Discrete-Time Markov Chains and Processes
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Lecture 28
Poisson Process

Hello everyone, welcome to the 28th lecture of the course Discrete Time Markov chains and Poisson processes. So, today we will start with Poisson processes. So, in last three lectures, we completed all the prerequisites for the Poisson processes, and today we will finally start with it. So, let us begin.

So, before going to the Poisson process first we need to know what is a counting process? So, a stochastic process N_t is said to be a counting process, if N_t represents the total number of events that occur by time t or in other words, the number of events that occurred within the time $[0, t]$. And like events like you, when you see example, you see it can be various kinds of events. So, a stochastic process N_t is said to be a counting process, if N_t represents the total number of events that occur by time t.

So, it is basically counting the number of events that has happened up to time t or in the interval or in $[0, t]$. Now, from this definition, few things are clear. So, a counting process possesses the following properties first of all, N_t is a continuous-time stochastic process that is, because you see here, here the parameter t varies over $t \geq 0$.

Like we saw in discrete time Markov chain that was a discrete time stochastic process, because it was indexed by N or we were looking at the process at discrete time points, but for a counting process, it is a continuous-time stochastic process, because we are looking at the process or observing the process for all time greater than or equal to 0. So, we are continuously observing the process. Now N_t is greater than or equal to 0 for all t again, that is obvious, because why?

It is counting the number of events. It represents the total number of events and so, number of events cannot be less than 0. So, obviously, N_t is greater than or equal to 0. So, it is not just greater than or equal to 0, it is integer-valued. Why? Because it is counting and when you are counting, it is like 1 2 3 4, so in the when you count the now it is you count integers. So, N_t is non negative integer valued, because it is counting something.

It is counting the total number of events that has occurred by time t, so it is integer value. Now, if $s < t$, then $N_s \leq N_t$. Again, that is very simple to see. So, suppose s is here and t is here, the number of events that has happened up to time s will obviously be less than the number of events, that has happened up to time t. So, the number of events which has happened over a longer time duration will be greater than the number of events that has happened on a shorter time duration.

So, this, again, it is possible that within s to t, there are no events, so that is why you have equality, it is not that $N_s < N_t$, but at least N_t should be greater than or equal to N_s. So, the number of events that happens on the time interval $[0, t]$ will be greater than or equal to the number of events that has happened on $[0, s]$, if s is strictly less than t. Again, a very

intuitive property.
And finally, so for $s < t$ if you look at $N_t - N_s$, that equals the number of events that occur in the time interval, $(s, t]$ because N_s is counting the number of events on the closed interval $[0, s]$. And Nt is counting it on $[0, t]$. So, when you subtract, you get $(s, t]$. So, $N_t - N_s$ equals the number of events that occur in the time interval $(s, t]$ or that occur after time s and up to time, capital up to time t. So, these properties are very easy or very intuitive to see just from the definition of counting process.
Moving on to some examples. So, if N_t is the number of persons who enter a particular store say a shop or a departmental store up to time t then that is easily a counting process. Because it is so, here what is the number what is events? Events is arrival of a person. So, it is counting the number of persons who are or a entry of a person is the events, is the event that this process is counting. So, the number of persons who enter a particular store up to time t.
Another example, total number of people who are born up to time t. Again, so, here what is the event? Event, what is the event here? So, event here is birth of a person. So, N_t is counting the total number of people who are born up to time t. So, again it is easy to see that it is a counting process. Now, if I say N_t is the number of persons in a store at time t. So, a given a time t how many people are there in a particular store or in a particular shop, then that is not a counting process.
Why? Because, again it so, first one important property for counting process is, it should be like if $s < t$ then N_s should be less than or equal to And then. But see, the number of people can, it is or in other words, so, a counting process should be what is called non-decreasing. But, if you are looking at the number of people in a store at a particular time T so, now say at 10am there can be 100 people, but at say 11am there can be 90 people.
So, the number of people at a given time that can go up as well as can go down. So, it is not a counting process, because one important property of counting process is this property four which says that if $s < t$ then $N_s \leq N_t$. So, it is what is called a non decreasing process. But, if you are counting the number of persons in a store at a particular time t, it is not the number of people.
So, like for example, if you look at the first example, it is number of persons who enter a particular store. So, you are just keeping a, you are counting it cumulatively. But if you are counting the number of persons in a store at a particular time t then that number can go up as well as down. So, this is not a counting process. Another example say N_t, the number of phone calls to have arrived at a call, call center up to time t that is again a counting process. So, here event is arrival of phone call. So, this, this and this are counting processes, but this is not a counting process. One simple way to see that, that this example 49 is not a counting process is that it can, it is not a non decreasing process. So, the number of persons in a store at a particular time t so, if you are looking at a time t_1 and time t_2, where $t_1 < t_2$, so, it is very much possible that $N_{t_1} \geq N_{t_2}$.
So, the number of, persons in a store at time t_1 can be greater than the number of persons at time t_2, even if $t_1 < t_2$. So, this example 49 is not a counting process. But all the remaining

