Discrete – Time Markov Chain and Poisson Processes
Professor Ayon Ganguly
Department of Mathematics,
Indian Institute of Technology, Guwahati
Lecture 15
Module: Classification of States
Lecture: Class Property

Welcome to the lecture number 15 of the course, Discrete-time Markov chains and Poisson processes. Recall that in the last lecture, we have talked about what is called number of visits, we denoted by V_i and then we have seen what is the distribution of V_i and finally, we have seen a very critical result, which can be used to classify a state into recurrent or transient. Which is in terms of $p_{ii}^{(n)}$, It says that i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$. We have seen this result that i is recurrent if and only if summation $p_{ii}^{(n)} = \infty$ had the sum runs from 0 to ∞. And we have seen i is transient if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$, these two very important results we have seen. And then today we will going to discuss about what is called class property. So, we are going to first show that transient or recurrence are class property, before going into this one just recall that what is a communicating class. So, we just recall that if this is our state space S, then if I take all the if I consider the relation that i and j are communicating each other that means, j is accessible from i, and i is accessible for j and in layman's term that mean that in some step with positive probability I can move from i to j and similarly, in some step with positive probability I can move from j to i. So, we have seen the definition was that there exist in n and m integers such that I can move from i to j in n steps. So, this probability is strictly greater than 0. Similarly, from j to i in m step is strictly greater than 0. And we have seen that if I talk about this relation then this relation actually partition the state space. That all-state here communicates, all state here communicates, similarly, it communicates here and it communicates here, and no state from one partition to another partition communicates, I can move from this to this but I cannot, in that case I cannot come back from here to here. So, moving out is possible, but in this case moving in is not possible, if moving out and moving in is possible, then this completely is a one class. So, that is why this way we partition and now we look into the theorem, this theorem tells us that, well C be a communicating class, suppose C be a communicating class, suppose C is this set, which is a communicating class, then either all states in C are recurrent or all are transient. So, in C, if either all of the states in C, suppose C has several states, all of the states in C either recurrent all of them will be recurrent, or all of them will be transient. There cannot be any communicating class in which one state is recurrent and another state is transient, that cannot happen. Only the only possibility is that in a in a communicating class, all the states are of the same type either they are all recurrent or they are all transient. And that is why this practical property is called class property that basically mean that, in a class that all the states are either transient or recurrent. All of them will be transient or, all of them will be recurrent. It is like that. So, let us move on to the proof and to prove this one we are going to use the definition of communication as well as we will use the Chapman-Kolmogorov equations, and we also going to use what we have seen in the last class that means, these two things, that we will going to use.

