Discrete-Time Markov Chains and Poisson Process
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Module Poisson Process
Lecture 32
Independent Poisson Processes

Hello, everyone. Welcome to the 32nd lecture of the course Discrete Time Markov Chains and Poisson Processes. So, in the last couple of lectures, we learned about these conditional arrival time distributions and we also saw a few problems using that concept. So, today, we will start with like we will continue with a few more properties of Poisson processes. So, let us start.

So, before starting with this theorem, let us recall something. So, we saw that if $N(t)$ is a Poisson process with rate λ and if you can classify each arriving event into two types, say type 1 and type 2. Further, suppose that each arriving event or each event is type 1 with probability p and type 2 with probability $1-p$ independent of everything else, then if we count the number of type 1 events and count the number of type 2 events, then again these two are independent Poisson processes with rate λp and $\lambda(1-p)$ respectively.

So, basically, if you split a Poisson process, then again, those two processes again remain Poisson processes with different rates but also, they are independent. So, today we will see kind of an opposite result like what happens if you merge or superpose two Poisson processes. So, let us start. Let N_1 and N_2 be two independent Poisson processes with rates λ_1 and λ_2 respectively. So, N_1 and N_2 are two independent Poisson processes with rates λ_1 and λ_2 respectively.

Now, for $t \geq 0$ define this $N(t) = N_1(t) + N_2(t)$. So, we are adding these two independent Poisson processes or we are combining the arrivals from these two independent Poisson processes N_1 and N_2. Then this combined process $N(.)$ is again a Poisson process but now the rate just gets added up. So, the new rate or the rate of the combined process is the rate of process 1 plus the rate of process 2, so, $\lambda = \lambda_1 + \lambda_2$. So, you see, this is just kind of the opposite result which we saw last like few lectures back like if you split a Poisson process then it remains a Poisson process.

Now, here we are saying that if you superpose or add to Poisson process, then again, the resultant is a Poisson process there the rates decreased. So, it was from λ it became λp and $\lambda(1-p)$ respectively. Here the rates increased because we are adding two processes and the new rate is new rate $\lambda = \lambda_1 + \lambda_2$. So, this is like the result which we saw some lectures back and these results these are kind of complementary to each other. So, if you add two independent Poisson processes by adding I just simply mean in you define this new process $N(t) = N_1(t) + N_2(t)$, then this new process N is again a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.

Next, we have one more theorem about independent Poisson processes of two independent Poisson processes. Let $N_1(.)$ and $N_2(.)$ be two independent Poisson processes with rates λ_1

and λ_2 respectively. So, the same setting as the previous theorem. Then the probability that exactly n events of the process $N_1(.)$ happen before the nth event of the process $N_2(.)$ is given by this quantity. So, you have two independent Poisson processes N_1 and N_2. The previous theorem told you that if you combine these two processes and look at the count the combined arrivals then that is again a Poisson process with rate $\lambda_1 + \lambda_2$.
Now, this theorem is talking about something else. It is saying okay what is the probability that there will be exactly n arrivals of the first process before the mth arrival of the second process. So, like in particular you can ask okay what is the probability that there will be two arrivals of the first process before the first arrival of the second process. This kind of question again very like real-life questions.
And today will see like we will see some examples, then you will figure out how these kind of questions are very natural to ask and now, the answer is provided by this theorem. It says that the answer is given by this quantity that the probability that exactly n events of the process or N_1 happens before the mth event of the process N_2 is given by this where $n \geq 0$ and $m > 0$ is given by this expression.
Now, let us see the proof of this. So, the first theorem, we are not going to see a proof of let us see proof of this theorem. Now, by properties of the exponential distribution, each event that occurs is an event of the $N_1(.)$ process with probability this and it is an event of the second process or N_2 process with probability this.
Now, how are we getting this? Now, see, now, once an arrival happens the process restarts because both the individual processes N_1 and N_2 are Poisson processes and also the previous theorem told us that if we look at the combined arrivals, like both arrival from process 1 and process 2, that is, again a Poisson process.
So, once an arrival happened, say it does not matter what arrival it is, say of process 1 or process 2, then you can think for the combined process, you can think that it restarts itself. Why? Because we know Poisson process has this Markov property that it restarts itself probabilistically restarts itself at every point of time and the main reason for that is that the inter-arrival times have an exponential distribution.
So, say suppose the first arrival is from process N_1. Now, from here on again, you can think of that everything restarts now, whether the second arrival will be N_2 or N_1 how do you know that? Now, remember, N_1 has inter-arrival times which is exponential λ_1 because it is a Poisson process with rate λ_1. So, this is for N_1 process and for N_2, it is Exp λ_2.
Now, N_1 or the next arrival will be N_1 if this exponential is smaller than this exponential. So, let me write this. So, the next arrival will be of $N_1(.)$ if Exp λ_1 is smaller than Exp λ_2 and other it will be from N_2 if Exp λ_2 is smaller than then Exp λ_1 correct. So, the first arrival suppose assume that it is N_1. Now, this points out once an arrival happens the process restarts itself.
Now, the next arrival will be N_2 or N_1 depending on whether this so remember N_1 corresponds to Exp λ_1, N_2 corresponds to Exp λ_2 because these are the inter-arrival times. Now, the next arrival will be N_1 if Exp λ_1 is less than Exp λ_2 and the next arrival will be N_2 if Exp λ_2 is smaller than Exp λ_1. Now, you have seen this property of exponential distributions.