are counting processes.
Now, again two more definitions, a counting process N_t is said to have independent increments, if the number of events that occur in disjoint time intervals are independent. What is the meaning of that? That means for any $n = 2, 3, ...$ or for any $n \geq 2$ and if you take any n time points t_1,t_2,t_3 where $0 \leq t_1 < t_2 < t_3$.
Then, if you look at these random variables $N_{t_2} - N_{t_1}$, $N_{t_3} - N_{t_2}$, $N_{t_n} - N_{t_{n-1}}$ these are independent. So, what does it mean? Say, suppose this is your t_1 this is your t_1 this is your t_1 and so on. Say, this is your tn then what is it saying? Like if I look at on this interval on this interval, the number of events that happen in this interval, the number of events that happen in this interval are independent.
So, if you look at two disjoint intervals, time intervals, and then the number of events that happen in this interval, and the number of events happen in this interval, this should be independent. And this is not just true for two intervals. So, for example, if you take n number of disjoint intervals, then the number of events happening in each of those intervals are independent. And this is true for any n greater than or equal to 2.
So, events happening in disjoint intervals are independent. That means, say for example, if I tell you the number of people who arrived in a store between 10am and 11am is, say 5, and I asked you, number of people who will arrive in the store, say between 1pm and 2pm, then this first information will not give you any additional information about the number of arrivals between 1pm and 2pm.
So, the number of arrivals between 10am and 11am does not give you any information about the number of arrivals between 1pm and 2pm. So, the number of arrivals in two disjoint time intervals are independent. That means, information about one does not give you any information, extra information about the other. That is in like very roughly what independence means, you know what the mathematical definition of independence is.
But in words what independence is x and y are independent, if information about x does not give you any additional information about y. So, that is independent increments. Now, the second definition is, a counting process N_t is said to have a stationary increment, if the distribution of $N_{t+s} - N_t$ depends only on s for all t greater than or equal to 0. That means, the number of events that happen in a particular time interval depends only on the length of the interval and not on where that interval is situated.
So, the number of arrivals between 10am and 11am. And the number of arrivals between 1am and 2pm, that will be, so these two distributions will be same. So, when I say so the distribution of N_{t+s}, it is not that I am saying the number of arrivals will be exactly same. But see the number of arrivals say between 10am and 11am, and 1pm, and 2pm, these are two random variables.
What this stationary implement says, increment property says is that the distribution of these two random variables are same. So, the distribution of the number of arrivals in a particular time interval depends only on the length of the interval and not exactly where it is situated. So, $N_{12} - N_{10}$, will have the same distribution as, say $N_{15} - N_{13}$. Because here, the length of the interval is 12 minus 10 equal to 2, here also, it is 15 minus 3 equal to 2.