So, let us proceed to the proof, so, C is a communicating class, so, I take two states in C so, if this is my S, inside that I have C. And suppose i and j are two communicating class here, and i and j are two different, then by definition of the communication we know that there exist $n, m \geq 0$, such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$. So, that means from i , if this is i, this is j , from i I can move to j with positive probability in n steps and from j , I can move to i in with relative probability in m steps. Now, suppose that i is transient we want to show j is transient, if we can show the j is transient, then I am done because here I am taken any pair of states is ij in C so, in any pair of state if one of them is transient then other is transient, that means all the states in C will be transient. I take any pair of state if one is transient then other is transient, if I can able to prove that, then that basically mean that all the states in the communicating class C will be transient. So, we assume that i is transient, we want to show that j is also transient. So, how can we prove? I can how can we proceed? The way of proceeding is simple that just recall that Chapman-Kolmogorov equation. So, Chapman-Kolmogorov equation tells us that, if I want to find out $p_{ij}^{(n+m)}$ that I can write as $\sum_{k\in S} p_{ik}^{(n)} p_{kj}^{(m)}$. That is why I can able to write so, that the idea is nothing but if I if this is the state i, this is the state j . I want to move from here to here in $m+n$ step, I want to find out the probability of moving from this to this in $m+n$ step, then what I am doing? I am taking a intermediate state k, such that I am moving from here to here in n step and here to here in m step, and then I am taking this k can take any values in the state space S, that is the idea. So, the path I am considering here passes through k, and I reach from i to k in in n step and then I just move the k over all the possible state in S that means basically the path actually moving to all possible values of k in between and from i to k I am going in n step, and then k to j in m step, that is the idea. So, here I have a sum. Now, notice that if I take a particular value of k here suppose I take k equals to k_0 which of course belongs to the state space S, then I can write $p_{ij}^{(n+m)} \geq p_{ik_0}^{(n)} p_{k_0 j}^{(m)}$. So, what basically mean that means I am taking one of the sum from here, special value of k as k_0 here. So, one of the components from here, not the whole sum, and because this product is always greater than equals to 0, because they are the product of two probabilities. So, they are always greater than equals to 0. So, if I take a particular term that will be even smaller than this sum. So, I have this particular inequality. And the way to interpret it physically means that I am moving from i to j in $n+m$ step, but I am taking only one path which is taking me through the state k. So, there could be any other possibilities, that I move to n and m, maybe some other k_1 here. I move to k_2 here in n, and m in this side, so, I am just removing all these possibilities I am taking only one possibility and that means that probability will be lower. I am taking only one path which actually leads me to j from i through k_0, at the n step. Instead of taking all the possible k, I am taking one so, this will going to happen. So, in the similar manner if I have something like that I am from i , I come back to i in $(n+r+m)$ step, I want to come back, that probability is this. Now, this can happen in several ways, one of the ways is that from i, I move to j in n step, then from j, I move to j in r step, and then for j , I move to i in m step that can happen. Now, this for any value of j this can happen, in any value of j this can be written but I have taken a particular value of j with j I have taken here. So, that is why again using the similar thing that this will be that instead of taking all the paths that leads me to i to i in $n+r+m$ step, I am taking one particular path which is taking i to j in n step, j to j in r step, and j to i in m step. So, that means that this probability that moving from i to i in $n+r+m$ step is same, is greater than or equal to moving from i to j in n step, j to j in r step and j to i in m step. So, this way I get this particular inequality, and this inequality holds true for any value of $r \geq 0$, this inequality holds true. Now, you will see that we have heard that this quantity strictly greater than 0, this quantity strictly greater than 0. So, this one is greater than 0. Similarly, this one is greater than 0.

So, that means that now, I can able to write that p_{ij}. Or I can able to write that

$$p_{jj}^{(r)} \leq \frac{p_{ii}^{(n+r+m)}}{p_{ij}^{(n)} p_{ji}^{(m)}}.$$

Note that in this case, this n and m I have fixed earlier that m and n and I am taking for which this quantity strictly greater than 0, and this quantity strictly greater than 0. So, m

and n are fixed only through this condition then I am just taking r which can take any value greater than equals to 0, and I have this particular inequality. Now, let us take the sum on both sides, sum over $r = 0$ to ∞ this side, sum over $r = 0$ to ∞ on the other side. So, both side I take the summation with respect to r and when I take that I get actually this one that $p_{jj}^{(r)}$ taking the sum of over $r = 0$ to ∞ is less than or equals to these two quantities do not depend on r so, I take them outside of the summation sign and finally, I have summation $p_{ii}^{(n+r+m)}$. Now, we have assumed that i is transient. So, i is transient basically mean that, i transient that if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$. Now, look into this sum, this sum is $p_{ii}^{(n+m)}$, if I take $r = 0$, then $p_{ii}^{(n+m+1)}$ if $r = 1$ plus so on so forth, it is going on. And this sum is nothing but $p_{ii}^{(0)}+, p_{ii}^{(1)} + \ldots + p_{ii}^{(n+m)} + p_{ii}^{(n+m+1)} + \ldots$. So, clearly if this sum is finite, now, if I take only this part, this part sum has to be finite because with this part is adding something non negative even I am getting finite. So, if I remove this part the rest of the part has to be finite. So, here this part is finite and these two quantity strictly greater than 0. So, if I divide the finite quantity by a positive constant that will automatically will be a finite quantity. So, we finally put that $\sum_{n=0}^{\infty} p_{jj}^{(r)} < \infty$ and this implies that j is transient and we are done with the proof, we start with i is transient we prove that j is transient. We are done and to prove this one what we have used? We have used the definition of communication, we have used the Chapman-Kolmogorov equations and from that I derive this inequality and finally, we have used the necessary and sufficient condition for transient and recurrent, the necessary and sufficient condition for transient we have we have used. So, that means that one side we have shown that if i is transient, j is also transient. Now, I have to show the other one that if i is recurrent, then j is also recurrent.