Say suppose if you have two exponential distributions, then we one will be the minimum among the two, that probability is given by the rate of that particular exponential distribution divided by the sum of the rates. That is how you get that each event that occurs is an event of $N(.)$ process with probability this because for N_1, the rate is λ_1.
So, the probability that exponential λ_1 will be minimum among exponential λ_2 and among exponential λ_1 and exponential λ_2 is given by this quantity and an event of N_2 that is when exponential λ_2 is minimum of exponential λ_1 and exponential λ_2 and that probability is given by this and why like this is true for each event because once an arrival happens, you can forget everything else and you can think that you are starting afresh and that is because of Markov property of Poisson processes which in turn comes from the fact that the exponent and that the inter-arrival times are exponentially distributed and exponential distribution has this memoryless property.
So, the important thing is each arrival will be an arrival from process 1 with this probability and arrival from process 2 with this probability and this is independent of all that has happened previously by the Markov property or memoryless property. Thus, for the required event so, what you want let us see what we want. We want that probability that exactly n events of the process N_1 happens before the mth event of the process $N_2(.)$ is given by this.
Now, so, the probability that there will be exactly n events of type 1 before the mth event of type 2 is given by so, if that has to happen, then among the first n plus m minus 1 events, here, when I say first n plus m minus one events are looking at the combined arrival process both from process 1 and process 2 or in other words, we are looking at the process $N(t) = N_1(t) + N_2(t)$ which we know from the previous theorem is again a Poisson process with rate $\lambda_1 + \lambda_2$.
So, for the required event to happen among the first n plus m minus 1 events. And again, let me repeat here I am looking at the combined arrival process or I am looking at the sum of $N_1(t) + N_2(t)$ there should be n events of the process $N_1(.)$ and m minus 1 events of process $N_2(.)$. So, if I am looking at the first n plus m minus 1 events, so, if there is any, so, there should be n events of process N1 dot and m minus 1 events of process $N_2(.)$.
So, if there is any less than n minus 1 events are processed $N_2(.)$ then there will be more arrivals from process 1. So, again, this part is easy to see that if the required event needs to happen that exactly n events of process 1 before the mth event of process 2 then if you are looking at the first n plus m minus 1 events, there should be n events of process of N_1 and m minus 1 events of the process N_2 and the last event of the m plus nth event should be from process N_2.
So, the first n plus m minus 1 events there should be n from sorry, there should be n from N_1 and m minus 1 from N_2 and the last one the last event should be from N_2. Then you will have exactly n events before the n events from first process before the mth event from process 2 because see if the last event has to be from process 2 because if the mth event was before then that means there will be less number of arrivals before the mth event. So, the last event should be the mth event and before that there will be n plus m minus 1 events among that n events should be from process 1 and m minus 1 events should be from process