So, you see this, it basically depends on the length of the interval, because here, the length of the interval is s, and not where the interval is actually situated on the timeline. So, that is what is stationary incremental. So, the independent property of independent increments, tells you that if you are looking at two disjoint time intervals, then the number of events happening in these does, again, it is not necessary for just two.

if you take, say, in disjoint intervals, then the number of events happening in this N disjoint intervals, these are independent, that is independent increments. And stationary increments tells you that , the number of events happening in an interval depends on the length of the interval and not exactly where that interval is situated on the timeline. So, the length of the interval is important.

So, the number of arrivals between 10am and 11am and the number of arrivals between 1pm and 2pm, they will be, the distribution will be same. I am not saying they will be equal, but distribution will so, will be equal. So, these are two random variables, their distribution is same. So, the probability say that the number of arrivals between 10am and 11am will be k equal to k will be same as the number the probability that number of arrivals between 1pm and 2pm equal to k.

So, the distribution is same. The distribution of the number of events in a particular interval depends only on the length of the interval, and not exactly where the interval is situated on the timeline. That is stationary increments.

Now, finally, we come to the definition of Poisson process. A counting process N_t is said to be a Poisson process with rate $\lambda > 0$. So first of all, Poisson process is a counting process. So, it counts the number of events up to, sorry within the interval $[0, t]$. And that is what N_t counts. But it is not just any counting process, it is it has some further properties. First of all, N_0 equal to 0 with probability 1. So, when you start you have 0 events.

Now, it has independent increments, that means number of events in disjoint intervals are independent. It has stationary increments, which means the number of events in a particular interval depends only on the length of the interval, and not exactly where it is situated. And finally, remember, what is N_t? N_t counts the number of events on the time interval 0 to t, or the number of events up to time t. Now, so that is a random variable.

Now, what this property four is saying is that N_t has poison λ distribution. So, the, it is a discrete random. So, you already know since it is a counting process, it will be a non negative integer valued random variable or in other words, it will be a discrete random variable. And the probability mass function of that discrete random variable is given by this.

So, probability N_t equal to k is equal $\frac{e^{-\lambda t}(\lambda t)^k}{k!}$, so, for k positive integers. So, and this is precisely the of a poisson λt distribution. So, you know what upon you, again you must be familiar with from your basic course in probability you must be familiar with poisson distribution what its probability mass function is.

Then you can easily recognize that this is the probability mass function of a Poisson λt distribution. So, the number of events that happen, that occur up to time t that has poisson λt distribution that is why you call it a Poisson process. So, this Poisson process that name comes from this fact that N_t has Poisson distribution and the parameter is given by λt. So, this λ is basically the rate λ.

So, that is the definition of a Poisson process. So, a Poisson process is a counting process, which starts from 0 has stationary and independent increments. And the number of events on between $[0, t]$ or on the closed interval $[0, t]$ has poisson distribution and the parameter of Poisson distribution is λt. In that case, you say that the continuous-time stochastic process is a Poisson process.

Now, some remark, so, when you give this definition, so, the definition fixes all finite-dimensional distributions of the stochastic process, what do I mean by that? Let me explain that via this example. So, suppose you take 2 time points t_1 and t_2 where $t_1 < t_2$ and you take these two non negative integers $k_1 \leq k_2$. Let us say if you ask what is probability that N_{t_1} is equal to k_1 and N_{t_2} is equal to k_2.

So, you can see why I am taking $k_1 \leq k_2$, because $t_1 < t_2$. So, you know, since this is a counting process, so, $N_{t_1} \leq N_{t_2}$. So, if you take some other. So, if you take say $k_1 > k_2$ then this probability is actually 0, because that is not possible. Because this is a counting process and $t_1 < t_2$. So, $N_{t_1} \leq N_{t_2}$.