So, let us move to this one and that prove is quite simple. So, now, suppose that i is recurrent, that we want to show that j is also recurrent. If I am able to show that I am done, because in a communicating class this one state is transient, we have shown that the other state is transient. So, that means all if one of them is transient, all others are transient. On the other hand now, we are going to show that if one of if in a pair one of them is recurrent, than the other is also recurrent that means, if one of the class one of the state in the communicating class is recurrent, then all of the rest of the state in the communicating class has to be recurrent, there is no other option. So, let us proceed in this way let us prove this one by contradiction, if possible suppose that j is transient I want to show that j is recurrent. So, suppose if it is possible, if it is at all possible suppose that j is transient. And finally, what I am going to show, we are going to show that if we assume that j is transient, then there is a contradiction. So, if there is a contradiction that means, what I am assuming that is false and that means this statement is false and j is not transient that means j is recurrent, because a state is either transient or recurrent. So, j is not transient means that j is recurrent, that is the way to prove. Now, let us find this contradiction why do we get the contradiction? So, j is transient and i and j both belongs to a communicating class. So, i and j communicates, and because that we have pointed out that communication relation is a reflexive one. So, i, j communicates means j, i also communicates. j is transient that means i has to be transient, but our main hypothesis is that i is recurrent. So, clearly this is a contradiction. So, j cannot be a transient So, j is a recurrent state, and that completes the proof. So, the first part where i is transient, to show that j is transient we have used several concepts, but when we try to show i is recurrent we want to show that j is also recurrent, by contradiction very easily we prove that we can prove that once we prove this part. So, that completes the proof that in a for a Markov chain if I have a communicating class C, then inside states, all of them will be either transient or all of them will be recurrent, that is the scenario in state inside a communicating class. And that is why we have given some names to them for example, a communicating class is called recurrent class if all the states of the class are recurrent. So, we know that all the state if one of them are recurrent all the state are, so in a communicating class all the states are either recurrent or transient. If the classes are recurrent, if the states are recurrent, then we say that this communicating class as a recurrent class, if the states are transient, we call that is a transient class. So, if all the states in a communicating class are recurrent, we call this is a recurrent class if all the states in a communicating class are transient, we call it is as transient class. Similarly, we do it for Markov chain also, if a Markov chain is irreducible recall the definition of the irreducibility, irreversibility basically mean that all the states of the Markov chain communicates among themselves. So, there is only one communicating class which is the state base. So, in case of irreducible Markov chain, it mean that all the states of the Markov chain communicates among themselves. So, that means that there is only one communicating class and that one communicating class is nothing but the state base. So, now, for a irreducible Markov chain that means that all the states will be either recurrent or all the states will be transient, either all the state will be recurrent or all the state will be transient for a irreducible Markov chain because there is only one class and as we have seen that in a class in a communicating class, either all the states are recurrent, or all the states are transient. So, in case of irreducible Markov chain all the states are either recurrent or all the states are transient. And that is why we sometime calls the Markov chain is recurrent, if the Markov chain is irreducible, and all the states are recurrent, or we say that Markov chain is transient that mean that Markov chain is irreducible and all the states are transient. So, these two terminologies we are going to use that a class is recurrent means all the states inside the class are recurrent, and a Markov chain is recurrent that basically mean that Markov chain is first irreducible and then all the states are recurrent states.