N_2.
Now, what is the probability of such a thing happening to see in the first n plus m minus 1, you need that n event should be from process N_1 and m minus 1 events should be from process N_2. Now, for that what you need to do? You just simply need to choose. So, it like in the first n plus m minus 1 which places will or which events should will be from process N_1. Now, from n plus m minus 1, number of ways you can choose n places is this many ways n plus m minus 1 choose n ways.
So, in the first n plus m minus 1 among the first n plus m minus 1 events there should be n events from process N_1. So, you just choose from n plus m minus 1 places, the places for events from process 1 that can be done in n plus m minus 1 ways and what is the probability of event from process N_1, that probability is $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and that should happen n times so, that probability is n.
Now, from process 2 the number of events should be n plus m minus 1 correct sorry it should be m minus 1. So, in the first n plus m minus 1 there should be n from type 1 and m minus 1 from type 2 or from N_1 and N_2 but now, the last event or the m plus nth event should be a process N_2. Again, that probability is just $\frac{\lambda_2}{\lambda_1+\lambda_2}$. So, what you get is so another this thing will get multiplied and you get $(\frac{\lambda_2}{\lambda_1+\lambda_2})^n$.
So, again let me repeat how am I arriving at this quantity. So, in the first n plus m minus 1 events, there should be n events of the process n plus N_1 and m minus 1 events of the process N_1. So, how can you choose n places from n by n plus n minus 1 places that is precisely in these many ways. Now, that is how many ways such a thing can happen.
Now, look at let us find out the probability of such events. So, there should be n events of process N_1 and you see an event is from a process N_1 with this probability. So, you get this quantity $\frac{\lambda_1}{\lambda_1+\lambda_2}$ to the power n and m minus 1 events should be of N_2 whose probability is this, so you get m minus 1.
But now, the last event again should be a process 2. That probability is $\frac{\lambda_1}{\lambda_1+\lambda_2}$. So, when you combine these two, you get you do not need this m minus one term and you just get $(\frac{\lambda_1}{\lambda_1+\lambda_2})^n \times (\frac{\lambda_2}{\lambda_1+\lambda_2})^m$ and that is precisely this point.
So, the main thing that we are using here is this first line that by properties of the exponential distribution, each event that occurs is an event of N_1 process with probability this and an event of N_2 process with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$. And then we just simply count how many ways the required event is possible and the required event means among the first n plus m minus 1 events, there has to be n events of process N_1 and m minus 1 events of process 2. That probability is n plus m minus 1 choose n times $(\frac{\lambda_1}{\lambda_1+\lambda_2})^n \times (\frac{\lambda_2}{\lambda_1+\lambda_2})^{m-1}$ and then finally, the last event the n plus mth event should be from process N_2 and that probability is N_2 $\frac{\lambda_2}{\lambda_1+\lambda_2}$ and hence we get the final result. So, that is the proof of the theorem. And now, we are going to see some applications of these two theorems.
Now, the first example, this will be an application of the previous two theorems that we just learned. So, cyclones arrived according to a Poisson process with rate 8 per years that in 10 years 8 cyclones come. So, that is the rate. Earthquakes arrive according to a Poisson process with rate 5 per 10 years. So, here the unit is per 10 years.