So, this probability will be nonzero only when $k_1 \leq k_2$. So, what is probability? N_{t_1} is equal to $k_1 \cap N_{t_1}$ is equal to k_2, that means, what is the probability that up to time t_1 there has been k_1 events and up to time t_2 there has been k_2 events where $t_1 < t_2$ and $k_1 \leq k_2$. Now, how do I calculate that from the definition?

So, this is same as probability N_{t_1} is equal to k_1 and $N_{t_2} - N_{t_1}$ is equal to $k_2 - k_1$. So, these are these are same. So, if N_{t_1} is equal to k_1 and $N_{t_2} - N_{t_1}$ is equal to $k_2 - k_1$, but as soon as I write it in this way, so, now, t_1 is here, t_2 is here. So, when you look at this $N_{t_2} - N_{t_1}$ is looking at the number of arrivals in this interval or the number of events in this interval. And $N_{t_2} - N_{t_1}$ looks at the number of events in this interval. So, these two are disjoint intervals a poisson process has independent increments. So, the probability of this intersection will be the product of the probabilities. So, it will become probability N_{t_1} equal to k_1 and since N_t has Poisson λt distribution, so, it will be this and then probability $N_{t_2} - N_{t_1}$ equal to this.

So, you will get the other term will be probability $N_{t_2} - N_{t_1}$ is equal to $k_2 - k_1$, but the Poisson processes, a Poisson process also has stationary increments. So, $N_{t_2} - N_{t_1}$, so, this has same distribution as in $t_2 - t_1$. Because this $N_{t_2} - N_{t_1}$ what is the length of this interval? The length of this interval is $t_2 - t_1$. So, the number of arrivals between the t_2 and t_1 is same as the number of arrivals on the interval t_2 minus on the interval of length $t_2 - t_1$.

So, probability $N_{t_2} - N_{t_1}$ is equal to $k_2 - k_1$ is same as this is probability, this is equal to $k_2 - k_1$. Now, again you know that N_t has poisson λt distribution. So, this becomes $\frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^{(k_2-k_1)}}{(k_2-k_1)!}$. So, you see we have used all the properties here.

We have used independent increments to break this intersection, the probability of intersection into product probabilities. We used these stationary increments to claim that this probability of $N_{t_2} - N_{t_1}$ is equal to $k_2 - k_1$ is same as probability of $N_{t_2} - N_{t_1}$ is equal to $k_2 - k_1$. And these probabilities we have written down using the fact that N_t has poisson λt distribution.

So, you see using this property so, here I am just I have just done it for two time points, you can do it for any n time points for any $n \geq 2$. Provided like you if the time points are increasing then this k_1,k_2 should also be non decreasing otherwise the probability will be 0 that is because a counting process is non decreasing. So, you see from the definition you can answer questions about all such probabilities.

Now, another remark now fix any $T > 0$. Now, define a new process in superscript T subscript t by so, $N_{T+t} - N_T$ And then. Then N_t^T is again a Poisson process with rate λ. Thus a Poisson process probabilistically restarts itself at any point of time. So, what is basically this N_t^T?

So, suppose this is your T. So, now, what this N_t^T? this N_t^T is counting is basically counting the number of events from this time T onwards. So, when that is what you get by this definition. So, $N_{T+t} - N_T$ so, you subtract this so, that you are now counting the number of events from T onwards.

Now, what this remark is telling you that this new process which is counting the number of events from capital T onwards is again a Poisson process with the same rate λ if the original process was a Poisson process with rate λ. So, what does that mean that? Means, they see if you look at the process from any time T then what you can do is you can forget what has happened up to time t.

Now, if I ask you, so, say after t amount of time how many arrivals will happen, then you can just assume that the process starts from T. So, If I says, so, you do not know when the process actually started, you go and start observing the process. Now, if you have to say something saying then next t units of time, what is the probability that there will be no arrival? For that, you can simply assume that when you just started observing the process, the process has started from that point itself.

