Discrete - Time Markov Chains and Poisson Processes

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Module: Poisson Processes
Lecture 30
Poisson Thinning II

Hello everyone, welcome to the 30th lecture of the course Discrete-Time Markov Chains and Poisson Processes. So, in the last lecture, we saw this concept of Poisson thinning which said that suppose you have some events arriving according to or some events happening according to a Poisson process with rate λ. And each event you classify as type I with probability p and type II with probability $1-p$ independent of everything else, then if you count the number of type I events and call it as $N_1(t)$ and the process counting the number of type II events you call it as $N_2(t)$ then both this $N_1(t)$ and $N_2(t)$ are again Poisson processes with rates λp and $\lambda(1-p)$ respectively.

Moreover, these two processes are independent. So, this is what is called Poisson thinning, and using that concept, we also solved a problem in the last lecture. So, today we will be seeing another problem and a very interesting problem where we will use this concept of Poisson thinning again, so, let us start.

So, the problem is quite long. So, let us read it carefully. Suppose non-negative offers to buy a house that you want to sell arrive according to a Poisson process with rate λ. Now, once the offer is presented to you, you must either accept it or reject it and wait for the next offer. So, you want to sell a house. So, offers are coming for buying that house. Now, once an offer comes, you will either accept that offer or rejected it and wait for the next incoming offer. Now, suppose you incur a loss at the rate c per unit time until the houses are sold. So, until you are able to sell the house you incur a loss at the rate c per unit time.

So, you can think of it as maintenance cost. Now, assume that each offer is the value of a continuous random variable having probability density function s such that $f(x) \neq 0$ for all $x > 0$ obviously, since this is about offered $f(x) = 0$ for all $x \leq 0$ because an offer cannot be negative. And also you have this assumption that $\int_0^\infty x f(x)dx > \frac{c}{\lambda}$. So, this is just a technical assumption you need in order to solve this problem.

So, the offers are actually random. So, what will be the amount of the offer that is random? So, each offer is a random variable, each incoming offer is a random variable, which has it is a continuous random variable, which has a probability density function s. So, each so, these offers are coming independently of one another and each offer the value of each offer is a continuous random variable having probability density function, some f, if certain assumptions that for every $x > 0$ it is non-zero and another constraint which is this that $\int_0^\infty x f(x)dx > \frac{c}{\lambda}$ where c is this rate at which you incur a loss.

So, c is the rate at which you incur a loss, and lambda is the rate of arrivals of the offer. Now, what is your objective or your aim? Your objective is to maximize your expected net return, which is equal to the amount you receive by selling the house minus the total cost incurred. So, you are incurring a cost at a continuous rate, which is c, and then finally, when you are able to sell the house you get a certain amount. So, what is your net return or net gain? So, it is the amount you get by selling the house minus c times the amount of time you had to wait for selling the house.

Now, suppose that you employ the strategy of accepting the first offer that is greater than some specified value y. So, you are following a certain strategy, what is the strategy that you will wait until the first offer which is greater than y. So, you fix a value y and your strategy is you will wait for the first offer which is greater than or equal to y if an offer less than that comes, you will not sell the first time an offer of size greater than or equal to y comes you accept the offer and sell the house? So, that is your strategy.

Now, what is the base value of y? So, how do you fix your y that says that you can maximize your return? So, your strategy is you wait until the first offer of size greater than or equal to y. So, what you are trying to find out? You are trying to find out what should be that value of y that I should wait until fine. So, what is the optimal value of y optimal in what says such that you get the maximal expected net return where the net return is the amount you receive minus the total cost incurred. So, you are trying to maximize your net return.

So, again since everything here is random, so, you are basically trying to maximize your expected maximal return. So, for what value of y will that be maximized? So, the questions find that value of y and what is the maximum expected net return for that, but for that optimal value of y? So, the question is clear, so now w will try to solve it.

Now, an offer is of size greater than y with probability this why, because you see that offers. So, here you have seen that each offer is the value of a continuous random variable having probability density function $f(.)$, so, the probability that the offer will have a size greater than y is $1 - F(y)$ because, where if I am thinking of as the, so, if y is probability. So, if I say so it is here, so, if say X is the size of the offer, so then that is $\mathbb{P}(X \leq y)$. So, $1 - F(y)$ is basically the $\mathbb{P}(X > y)$.

So, as I am thinking it as the CDF of the offer, the Cumulative Distribution Function of the offer which we know is each offer is a random variable with a density function f, I am calling capital F as the CDF of that. So, an offer is of size greater than y with this probability, which is $1 - F(y)$, now, since this has a density, so, if you write it in terms of density it is $\int_y^\infty f(x)dx$. Thus, offers of size greater than y arrive according to a Poisson process with rate this.