So, again, I have always said this thing that you should be careful about the unit, time unit. Further, assume that the arrival of cyclones is independent of arrival of earthquakes. So, you see, this is precisely the setup of the two theorems that we saw in today's lecture that you have two independent Poisson processes. So, here one process is the arrival of cyclones and the other process is the arrival of earthquakes they are independent of one another.
Now, what are the questions? The first question is, find the probability that there will be no disaster in a given span of 5 years and disaster means either a cyclone or earthquake. So, now, we are looking at the combined process. So, a disaster can be either a cyclone or earthquake for the given problem.
So, the question is find the probability that there will be no disaster which means there will be no cyclone, no earthquake in a given span of 5 years. We know that this arrival of disaster is again a Poisson process with the combined rate and the second question is find the probability that there will be exactly 2 cyclones before the next earthquake. So, for part b, we will need to use theorem two that we saw in today's class.
So, let us solve it step by step. So, let $N_1(t)$ count the number of cyclones up to time t and $N_1(t)$ count the number of earthquakes up to time t. Now, it is given that both of them are Poisson processes. Now, define $N(t) = N_1(t) + N_2(t)$. So, I am looking at the combined process or the process of arrival of a disaster which can be either a cyclone or an earthquake. Now, then $N(.)$ is a Poisson process with rate $\frac{8}{10} + \frac{5}{10}$ which is $\frac{13}{10}$ per year. So, the initial lead was given in per 10 years, so, I have just converted it to per year, that is why I have divided by 10 here.
So, the combined process which I defined by N is again a Poisson process with rate $\frac{13}{10}$ per year. So, disaster arrives according to a Poisson process with rate $\frac{13}{10}$ per year and here disaster means either a cyclone or an earthquake. And now, what you are asked? You are asked the probability that there will be no disaster in a given span of 5 years. So, we know Poisson process has this property of stationary increment. So, the number of events in an interval just depends on the span or the length of that interval. So, here the span is 5 years. So, you just need to find out what is the probability that in a span of 5 years, there will be no event from the combined process right.
So, the probability that there will be no disaster in a given span of 5 years is $e^{-\lambda t}$ right. That is basically what is the probability that a Poisson process with parameter λt has 0 events where λ is given by $\frac{13}{10}$ and t is 5. T is 5 because you are looking at a span of 5 years and λ is given by $\frac{13}{10}$ which we have already calculated. So, the probability that there will be no disaster in a given span of 5 years is $e^{-5\frac{13}{10}}$ which is just $e^{-\frac{13}{2}}$.
So, once you can mathematically formulate the problem solving is very easy, it is just one line. And now, what about this second question the probability that there will be exactly two cyclones before the next earthquake? So, here in terms of the in terms of theorem 2, n is equal to 2 and m is equal to 1. So, that gives you this answer.
So, the probability that there will be exactly two cyclones before the next earthquake is given by this, Why? That is because, so again I have just in this formula, my n is equal to 2, my m is equal to 1, my $\lambda_1 = \frac{8}{10}$ and my $\lambda_2 = \frac{5}{10}$. So, when I plug in all these values in

this formula, I just get $(\frac{8}{13})^2 \times \frac{5}{13}$ and hence we get the answer.

So, you see again once we have formulated everything mathematically finding the answers is very, very trivial because of the two theorems that we have learned in the beginning of today's lecture. So, we see a nice application of the two theorems we via this example and you see again the example talks about some very real-life phenomena like the arrival of earthquakes, disaster cyclones, all these are very real-life things and you are always interested in predicting how many such disasters you will have in a given span or in future. So, again, these are the real-life questions that you can easily answer if you can properly mathematically model them.

So, now, moving on to the second example. So, this example is not just, particularly for what we learned in today's class but it will use all kinds of properties that we have learned till now. So, it will not just use the two theorems which we have learned in today's lecture but all the theorems that we have or all the properties of Poisson processes that we have learned till now.

So, starting from 7 am buses arrive at a bus stop according to a Poisson process with rate λ per hour. So, there is a bus stop and starting from 7 am buses arrive at that particular bus stop according to a Poisson process with rate λ per hour.

Again, starting from the same time 7 am passengers arrive according to an independent Poisson process with rate μ per hour. So, again you see here the setup is very similar to what we saw in the two theorems that we saw in today's lecture. So, you have two independent Poisson processes. So, here what are the processes? One is the arrival of buses which happens according to a Poisson process with rate λ and the other one is the arrival of passengers which happens according to an independent Poisson process with rate μ per hour.

Now, when a bus arrives, all waiting passengers instantly board the bus and the bus leaves and subsequent passengers wait for the next bus. So, once a bus arrives, all the waiting passengers instantly board that bus and the bus leaves and the passengers who come after that will wait for the next bus. So, the situation is very simple. Again, a very real-life situation. Now, there are again two questions.

Find the probability mass function of the number of passengers who board an incoming bus. So, here the number of passengers who board an incoming bus that because this is number of passengers, it will be a discrete random variable. So, we can talk about its probability mass function. So, the first question that is being asked is find the probability mass function of the number of passengers who board an incoming bus.

So, here which bus it whether it is the first bus second bus is not important, again, because of Markov property of Poisson processes. Because once a bus arrives, passenger boards and the bus leaves, so you can think of them from that point onward, again, the process restarts. So, once a bus arrives and leaves the process restarts.