So again, you see, you see what is a, you see a memoryless property or Markov property. So that is why this property is called Markov property. Now, a Poisson process is one of the simplest example of a class of stochastic processes, which is called continuous-time Markov chains. So, you have learned what Discrete Time Markov chains is. So, there it is, it is a process or that, where the time parameter is discrete.

And there also if you know, the current state, then you can predict the future evolution, you do not need to know how it arrived at that current state or given the present, the past and the future independent that was Discrete Time Markov chains. So, you see, even for a Poisson process, say, if you go and start observing the process from T onwards, then you can say, like, answer any question for about the future by assuming that the process has just started there.

So, for example, if I asked, in the next 100 units of time, what is the probability that there will be two arrivals, so it is just the probability that the Poisson process will have two arrivals in 100 units of time, that is the answer. So, that is, again, that is the meaning of that at each time, the process probabilistically restarts itself. So, you do not need to know about the past as as long as you know, the present you can predict or you can make probabilistic statements about the future.

So that is, so this memoryless property is what we call as Markov property. So, this property of poisson process is called the Markov property that it probabilistically restarts itself. Again, this is just a word what it means is that if you look at this process in $T + t -$ T, so you do this subtraction, so that you start from 0. Then that is again a Poisson process with rate λ, that is the meaning of probabilistically restarts itself at any point of time.

So, the from the time you start observing the process, you can very well assume that the process has just started from that time point. It does not matter when it actually started. If you have to answer questions about the future it is enough to assume or it is fine to assume that the process has just started from that time point, this is what is called Markov property. And then, so keep this statement in mind.

So, in future if you do a course or if you study some continuous-time Markov chains, then you will see that this Poisson process is one of the simplest example of this class of stochastic processes, which is continuous-time stochastic Markov chains. So, this continuous time Markov chains is continuous-time analogue of discrete time Markov chains. So, there the time parameter is continuous.

And this Poisson process is one of the first examples and probably the most simplest example of continuous time Markov chain. Anyway, so you do not need to bother too much about what continuous-time Markov chain is. So, just keep this remark in mind that this Poisson process is a specific example of a much larger class of continuous-time stochastic processes, namely continuous-time Markov chains.

Now, how do we prove it? We will have to prove the full properties of Poisson process. Again, it is easy to see that this is a counting process. And what is N_0^T that is just simply $N_T - N_T$. So, that is equal to 0 with probability 1. Now, if you take $t > s$ and if I look at say, $N_t^T - N_s^T$, that by definition will be this. So, you see, so, this is basically the number of arrivals for this new process within the time interval s to t.

That is the same as the number of arrivals of the original process on this time interval $T + t$ or so, let me draw a picture. So, if you are looking at, so here is T. So, if you are looking at $T + s$ to $T + t$ then the number, so, what is this$N_t^T - N_s^T$ that is looking at the number of arrivals in this interval. But this is just the number of arrivals of the original process in the interval $T + s$ to $T + t$.

Now, you know that, so, now, you know the original, so, you see the number of arrivals of this new process is just the number of arrivals of the original process, but on a shifted interval. So, once you have that observation, then the stationary independent increments follow from that of the original process. So, if you take two disjoint intervals of this mod, modified process that will be two disjoint intervals of the original process, but it will be shifted by T.

But since the original process has these independent increments properties, this modified process will also be have the independent increments property. Again, if you look at an interval, since, of the modified process, that is again an interval of the original process, but shifted. So, the number of arrivals in that interval will only depend on the length of that interval, because the original process has that property.

So, the main thing is, if if you look at an interval for this modified process, that is again an interval for the original process, but the interval is shifted by T. So, what, the properties which the original process had this new process will also have the same properties namely the properties of stationary and independent increments. And finally, so, if I look at probability N_t^T equal to k that is just by definition this.