With that, let us proceed and let us now see another theorem which helps us to tell whether a state is transient or recurrent. So, this says that every recurrent class is closed. So, what does mean what does closed mean, just recall that closed mean that if I have a class C, and closed mean that I cannot come out of the class, I will be inside the class always and I cannot come out of the class. So, that means that if I have i here, I have some j outside C then j is not accessible from i , that mean that from i , I cannot go to j for all $j \notin C$ and for all $i \in C$, for all $i \in C$ and all j does not belongs to C, I cannot go from i to j . That mean the closed class and that mean that from the class I cannot go outside. So, what we are going to prove now is that recurrent class is always closed. So, if I have a communicating class, who all the members all the states inside that communicating class are recurrent, then we call corresponding class is a recurrent class, and this result says that such classes are closed. So, from such class if I start from such class, I have to be in that class forever, I cannot go outside that class in future. Or if I come to this class, from outside if I come to this class that is of course possible from outside I can come to this class, once I come to this class, I am completely I will be absorbed in this class, I cannot come out of this class again, I have to be in this class forever. So, let us proceed. So, suppose that C is recurrent class, C is recurrent class that is the hypothesis given in the theorem. And suppose now, I assume that C is not closed, I assume that C is not closed what does this mean? That means that if i belongs to C, I am in C, and $j \notin C$ I am outside C, then I can move from i to j in some finite time. So, is not close that means that from i belongs to C, and $j \notin C$, I can go from i to j for some $i \in C$, and some j does not belongs to C, I can move from i to j that is the not closed, closed mean I cannot go outside this state, outside this class and not close mean that I can actually go outside this class. So, from i , I can move to j for some $i \in C$, and some $j \notin C$. And so, that mean that from i to j , I can go that mean that p_{ij} in some case step is greater than 0, this is nothing but $p_{ij}^{(k)}$ in some k step I can go outside so, this is can be done in k step at least. So, that means that probability has to be strictly greater 0. Now, look into this. So, now, we are going to show that under all this condition this one going to be true, why? The reason being that see the first one is nothing but I am starting from i , and first one is nothing but at the k step I am in j . So, that means that I start from 0, I have time k here. Suppose, this is the state i, this is the state j . $i \in C$, $j \notin C$. Now, I am starting from i , and at the time k I am at j . So, it moves like that and finally at the time k it reaches there. Now, what does this mean? This means that $X_n = i$ for infinitely many n. Notice that this intersection. So, I just move like that and I finally reach here, notice that I can cut that I can reach i for several times, and from here again it moves on like this, it again come back to i again and again, I am talking about this. Now, what does this mean? Notice that this is j , what does this mean? This means that from j I can actually go to i, there is a positive probability from j , I can come back to i. So, that means that if this happens, then if this probability is greater than 0, that mean that from j , I can go to i. Because this is j , and then at the k step I am in j , from starting from i at the k step I am in j . And then if $X_n = i$, if I have to visit the state i, for infinitely many times that means, I have to from j I have to again come back to i, there is no other possibility. So, that means from i , I can go to j , and from j I can go to i. from i , I can go to j and if this probability has to be strictly greater than 0, then this has to be from j I can go to i. So, that means, now, if I clap this one with this one, that mean that i and j communicates and that cannot happen because I have taken that j is not in C, and if i and j communicates then j has to be belongs to into the class C, then j has to be belongs to the under class C. So, this cannot happen. So, that means this cannot happen. So, only possibility is that this probability is equals to 0. So, I hope this expression is fine with everyone that this probability is 0, we are we can show by contradiction that it possible suppose this probability strictly greater than 0, and if this probability strictly greater than 0, then i is accessible from j , and beforehand I have j is accessible from i . So, that means i and j communicates and that cannot happen because $j \notin C$ and C is the communicating class. So, that means, if i, j communicates then j has to be belonged to into the C, and about we have to assume that $j \notin C$. So, this cannot happen. So, that means this cannot happen, that means this cannot happen, that means this cannot happen. So, only possibility is that this probability is equals to 0. So, we have with these all assumptions, we show that this probability is equals to 0.

