and only if, the same sum is less than infinity. So, they sum in this time is finite. i is recurrent means that the sum has to be infinite, on the other hand i is transient implies that the sum has to be finite on the other hand if the sum is finite then the i is transient. So, this is one way to show, to check whether i is recurrent or i is transient or not and the way basically we need to find out all the values of $P_{ij}^{(n)}$ and then I need to take the sum and if the sum is finite, we can directly tell that i is transient and if the sum is infinite I can tell that i is recurrent. Now, let us go to the proof. Proof is not very difficult actually, what we going to do is that we are going to prove the first one and if I can prove the first one the second one automatically follow from the first one, how?. Suppose that we have already proved the first one. Now, I have to prove two things that if given condition is that i is transient we have to show that this sum is finite; that is to show that summation $P_{ii}^{(n)}$ is finite and n runs from 0 to infinity. Now, I can go in this way. i is transient given I have to show this one suppose if possible assume the sum n equals to 0 to infinity $P_{ii}^{(n)}$ is less than infinity. What I am doing is that if possible suppose this is not true. That means this quantity is infinite and if now, I can prove the first one this one is infinite implies that i is recurrent. And that is the contradiction because I start with i is a transient I get i as a recurrent. So, that is a contradiction that means what I am assumed here is not true and that means this one is true. So, i is transient implies that this one is finite. Similarly, if I assume this one is finite, then if possible I take i is recurrent and if i is recurrent then this one has to be infinite and that is a contradiction. So, clearly if this one is finite then i transient. So, that means that if I can able to prove the first statement, I am done with both the statement, because the second statement automatically follows from the first statement you can prove it by the contradiction . Let us look into the proof of the first one. First part we have start with that suppose it is given that i is recurrent and I want to prove that the sum is infinite; suppose that i is recurrent, we want to show that $P_{ii}^{(n)} = \infty$ when I am taking the sum n equals to 0 to infinity. Let us recall the definition of recurrent state which is nothing but $f_i = 1$. Now, you see that in this case $P_i(V_i = \infty)$ that means, I keep on coming to the state

i again and again again and again that probability is 1. That means V_i takes the infinite value with probability 1. So, if I now try to calculate what is the expectation of V_i under $X_0 = i$ that is nothing but the value multiplied by the corresponding probability which is basically infinity. So, conditional expectation of V_i given $X_0 = i$ is infinity if i is recurrent. So, I write this one exactly here that infinity is equals to $E_i(V_i)$. Now, I just plug in the definition of V_i here and because expectation is a linear operator I can take this expectation inside the summation sign and I get this one this is nothing but the generalization of this kind of thing that if I have $X_1 + X_2$ that I can write as expectation of X_1 plus expectation of X_2 and this is of course, for finite one and this is for infinite one. So, you can take this particular thing is basically nothing but some kind of generalization of this particular kind of statement what we know. Now, if I take the expectation here recall what is $\delta_i(X_n)$?. $\delta_i(X_n)$ is nothing but taking value 1 if $X_n = i$ and it takes value to 0 if $X_n = 0$. Now if I try to find out what is the expectation of $\delta_i(X_n)$ that is nothing but 1 multiplied by the probability that $X_n = 1$ because the condition will be the $X_0 = i$ and then $X_n = i$ plus 0 times probability that $X_n \neq i$. This $\delta_i(X_n) = 1$ if $X_n = i$ and it is 0 if $X_n \neq i$. That means that when if I try to find out the expectation of $\delta_i(X_n)$ which is the value of $\delta_i(X_n)$ multiplied by the probability plus value of $\delta_i(X_n)$ multiplied by a corresponding probability. So, that terns out to be this and this tern do not contribute anything this 1, I can remove. Finally, I get it the conditional expectation of $\delta_i(X_n)$ given $X_0 = i$ is same as $P_i(X_n = i)$ which is written here. And this quantity is nothing but our $P_{ii}^{(n)}$. This shows that if i is recurrent we have shown that $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$. So, one side is done that if i is recurrent, we have shown that the sum is infinite. Now, I have to show the other side that if this sum is infinite I have to show that i is recurrent let us look into that.