But now, you know that the original process N_t is a Poisson process that has stationary increment property. So, this has the same distribution as N_t. Because again, the number of arrivals in a, in an interval of a particular length depends only on the interval of the length and not exactly where it is situated. Again, when I the distribution is really same. So, probability that $N_t^T - N_T$ equal to k that is just simply probability N_t equal to k which is equal to this.

So, you see, the fourth property of Poisson process is also satisfied by this modified process N_t^T. Hence, this is we have shown that this is again a Poisson process with rate λ. So, the proof is not very difficult, but it is a very important property that a Poisson process probabilistically restarts itself at every point of time. So, the from the point you start observing the process, you can assume that the process has started from that point onward.

You do not actually need to bother from which point the process actually started. The point, the time from when you start observing the process, you can pretend that that is the start of the process. And if you do that, everything will be fine. That is what this remark is saying.

Now, we come to another topic which is inter-arrival times. Now, let t_1 denote the time of occurrence of the first event. Now, for $n \geq 2$, let T_n denote the time elapsed between the $(n-1)$th and nth event, then if you look at this sequence of random variables T_n, this is called the sequence of inter-arrival times. So, the picture should help in understanding what this definition is about.

So, T_1 is this first interval and in general what is T_n? T_n is the time between the nth event and the $(n-1)$th, n minus first event. So, for example, T_2, so, this is the first occurrence, this is the second occurrence. So, this is basically T_2 sorry, so, this should be T_3 not T_2. So, now, this is the third occurrence and the second occurrence. So, this difference is T_3. So, in general T_n is the time between the occurrence of the nth event and $(n-1)th$ event.

So, that is called the inter-arrival times. Now the terminology inter-arrival because generally you see like the counting process you basically model arrival of certain things, arrival of calls, arrival of people. So, that is why this terminology inter-arrival. So, it is basically, it looks at time between two successive arrivals. So, what is the nth inter-arrival time? So, that is the time between the arrival of the nth event and the $(n-1)th$ event. So, that is inter-arrival time. So, this picture should explain things clearly. So, you can see it should not be T_2 but rather T_3. So, that is inter-arrival times.

Now, again, these are random variables. So, what is the distribution of these Ts, so, that, this theorem answers that, so, for n equal to 1 2 3 and so, on T_n's are iid that means, they are independent and identically distributed exponential λ random variables. So, if your N_t is a Poisson process with rate λ, then the inter-arrival times are independent and identically distributed random variables and has distribution exponential λ.

So, you see exponential random variable making its appearance in the context of Poisson processes. So, that is why we studied all those properties of exponential distributions as a prerequisite for Poisson processes. So, you will see again not maybe not in this lecture, but in the coming lectures, all those properties coming in handful, when you try to analyze Poisson processes. So, the inter-arrival times are iid and they have distribution exponential lambda.

Now, proof so, first let us try to find the distribution of T_1. Now, for $T > 0$, $\mathbb{P}(T_1 > t)$ what is the meaning of that? That the first arrival has happened again, let us go back to this picture. So, you, $(T_1 > t)$ that means, so, say this is T that means, in this interval there is 0 arrival or the first arrival has happened after time t. So, what is the probability of that? So, the first arrival has happened after time t.

So, this is 0 this is T this is say the time of first arrival T_1. Remember this T_1 is a random variable. So, $(T_1 > t)$ means in this time interval which is 0 to t, there has been no arrivals. So, that is so, $\mathbb{P}(T_1 > t)$ is same as probability N_t equal to 0. And similarly, if you know that there has been no arrival up to time t. That means, the time of first arrival is greater than t.

So, probability of $\mathbb{P}(T_1 > t)$ is same as probability N_t equal to 0, but as soon as you write it in terms of N_t , you know that N_t has Poisson λt distribution. So, probability N_t equal to 0 is just simply $e^{-\lambda t}$. But that tells so, that means, $1 - F_{T_1}(t)$ is equal $e^{-\lambda t}$ that tells you that $F_{T_1}(t)$ is $1 - e^{-\lambda t}$, which is precisely the cumulative distribution function of an exponential distribution with parameter λ.