So, here we are using again the concept of Poisson thinning because, each offer will be an arriving offer that will be greater than y with probability $1-F(y)$. So, now offers are arriving according to rate λ. So, here basically your say think of you will say an offer is of type I if it is greater than y.

So, the rate at which offers of type I arrive is $\lambda(1 - F(y))$. So, here the p is $1 - F(y)$. So, λp. So, this is where we are using the concept of Poisson thinning that offers of size greater than y arrive according to a partial process with rate $\lambda(1 - F(y))$ because, an offer is of size greater than y with probability $(1 - F(y))$ where $(1 - F(y))$ is given by this quantity.

Now, therefore the time until ‘the first arrival of an offer of size greater than y is exponentially distributed with parameter $\lambda(1-F(y))$, because offers of size greater than y arrive according to a Poisson process with rate λ and that it inter-arrival times are exponentially distributed. So, the time or the so T_1, so you are basically interested in T_1 when the first offer of size greater than y will arrive. So, that will have exponential distribution with parameter $\lambda(1-F(y))$. Now, let x be the size of an offer, and let $G(y)$ denote the net return from a policy that accepts the first offer with a size greater than y.

Remember, we are interested in the expected net return. So, what is $\mathbb{E}(G(y))$ that should be the expectation of the accepted offer minus c times the expected time to accept, why, because by selling a house, you get a certain amount, but you also incur a loss, what is the loss, the loss is c times the amount of time you take to sell the house? So, what is the expected net return? The expected net return is the expected amount you get by selling the house minus c times the expected time to accept the offer.

Now, what should be the expectation of this accepted offer? So, it should be the expectation of X but not just the expectation of X under the information that $(X > y)$, because you have accepted the offer which means the size of the offer was greater than y. Because that is the strategy that you are following, you will accept an offer only when it is greater than y or you will accept the first offer which is greater than y. So, now since you are given the information that you have accepted the offer, that is equivalent to giving the information that that particular offer was great of size greater than y.

So, when you try to find the expectation of the or the mean of the accepted offer, so it will be a conditional expectation. So, the $\mathbb{E}(X|X > y)$ that follows because you accepted the offer. Minus c times remember what was the time to accept. So, the time to accept is basically the time until the arrival of the first offer of size greater than y now that has distribution exponential with parameter $\lambda(1-F(y))$. So, the expectation of that will be just one over the parameter. So, you get $\frac{c}{\lambda(1-F(y))}$, that is the expected time to accept c is just the rate at which you incur the loss. Now, how do I calculate this?

Now, what is the expectation of x given some event A it is basically so you integrate the density or the of x on the set A on the event A divided by the probability of A, so here what is the event? The event is $(X > y)$. So, here, you should integrate it from y to infinity and not from 0 to infinity, because of this conditioning that $(X > y)$. So, because you see when you while calculating a conditional expectation, you integrate only over the given event. So, here the event is x $(X > y)$ greater than y.

So, you have integrated it from y to infinity, and then the probability of A so here is $(X > y)$ probability $(X > y)$ is just $(1-F(y))$ because we have assumed that F is the CDF of X. So, you get this. Now, if you just write it in a compact form, you get this. So, that gives you the expected net return when you are following a policy that you accept the first offer of size greater than y. Now, what you are trying to do in this problem? You are trying to find the optimal value of y optimal in what says the, optimal in the sense that the value of y which will maximize your expected net return.

Now, so if you have to maximize a function, then or if you want to find the point where a function is maximized, what do you do? You differentiate the function and put equal to 0 put the derivative equal to 0. So, we do that same thing. So, this is an elementary fact from calculus from we have you must have learned this when you learned about maxima, minima. So, if you find want to find the point where a function is maximized, you differentiate the function and then equate it to 0 then the value or the argument where it is equal to 0. That is the point where it is maximized or minimized.

But anyway, so if you have to find the maximum, then you need to differentiate the function and put equal to 0. So, that is what we do here. So, we differentiate this $\mathbb{E}(G(y))$ and equate it to 0. Now, if you do the derivative and then equate it to 0, you will see that this is equal to 0 if and only if this quantity is equal to 0. So, again, this calculation you can easily do, you have to just simply differentiate this quantity with respect to the variable y. So, here y is the variable. So, if you so the derivative of $\mathbb{E}(G(y))$ is equal to 0 if and only if this condition is true, so the optimal value of y will satisfy this relation, so you do not get it very explicitly, but you get it in terms of this relation. So, the optimal value of y should satisfy this relation.