And now, you see that now, we see that it implies that this probability equals to 0, implies that this quantity is strictly less than 1, why? Again, by contradiction will prove, suppose it possible this quantity equals to 1. Now, let us call this is A, this is set B, then here I have probability that $A \cap B$, and here I have that it is nothing but if possible, suppose this probability is 1, that mean that $P(B) = 1$. And we know that if $P(B) = 1$, then $P(A \cap B)$ is same as $P(A)$. If $P(B) = 1$, then $P(A \cap B)$ is same as $P(A)$. So, that means if possible if I assume this probability is equals to 1, this actually implies this, and so, if this probability is 1 that means this probability has to be same as starting from i, $X_k = j$, which is $p_{ij}^{(k)}$. So, if this probability is 1 that means, this probability is same as $p_{ij}^{(k)}$, probability of moving from i to j in k step, that is this probability will be, if this probability is 1. And here we are pointed out that this probability is strictly greater than 0. So, but here we have already pointed out this is 0. So, this cannot happen, because we have already shown this is 0. So, this is a contradiction and this contradiction has come because of this assumption. So, this assumption is also not true, and that implies that this probability has to be less than 1, strictly less than 1 there is no other option. So, now, we are done because we know that if i is recurrent, if and only this probability is 1. So, this probability is strictly less than 1, that means that i is not recurrent. So, this probability strictly less than 1, that means that i is not recurrent because we know that i is recurrent if and only if this probability is exactly equals to 1. So, it shows that i is not recurrent, i is not recurrent means that i is transient because a class has to be either recurrent or transient. So, i is not recurrent. So, that mean that i is transient and that mean that the class C is transient because $i \in C$, if C is transient all the states in C has to be transient. So, C will be transient class and that is again a contradiction to the fact that I have assumed that C is a recurrent class here. So, that means that if I assume that C is recurrent class, I start with the hypothesis recurrent class and then if I assume that C is not close, then that cannot happen because here if I assume that C is not close, I finally come with the fact that C is not recurrent class, so C is transient class. And that is a contradiction to the fact of the basic assumption of the theorem and so, that is why C is not closed is false. So, C is closed is true. So, we have proved the theorem that every recurrent class is closed. So, what we have still not prove? First we have proved that in a communicating class, all the states will be recurrent, or all the state will be transient if I have a recurrent class, the recurrent class is closed that mean that if I am in the recurrent class once, I cannot come out of that particular class forever, I will be inside that particular class forever. So, I cannot come out of that particular class in future.

So, with that let us proceed and now let us talk about finite, a kind of a partial converts of the previous theorem. So, previous theorem says that every recurrent class is closed. So, previous theorem says that recurrent class is closed, that is what the previous theorem says. Now, natural question is that, is the closed classes is recurrent? So, question is that, now the question is that is closed class is recurrent, is closed class recurrent? That is the question, and this answers partially, it says that not all closed class, but if the class is finite, if the closed class is finite, then it is recurrent. Every finite close class is recurrent it says that, so, it is not completely answered this question that is closed class recurrent? It completely do not answer this question, but it partially it answers this question that well, if the closed class is finite, then it has to be recurrent. So, every finite closed class is recurrent, which is the main point of this theorem, and let us now move to the proof of the theorem. So, proof goes like this, suppose C is closed and finite. So, C is closed and finite that is the hypothesis that I am taking which is the hypothesis given in the theorem, and suppose the Markov chain starting from C. So, that means, I am already in C, I entered into the C, the chain entered into the C. So, I am instead of that I am supposing that I am starting in C. So, C is here I am already the X_0 is inside somewhere, X_0 is there now, then for some $i \in C$, some $i \in C$, we must have the following, we must have this particular thing, why this is true? What this is saying? It is saying that starting from i , not starting from anywhere, I will visit the state i for infinitely many n , why this is true? Let us think in this way suppose that the class is finite, C is finite. So, let us take C has k members, and suppose the chain can visit the first state n 1 time, second state n 2 times and in that way k step in n k times. So, suppose, that first state is visited n 1 time, second state is visited n 2 times, k state is visited, suppose that i-th state, state i is visited n_i times, and of course, these are all finite quantities. So, now, suppose if I take the sum of all those things, if I call n equals two $n_1 + n_2 + \ldots + n_k$, then that means, chain will be here for N number of time, what will happen $(N+1)$-th time? Notice that C is a closed class. So, I cannot come out of the C, and at the $N+1$ time it will be not in any of these states. So, that cannot happen, clearly that cannot happen. So, it $N+1$ step, if I assume that i, state i is visited only n_1 time, state 2 is visited only n_2 times, state k is visited only n_k times, then if I take N is the sum of all values, then what will happen at the time $N+1$? Notice that at the $N+1$, the chain will not be in any of the states. So, clearly the chain has to be go outside the state, and that is not possible because this is finite. So, that means, at least one of the state here has to visit infinitely many times, at least one of the states here need to be visited infinitely many times, and that given here that starting from any state does not matter from why I start, at least one of the state has to be visited infinitely many times with positive probability. So, there exist a state in i, state $i \in C$, such that the state i will be visited infinitely many times does not matter from where I start. So, that means that $P(X_n = i \text{ for infinite many } n) > 0$. Now, let us use the strong Markov property, and let us do the conditioning with respect to