So, thus you get that T_1 has exponential λ distribution. So, you see the proof is very simple because $(T_1 > t)$ if and only if there has been no arrival up to time t. Now, suppose we want to find what is the probability that $(T_2 > t)$? Now, we use this conditional formula so, that is same as integration 0 to infinity probability $(T_2 > t)$ given that $(T_1 = t_1)$.

So, the first arrival has happened at time t_1 and then you multiply by the density and then integrate. But now, let us try to find this what is probability $(T_2 > t)$ given that $(T_1 = t_1)$. Now, if $(T_1 = t_1)$ that is just, that tells you that up to time t_1 there has been one arrival. Because, so, again let us see the timeline. So, this is T_1 so, there is an arrival at time t_1.

So, if I look at this interval $[0, t_1]$ there has been exactly one arrival. So, the number of arrivals or the number of events up to time t_1 is 1. And again if you know that, so, this is same as T_1 equal to this, but now, what is this? So, if N_{t_1} equal to again, N_{t_1+T} if now, there has been an arrival at time t_1. And the second arrival has taken, has not happened up to time t. That means what? In this time period there has been 0 arrivals. So, actually see here you do not need to write this, you write it as T_1 equal to t_1.

But now, see, you are saying that the reason arrival at time T_1 and there is no arrival in this interval from T_1 to T sorry. So, let me repeat this again. So, it is not, why it is $t_1 + t$? That is because you are saying T_2 is greater than t which means, the time between the first arrival and the second arrival is greater than t. So, this is actually $t + t_1$.

So, the probability that probability $(T_2 > t)$ given that $(T_1 = t_1)$ is same as probability $N_{t+t_1} - N_{t_1}$ equal to 0 given $(T_1 = t_1)$. Now, why I am saying that? Because it is just saying that $(T_2 > t)$ and there has been an arrival at time t_1. So, there has been no arrival in this interval t_1 to $t_1 + t$. Now, this thing is talking about something in this interval and $(T_1 = t_1)$, $(T_1 = t_1)$ is talking about something in this interval correct.

But that means that so, but these are two disjoint intervals Poisson process has the property of independent increments. So, this conditional probability is just same as this unconditional probability. But now again this has the same distribution as N_t, why? Because of stationary increments, because of stationary increment. So, this is same as probability N_t equal to 0, which is $e^{-\lambda t}$.

So, you see that $(T_2 > t)$ given T_1 the first arrival happened at t_1 is basically independent of T_1. So, you see this this final answer does not depend on t_1. So, when you are doing this integration, so, it will be just $e^{-\lambda t}$. But now, this does not depend on T_1, so, it will come out of the integration and it will be just the integration of this density, but if you integrate a density you just get 1.

So, the final answer is this is equal to this. But also says this conditional probability does not depend on T_1 that also tells you that T_2 is independent of T_1. So, now, by repeating this argument we can complete this proof. So, what we have got here is that these inter-arrival times are iid exponential λ are iid having exponential λ distribution. So, the inter-arrival times are exponentially distributed.

So, now, you see from where this Markov property or the memoryless property of Poisson distribution or Poisson process coming in. Because we saw in the previous remark that the Poisson process has this memoryless property or Markov property. Now, you know that this exponential distribution actually has these memoryless property.

So, you see everything is connected that, it is actually the fact is that this memoryless property or Markov property of Poisson distribution comes from the fact that the inter-arrival times are exponentially distributed. And you know, that exponential distribution has these memoryless property. So, all these things are connected.

Moving forward, now, if S_n denotes the time of the nth event. So, when the nth event happened then what is S_n? $S_n = \sum_{i=1}^{n} T_i$. Because say time of first event is T_1, time of second event is what? So, first event T_1 and then time between first and second event is T_2. So, what is the time of second event? It is $T_1 + T_2$. So, in general $S_n = \sum_{i=1}^{n} T_i$.