Now, suppose that summation is infinite we want to show that i is recurrent. The way to prove this one is again by contradiction that if possible, let us take that i is transient. So, if i is transient then $f_i < 1$ this is by the definition of the transient state. So, what I have now is this one that this quantity is same as this that, we have exactly shown here forget about whether it is a finite or infinite, we have shown that this is same as this does not matter whether i is recurrent or i is transient only thing is that if i is recurrent, then this expectation is infinite and that shows that this expectation is infinite. This is the same process I can use and I can claim that this equality holds true. Now, we have used a fact which the proof I am not giving here, but I use this one as a fact that tells that for a non negative integer values random variable X the expectation of X can be written in this way, this is nothing but $\sum_{n=0}^{\infty} P(X > n)$, that is what we are going to use here. Because V_i is the number of visit to state i and that has to be a non negative integer valued random variable this V_i is a non negative integer valued random variable. So, I can use this fact on

V_i only thing is that I have this condition with respect to i here, I have to keep that one here also. Now, it can be written that $P_i(V_i > r)$ and I have to take the sum over r in this case I have written in terms of n here I have written in terms of r. This probability I have already found out this is nothing but equals to f_i^r these probabilities basically f_i^r and when I do this sum that is basically nothing but a geometric progression and $f_i < 1$. So, it is the infinite geometric progression and the sum we know it is nothing but $\frac{1}{1-f_i}$ because f_i is strictly less than 1 and because f_i is strictly less than 1 this quantity is finite. So, this shows that if i is transient then this summation has to be finite and that is a contradiction because we have started with that summation $P_{ii}^{(n)}$ is infinite that is a contradiction of the fact that the summation $P_{ii}^{(n)}$ is now finite if I assume i is transient so i has to be recurrent. So, this assumption is wrong and i has to be a recurrent state. That completes the proof and as I mentioned that these two conditions are very very helpful to prove different kinds of results regarding Markov chain to check whether for a given Markov chain, given state is transient or recurrent these two results can be used.

Now, we are going to talk about a theorem which basically deals with finite state Markov chain. What this theorem says?, the theorem says that, suppose I have a Markov chain X_n which has finite states, S which is the state with which is finite. That means that number of states in the state space S is finite and I can write in this manner that they are maybe N number of states if I assume that there is N number of states, then basically the state space can be written in this particular form. Now, in that case, when I have a Markov chain with finite state space S then the theorem tells us that there has to be at least one recurrent state. So, that theorem says that, for a finite state space Markov chain, there at least one recurrent state and there could be many there could be more than one, but at least one state among the state space has to be recurrent. And the basic intuition behind this is very very simple. The intuition is as follows. See, when I have finite number of states, suppose, for example, suppose I have 5 states. Now, when the Markov chain actually evolving with time, then as the time increases Markov chain has to be in some one of the state at least it cannot be go out of this 5 state because this is the whole state space. So, it has to take value from this 5 states. So, that means, at least 1 state has to be visited infinite number of time, the reason is as follows that if all the states are visited finite number of times suppose the state 1 is visited in n_1 times step 2 is visited n_2 times and so on so, forth state 5 is visited in n_5 times, then if you think this number that $n_1 + n_2 + n_3 + n_4 + n_5$, then I know the state that the Markov chain will be the state space up to that time. Now, I add up 1 more time to that that means $n_1 + n_2 + n_3 + n_4 + n_5 + 1$ where the Markov chain will be Markov chain will have to be in some state into the inside the state space. So, that means, it is not possible that the all the states are visited only a finite number of times. So, that is basically the intuition behind this particular theorem. Now, we will see that how we can write this one mathematically. The proof goes like that, let us start with that the state base is given by this where the N is finite, N is integer and a finite positive integer, that is our