So, all these things are like will have the same distribution. So, the number of people voting a particular bus will have the same distribution because of this Markov property of Poisson processes. So find the probability mass function of the number of passengers who board an

incoming bus. That is the first question.
Now, the second question is given that a bus arrives at 8:30 am and no bus arrives between 8:30 to 9 am, find the probability mass function of the number of passengers who board the next bus, the next bus is the first class which comes after 8:30 am. So, here the situation is slightly different, you are given slightly more information. The first one is number of PMF of the number of passengers who board an incoming bus but the second one is saying suppose you are told that at 8:30 am a bus arrives. So, again from 8:30 am onwards, you can think that the process restarts. So, you can basically think of 8:30 as time 0 but now you are told something more.
You are told that okay within from 8:30 to 9, no bus arrives. So, you are given this extra information and under this information, you are asked the conditional. So, here basically you are given so the probability mass function that you are going to find will be a conditional probability mass function because you are given this information that the number that there has been no buses has arrived between 8:30 am to 9 am and some bus will arrive after 9. So, the question is how many passengers will board that particular bus?
So, 8:30 am there was an arrival between 8:30 to 9:00 no arrivals that information is given to you and you are asked what is the probability mass function of the number of passengers who will board the bus which comes after 9 because it 8:30 one bus came, 8:30 to 9:00 no bus came. So, after that some bus will come, how many passengers will board that bus you are being asked about that.
Again, let us try to solve the problem step by step. Let X be the number of passengers who board an incoming bus fine. So, again there are in this problem also there are two independent Poisson processes, one is the arrival of bus which has rate λ per hour and another is the arrival of passengers which have which is Poisson with rate μ per hour.
Now, n passengers will board a bus if there are exactly n arrivals of passengers before the arrival of the bus. So, say this is time 0. Now, a passenger arrives here, here, here, here, and now suppose a bus arrives here. So, the chance that this bus will be boarded by n passengers if there are n arrivals in this period.
So, n passengers will board a bus if there are exactly n arrivals of passengers before the arrival of the bus. So, now what is the question you have two independent Poisson processes one is the arrival of passengers which is Poisson with rate μ per hour and you have the arrival of buses which is Poisson with rate λ per hour and you are asked what is the probability that this there will be exactly n events of the process of arrival of passengers before the first event of the arrival of bus right. And now, you know, we have found out this general formula, we have found out this general formula.
So, in this case, n is n and m is 1. So, if you plug in that formula, you get this. So, this is for n arrival of passengers and this is for the arrival of the bus. So, the probability that n passengers will board a bus that is exactly when there are n arrivals of passengers before the arrival of the bus and that probability is given by this quantity by the second theorem that we learned in today's class. So, this is for all n. So, this is for $n = 0, 1, 2, ...$ because this is the number of passengers. So, these are the values which $X = 0, 1, 2, ...$ and the probability

mass function probability $X = n$ is given by this for $n = 0, 1, 2, ...$
But the main thing here is that we are again looking at two independent Poisson processes and we are interested in that there will be exactly n arrivals are one process before the first arrival of the other process and we know a general formula for this and we just plug in the values that we are given for this given problem. So, you see again once we can write it down or analyze the given problem mathematically then writing down the answer is very, very simple.
Now, for part b, now, part b is slightly different. So, suppose this is 8:30, so, which is the starting time. Now, you were told up to 9:00 there are no arrivals. So, some bus will arrive here and you want the total number of arrivals in this interval. Now, again you see, we can break this interval into two. So, this is one interval 8:30 to 9:00 and another interval is 9:00 to this. So, when the bus arrives, so, this point we do not know this is a random quantity. So, this is an interval where you are told that no bus arrives and then the other interval is from 9:00 to the arrival of the next bus.
So, for part b the number of passengers who arrived between 8:30 to 9:00 am has Poisson μ by 2 distribution. Why? Because the arrival of passengers is Poisson with rate μ per hour and we are looking at a 30-minute duration. So, since the unit is hour, so 30 minute is $\frac{1}{2}$, so, we are looking at 30 minutes. So, that is basically equal to half because the rate is given. So, you because the hourly rate is given or the time you need that you are given is hour, so, 30 minutes is $\frac{1}{2}$. So, that is why you get Poisson mu 2 because it is Poisson μ t, here t is $\frac{1}{2}$ and μ is the rate of arrival of passengers.
So, this many, so again the number of people who will arrive between 8:30 to 9:00 that is a random variable and its distribution is Poisson μ by 2 distribution but now again, so, up to this there has been no arrival but now again you can think of that the clock starts from here. Again, because interval times are exponential you know from here to here, there has been no arrival but you can forget that because of the memoryless property of exponential distribution. So, or which is which in turn gives rise to Markov property of Poisson processes. So, you can think that the arrival processes restart from 9 am. So, you can think of that the clock starts from this point but now then thus from 9 am the number of passengers who arrive until the arrival of the next bus has the probability mass function as in a because you can now think of 9 am as the beginning of time and then basically the number of passengers who arrive or who board the next bus you are thinking from 9 am as the starting time. So, the number of passengers who arrive until the arrival of the next bus and that PMF we have already found out here.
So, if X is the number of passengers who board an incoming bus, this is starting from time zero like from one bus to another bus, so that we have already found out. So, here because of Markov property or memoryless property, we can think that everything starts from 9 am and then the number of passengers who arrive until the arrival of the next bus has probability mass function as in part a. So, now if Y denotes the number of passengers who board the bus, then what will be its PMF?
So, Y basically is the number of arrivals in this interval and the number of arrivals in this