Now, each T_i from the previous slide, you know are iid with exponential λ distribution and we have seen that if you have n exponential n iid exponential random variables with the same parameter λ then if you look at their sum, then it has $\Gamma(n, \lambda)$ distribution. So, this is the first instance where we are using the property of exponential distribution that we learned previously.

So, you see the time of in order to find the distribution of the nth arrival or time of nth event of a Poisson process, you need to use this property of exponential distribution which says that if some of the independent and identically distributed exponential random variables having the same parameter is gamma, has gamma distribution. So, from that you get this as a corollary that S_n which is the time of the nth event as gamma n, λ distribution.

And you already know what $\Gamma(n, \lambda)$ distribution is, it is a continuous random variable having a certain probability density function. So, again this should be not T_2 but T_3. So, you see this is a S_1. So, this is the time when the second event happens. So, this thing is S_2, this is when the third event happens, this is S_3. So, again pictorially things should be very clear. So, the time of nth arrival is having $\Gamma(n, \lambda)$ distribution. And that is because of the fact that the inter-arrival times are iid exponential λ.

So, we will end today's lecture with an example. So, suppose that people immigrate into a territory according to a, according to some Poisson process, we rate λ equal to 1 per day. So, the time unit that we are looking here is day. So, this is again important. So, in problems you will see the time unit will be given. So, you will have to keep track of that time unit. So, here for example, the time unit is day.

So, if I ask you something about a week, you should keep that in mind a week is seven days. So, the time unit is important. So, suppose that people immigrate into a territory according to the Poisson process with rate λ equal to 1 per day. Now, there are two questions what is the expected time until the 10th immigrant arrives? So, that means, what is the expectation of the time of occurrence of the 10th event?

And what is the probability that the elapsed time between the 10th and the 11th arrival exceeds 2 days or in other words, the 11th inter-arrival time is greater than 2 days. So, first thing we need to find expectation of S_{10}. Because, you are asked expected time until the 10th immigrant arrived. So, that is the time of the 10th event which is denoted by S_{10}. So, expectation of S_{10} is the expectation of $\sum_{i=1}^{10} T_i$ but you know the expectation is linear.

So, expectation of sum is sum of expectations. Now, since the rate λ is 1, so, T_i has distribution, E, sorry, has distribution exp 1 so, expectation of T_i is equal to $\frac{1}{1}$, which is equal to 1. So, if you sum me up, you get 10 days. So, the expected time until the 10th immigrant arrives is 10 days. And the what is the second question? The second question is the probability that the elapsed time between the 10th and the 11th arrival.

So, the time between the 11th event, occurrence of 11th event and 10th event. So, that is precisely the 11th inter-arrival time. So, probability $T_{11} > 2$ we know that T_{11} has exponential 1 distribution. So, probability that $(T_{11} > 2)$, you know what, what is the cdf of exponential distributions from that you get that this is e^{-2}.

So, you see answering this once you have learned about what the distribution of inter-arrival time is, what is the distribution of the again, so here, you basically did not need the distribution of the nth event or time of occurrence of the nth event. So, here, you just needed expectations. So, you did not need the actual distribution, but you will see in future you will need the distribution in some other problem, you will need the distribution of time of occurrence of nth event. So, that has $\Gamma(n, \lambda)$ distribution and inter-arrival times have exponential lambda distribution.

So, here, λ was 1. So, the expected time of occurrence of the 10th event is just the expectation of S_{10}, which is just 10 days, because of the given data and the probability that the time between 10th immigration, 10th arrival and 11th arrival is greater than 2 that is just simply probability that an exponential random variable with, exponential random variable with parameter 1 will take value greater than 2 and that is just e^{-2}. So, again, this was a very simple problem. So, we will stop here today. Thank you.