So, thus the optimal value of y, let us call it as y star should satisfy this relation. So, which we get it from here, so I just take this thing to the other side, and then we get this. Now, what is 1 minus Fy star, remember, F is the CDF and the random variable has a density, so $(1 - F(y^*))$ is just this $\int_{y^*}^{\infty} f(x)dx$. And this side, you keep it as it is, now what you do is you take this thing to this side. So, remember, this y star is constant. So, this just becomes $\int_{y^*}^{\infty}(x - y^*)f(x)dx$, and you bring this $c\lambda$ to this side, so you get this is equal to 0.

So, , I have just done some very simple algebra here, so that you can easily check. So, finally, you get that the optimal value of y star will satisfy this relation. So, that is the optimal value or the so remember, so, what was the problem? What is the base value of y? So, the base value of y is given by this y^*, which satisfies this relation. Now, you want to find what is the maximal expected net return for that what you have to do you have to just find out the $\mathbb{E}(G(y))$ for $y = y^*$ because y^* is the optimal value.

So, if you have to now find the maximal expected net return, so, you have to just evaluate this $\mathbb{E}(G(y))$ at $y = y^*$r. So, we want to find the $\mathbb{E}(G(y^*))$. Remember, what was the expression for $\mathbb{E}(G(y))$? $\frac{\int_y^{\infty} f(x)dx - \frac{c}{\lambda}}{\lambda(1-F(y))}$, so, what you will do, you will just plug in $y = y^*$. So, we do that now, you see here, I have done a small thing that I have just subtracted and added y star. So, basically, I am not doing anything, so minus y star plus y star, so, this is just the $\mathbb{E}(G(y^*))$.

Now, what is the advantage of this trick of this subtraction and addition? Now, if I just look at So, I now break it, so, one thing is $\int_{y^*}^{\infty} f(x)dx$. Now, that is equal to this because y^* is the optimal value. So, this will remain in the denominator. So, this first part is giving you $\frac{c}{\lambda}$, because of this condition. Now, the y^* is constant. So, it comes out of the integration. So, it is just $\int_{y^*}^{\infty} f(x)dx$, which is just nothing but $(1 - F(y^*))$, that is because capital F is the cumulative distribution function and small f is the probability density function.

So, if you integrate the density function from y star to infinity that just gives you $(1-F(y^*))$, and this y^* is constant. So, it just goes out of the integration. So, the second term, you get

this and this c λ remains like this. So, this is equal to this. So, the advantage of writing it in this way is this $(x - y^*) + y^*$ that this first term, I am getting that this is equal to $\frac{c}{\lambda}$ using this fact, and that is because this y^* is the optimal value of y and it satisfies this relation.
So, we get this, now, you can easily check that this gets canceled and you end up with y^*. So, you get an interesting thing that the maximal expected net return is actually equal to the optimal value of y. So, the optimal size of the offer that you should wait for turns out to be also your expected or maximal expected net return. So, that is an interesting thing that comes out of solving this problem. So, this y star, which is the optimal value of y, until which you should wait before selling the house.
And it turns out that if you follow that policy, then the maximal expected net return is also equal to that value, y^*. So, that is an interesting thing, you can just think about how you can interpret this. So, I leave that on you. But again, it is an interesting thing that this maximal expected net return also comes to be equal to y^*. So, you see, like, using this concept of Poisson thinning, we have solved an interesting problem. So, this kind of problems are very real life again, this kind of situation arises very often when someone is wanting to sell a house.
So, what strategy he or she should follow? So, here it gives you a nice strategy that you wait until a particular value until an offer of size y^* arrives, and once it arrives, you just sell the house and where y^* is given by this particular relation. And what we needed to solve this problem? We just needed this concept of Poisson thinning, which gave us that offers of size greater than y, according to a Poisson process, we rate $\lambda(1 - F(y))$.
And then we just use this simple fact from calculus that if you have to find where a function is maximized, you just differentiate that function and equate the derivative to equal to 0 and find the value of x for which this derivative is equal to 0. And then that gave us the optimal value of y^* and then we plugged in the value of y^* in the expression for expected net return. And that gave us the maximal expected net return and which finally turned out to be actually equal to the optimal value of y, which is somewhat interesting, or, in fact, very interesting that it is coming out to be y^*. But again, I leave it on you to interpret like think you can think about why such a thing is happening that why is the maximal expected return is equal to y^*. So, that interpretation that leave it on you, but just I want to point out that here you get this interesting thing, so we will stop here today. Thank you all.