state space. Now, look into this expression, let us first decode this right hand side. Let us talk about the inside one. If I take i equals to 1 the inside summation is $\sum_{j=1}^{n} \delta_1(X_j)$. What is that recall that we defined $\delta_c(x) = 1$ if $x = c$ and is the 0 otherwise. That way that $\delta_1(X_j) = 1$ if at the time step j, I visit the state 1, then basically this quantity will take value 1 otherwise, this 1 to 2 take value 0. Now, when I take the sum that basically going to give me number of times the state 1 is visited after time 0, 0 excluded after time 0 till time n, in between time 1 to n, how many times the state 1 is visited that can be written in this particular form. Similarly, when I take i equals to 2 that is basically nothing but $\delta_2(X_j)$ which is basically again nothing but number of times the state 2 is visited between times $[1, n]$. And so on so forth finally, I have summation $\sum_{j=1}^{n} \delta_N(X_j) = 1$ that is basically the number of times the state N is visited between times $[1, n]$. So, these are the quantities inside for different values of i taking here the first quantity is nothing but number of times state 1 is visited second quantity is number of times state 2 is visited and so on and so forth. Finally, the last quantity here when $i = N$ this quantity is nothing but number of times state N is visited between the type $[1, n]$. Now, if I add up this thing that is nothing but total number of times the state 1 is visited state 2 is visited state 3 is visited state 4 is visited so on and so forth. Finally, the state N is visited between time $[1, n]$. And that is nothing but n because the state has to be any one of the state in the time $[1, n]$. So, that means if I add up how many times the state 1 is visited how many times the state 2 is visited dot dot dot how many times the state N is visited that will is going to give me the number of state that I have taken. So, that is why basically if I make this sum that double summation in this side that is going to give me n which is basically nothing but number of steps I have taken from time 1 after that time n which is basically nothing but n. So, we have this particular quantity. Now, you see that if I take the limit on both sides of this particular expression, I have limit n tends to infinity double summation the first summation is over i equals to $[1, N]$. The second summation is j equals to 1 to n for $\delta_i(X_j)$ i.e., $\lim_{n\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{n} \delta_i(X_j)$. Of course, this side is infinity because I am taking the limit of n, n tends to infinity so, this side is infinity now, look into this side you notice that N is finite here because the state space is a finite state space. So, N is a finite integer. So, easily I can interchange the limit and the first summation and I can write these quantities equals to limit n goes to infinity summation j equals to 1 to n for $\delta_i(X_j)$. That means that if you look into this part this part is nothing but I am taking the limit on each of these quantities as n goes to infinity. Now, this summation has to be infinite and this is a finite sum. So, that means, at least 1 of these quantity has to be infinite. This means that if I take the limit on both of them that at least one of the limit has to be infinite because this equality has to true. So, this whole summation need to be infinite and because this n is finite that says that at least one of the inside quantity has to be infinite. Now, let us call the corresponding i for which this quantity is infinite to be i_0. I say that it is at for at least one i is there for which this one is infinite and call this i is i_0 of course we can have multiple i for multiple i this quantity is infinite that can happen. But I can tell for sure that at least for one i this inside quantity has to be infinite and call this i to be i_0. It says that taking the limit on both the sides we get they are must exists an i_0 such that this quantity is equals to infinity the limit I just replaced this n with infinity here that by writing the limit there, so, basically this quantity has to be infinite. Now, what is this quantity?; this quantity is nothing but number of visits to the state i_0 starting from 1 up to the time infinity. So, total number of visits to the state i_0 this is basically this and that quantity is infinity with some positive probability that means, the state i_0 will be visited again and again again and again again and again. And that says that the state i_0 has to be a recurrent state. So, this theorem gives us a very nice thing that if I have a finite state Markov chain, I can directly tell that there exists at least one state in the Markov chain, which is recurrent. With that, I stop and thank you for listening.