interval right. So, if I call this as the number of people who arrive between 8:30 and 9:00 as X_1 and then 9:00 until the arrival of next bus as X_2, then $\mathbb{P}(Y = n) = \sum_{k=0}^{n} \mathbb{P}(X_1 = k, X_2 = n-k)$.
So, if the combine has to be n, then in one of these intervals there has to be k arrivals and in the other interval there has to be $n-k$ arrivals, only then because the total number of arrivals is arrival from 8:30 to 9:00 plus number of arrivals from 9:30 until the arrival of the next bus. So, again, these two are independent. Why? Because of memoryless property or Markov properties.
So, the number of passengers who arrive between 8:30 to 9:00 and 9:00 onwards because we are looking at Poisson processes. So, we know it has this property of what is called independent increments. So, we are looking at disjoint intervals. So, the arrivals or the events happening in these joint arrivals are independent. So, this X_1 and X_2 are independent. So, this joint probability will just simply be $\mathbb{P}(X_1 = k)\mathbb{P}(X_2 = n-k)$ and that is what this quantity is.
So, this is k running from 0 to n, this part is the probability that $X_1 = k$ or the number of arrivals from 8:30 to 9:00 is equal to k that is Poisson with parameter μ by 2. So, a Poisson distribution is equal the random variable distributed according to Poisson distribution, its value will be k that probability is $\frac{1}{k!}e^{-\frac{\mu}{2}}(\frac{\mu}{2})^k$ and then this is what we are getting from part a that there will be $n-k$ arrivals before the arrival of the next bus that is given by $(\frac{\mu}{\mu+\lambda})^{n-k}\frac{\lambda}{\mu+\lambda}$. So, this we get from part a.
This is the probability that starting from time 0, there will be $n-k$ arrivals before the arrival of the first bus or the probability that there will be $n-k$ arrivals of passengers before the arrival of the first bus. That probability is given by this. Now, this is just because of the property of Poisson distribution and these two are independent, again, because of the property of Poisson distribution because we are looking at disjoint intervals.
And so, the $\mathbb{P}(Y = n)$ if Y denotes the number of passengers who will board the next bus, then is Y will be basically $X_1 + X_2$ where X_1 is the number of arrivals between 8:30 to 9:00 and X_2 is the number of arrivals from 9:00 until the arrival of the next bus these two are independent. So, the $\mathbb{P}(Y = n) = \sum_{k=0}^{n}$. So, k from one part $n-k$ from the other part, so k and k can vary from 0 to n.
So, K running from 0 to n $\mathbb{P}(X_1 = k \cap X_2 = n-k)$ but because these are the independent probability of the intersection is just simply product of the probabilities and we get this formula. So, probability $\mathbb{P}(Y = n)$ is given by this quantity where the first part is because of Poisson distribution and the second part we get from part a. So, you see, again, we have used the two theorems that we have learned in today's class plus some additional properties of Poisson processes that we have already learned in previous classes.
So, again, you see, we have solved two problems which talk about some very realistic real-life or very realistic problems, or very real-life problems we have solved, but the main challenge was formulating those real-life problems mathematically, formulating them in terms of two independent Poisson processes and then the questions that were asked were very, very simple

applications of the properties that we have learned till now. So, we will stop here today. Thank you all.