$T_i < \infty$, if I do the conditioning with respect to $T_i < \infty$, then we get this one. So, we can write from here that it is nothing but $P(X_n = i \text{ for infinite many } n | T_i < \infty) P(T_i < \infty)$. So, $P(T_i < \infty)$, that means that for some n I will visit the state i. So, this one is written here. Now, here using the strong Markov property, it mean that once I am in i that the next part is this is T_i, this is does not depend on in this part and it is only that I start from i . So, now, I start from i , I visit the state i for infinitely many time. So, this probability is same as this part of the probability. So, this is this probability, this part is this probability. So, finally, I have that 0 is strictly less than this probability, which is same as product of two probabilities. So, this product of two probabilities is strictly greater than 0 because of these, so, both the probability has to be strictly greater than 0. So, that means this specifically the second part has to be strictly greater than 0. So, that means that i is not transient, because just recall that if i is transient, we have the i is transient, if and only if this probability is exactly equals to 0. So, that means, this probability strictly greater than 0 that means, i is not transient. So, i is recurrent, and i is recurrent means that one state in C is recurrent. So, all the state in C has to be recurrent. So, C is a recurrent class. So, we have shown that finite closed classes are recurrent. So, every finite closed class is recurrent. So, we have basically in this particular lecture till now, we have shown three main very, very important theorems. First one is that saying that in a communicating class either all states are recurrent or all states are transient. Second one tells, says that every recurrent class is closed. Third one gives us a partial converse of the previous theorem, which the so, the third theorem tells us that every closed finite class is recurrent. So, the question is still open that whether if I have a infinite closed class, infinite closed class for A always recurrent. So, this question the answer of this questions we will see as we proceed in the course, but, still now, we have just inserted partiality.

Now, let us move on to some examples. So, this example is a example which we have discussed in lecture number 8, and this is the example 14 of the lecture number 8, just go back and check, I have just only given the diagram here. So, from this diagram we have pointed out the fact that this is a irreducible Markov chain. That means all the state communicates among themselves, and there is only one communicating class which is $\{0, 1, 2\}$. So, that whole S is in this case the communicating class and this is a irreducible Markov chain. That is what we have discussed in the lecture number 8. Now, notice that this is a closed class of course, because this is S there is no point of going outside, all the state inside the S. So, this is of course, the close class, there is no problem with that, moreover, this is a finite class, because I have only three members in the class. So, this is a finite class, this is a closed class. And now, using the previous theorem that every finite close class is recurrent, I can tell that this class is a recurrent class, and that says that this Markov chain is a recurrent Markov chain, this Markov chain is a recurrent Markov chain. So, how we deduct that the state i is recurrent? The way of deducting this one in this case is that well, I have a that S is the communicating class, here is S consists of $\{0, 1, 2\}$. This is a closed class, this is a finite class and now, using the previous theorem, we know that every finite post class is recurrent. So, this class has to be recurrent, and this class recurrent mean that each of its members are recurrent. So, this means that $\{0\}$ is recurrent, $\{1\}$ is recurrent, $\{2\}$ is recurrent, and the class itself is also recurrent. So, that this example actually can be generalized to the fact that for a irreducible finite state space Markov chain, of course, in this case I have only one class Markov chain all the states are recorded. So, it is nothing matter that if I can have multiple, if I have maybe k states total k states, either if it is recurrent, if they are communicating each other, that means it is a irreducible Markov chain using the same argument I can able to prove that all the states are recurrent. So, there is only one class, in case of a irreducible finite state space with Markov chain, and this is a closed class this is a finite class. So, that means that all the states of that particular Markov chain has to be recurrent.

So, let us move on to the next example, again this example was discussed in lecture 8, example number 15. So, this is this was the diagram and, in this case, we have shown that in this case 0 and 1 communicates, 2 does not communicate with others, 3 does communicate with others. So, I have three communicating classes in this case $\{0, 1\}$ first one consists of 0, 1. Second one consists of $\{2\}$ only, and third one consists of $\{3\}$ only. In this case, all the communicating classes are finite, all of them are finite. And we have also pointed out this one that this one, and this one these two are closed, we are pointed out these two are closed so these two are closed and finite, and that says that the classes is this and this are recurrent. So, that means that $\{0\}$ is recurrent, $\{1\}$ is recurrent $\{3\}$ is recurrent, because these two are closed class, $\{0, 1\}$ and $\{3\}$, the these two are closed class, these two are finite class. So, that means that these two classes are recurrent, that means $\{0, 1\}$ and $\{2\}$, this, this and this are recurrent classes. On the other hand, $\{2\}$ is not closed, if $\{2\}$ is not closed, I can deduct that $\{2\}$ is transient, why? The reason is that, if possible, suppose $\{2\}$ is recurrent. If this $\{2\}$ is recurrent, then I can use that this theorem which says that every recurrent class is closed. So, this means that this implies that $\{2\}$ has to be closed. And that is the contradiction. This is the contradiction which is a contradiction to the fact that $\{2\}$ is not closed, and you can see that from 2, I can go to 1, or I can go to 0, I can go to 1, I can go to 2, 3. So, 2 is not closed. So, this is a contradiction, which is a contradiction and that tells me that this assumption is not correct. So, that means $\{2\}$ is not recurrent. So, $\{2\}$ has to be transient. So, in this case we have shown that this is a recurrent class, this is a recurrent class, this is a recurrent, this is a recurrent state, this is a recurrent state, this is a recurrent state, but this one is a transient state.

Let us proceed again same kind of example, in this case also we have shown that there are three communicating classes, each communicating classes are transient sorry, are finite each communicating classes are finite, we have also showed that $\{4, 5\}$ this one is closed and finite of course. So, that means, that class this class is recurrent. On the other hand, 0, 1 this class and this class they are not closed. So, they will be transient, just like the previous example, they are transient. So, we have $\{0, 1, 2\}$, this is a class, this one is a class, this class is recurrent class, this class is transient class, this class is also transient class. So, these all the examples, till now, we have talked about these 3-4 example, three examples, I think, three examples, we have talked about some finite state Markov chain.

Let us take one example here we have infinite state Markov chain. So, it is birth-death process, again we have discussed it in the lecture number 10, example number 19. So, in this case, we are pointed out that we have two communicating classes, one is $\{0\}$ only, and the others are consists of all the rest of the classes. So, there are two communicating classes one is $\{0\}$ and other is $\{1, 2, 3, 4, \ldots\}$, everything is here. So, in this case $\{0\}$ is closed, because, once I am in 0, I will be in 0 because this is a absorbing class absorbing state. So, $\{0\}$ is closed, 0 is only one member is there in this class, in this class, so, that means, this is finite. So, this is a finite and closed class. So, that means that this $\{0\}$ is a recurrent, 0 state $\{0\}$ is recurrent and class consists of 0 only is a recurrent class. Now, what about this part? This part is not closed, why? Because I have a probability, positive probability moving from 1 to 0, for moving from 1 to 0. So, this plot is not closed, and if this is not closed, clearly this is a transient class, this is a transient, this is not closed. So, definitely this has to be a transient class. So, in this case this one is recurrent and this one is transient. With this example, I stop here. Thank you for listening.