Discrete-Time Markov Chains and Poisson Process

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Module Poisson Process
Lecture 33
Some Problems

Hello everyone. Welcome to the 33rd lecture of the course Discrete Time Markov Chains and Poisson Processes. In today's lecture, we will solve a few problems on Poisson processes. So, let us start.

So, here is the first problem. Vehicles arrive at a toll gate according to a Poisson process with rate 2 vehicles per minute fine. Now, the driver pays tolls of rupees 50, rupees 100 or rupees 200 depending on whether the vehicle is of low, medium or heavyweight category. So, vehicles are of three categories namely low, medium and heavy. So, if the vehicle is of low weight category, then the driver pays a toll of rupees 50. If it is medium then rupees 100 and if it is heavy then rupees 200.

Now, if an arriving vehicle is lightweight with probability $\frac{3}{10}$, medium weight with probability $\frac{3}{10}$and heavyweight with probability $\frac{4}{10}$ then find the mean and variance of the amount of money collected in any given hour. So, every arriving vehicle is lightweight with probability $\frac{3}{10}$, medium weight with probability $\frac{3}{10}$ and heavyweight with probability $\frac{4}{10}$.

So, there are these three categories of vehicles light, medium and heavy fine and depending on the weight category drivers pay different amounts of tolls. So, the question is given, so, if you are given a 1 hour span, what is the mean and variance of the amount of toll tax collected fine.

Now, let $N_1(t)$ be the number of lightweight vehicles arriving up to time t. Let $N_2(t)$ be the number of medium weight vehicles arriving up to time t. Let $N_3(t)$ be the number of heavyweight vehicles arriving up to time t. So, again this is a case of Poisson thinning. So, the mainstream is the stream of the arrival of vehicles. Now, vehicles you can classify into three categories or three types, namely light, medium, and heavy.

So, basically the mainstream of arrival you are splitting into three. So, in the theorem we saw we are splitting it into two but if you can split it into two, you can split it into three. So, the way you can think of is first you split it into two and then one streaming again you further split. So, this is again a case of Poisson thinning.

So, there we saw that if you have a stream of Poisson arrivals and you split it into two separate streams, then again, those separate streams are Poisson processes with appropriate rates. So here, the probabilities are $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{3}{10}$ of lightweight, medium weight and heavyweight respectively. Fine. So, now I am calling $N_1(t)$ as the number of the counting process which is counting the lightweight vehicles. $N_2(t)$ the counting process, which is counting the medium-weight vehicles and $N_3(t)$ which is counting the number of heavyweight vehicles.

Now, by the theorem on Poisson thinning, $N_1(.)$ is a Poisson process with rate, so, the original rate was 2 now the probability that it will be lightweight is $\frac{3}{10}$. So, the rate of

this splitting process will be λp. So, it is $2\frac{3}{10}$ per minute. $N_2(.)$ will be a Poisson process again with rate $2\frac{3}{10}$ because it is also medium weight with probability $\frac{3}{10}$ and $N_3(.)$ will be a Poisson process rate $2\frac{4}{10}$ because it is heavyweight with probability $\frac{4}{10}$. So, all these N_1, N_2 and N_3 are again Poisson processes with rates $2\frac{3}{10}$, $2\frac{3}{10}$ and $2\frac{3}{10}$ respectively.
And also, Poisson thinning further tells you that these splitted processes are independent. So, N_1, N_2 and N_3 are independent. So, in the statement of the theorem for Poisson thinning, we saw that we are categorizing events into two types but here it is three times. But again, two or three does not matter it basically you can do it for k types because say first you are divided into two types then one type you can further divide into again two more types. So, you can get three types. So, by successively using the same theorem, you can extend the result to k types of events fine.
So, each arrival of each type of event will again be a Poisson process with appropriate rates and they will be independent of each other, right. That is what Poisson thinning told you. So, N_1, N_2 and N_3 are independent. Now, let X be the money collected in a given hour, then what should be X? Now, it should be so the number of type 1 or number of lightweight vehicles that arrive in an hour and they pay 50. The number of medium-weight vehicles that arrive in an hour and the pay rupees 100. And the number of heavyweight vehicles that arrive in a given hour and they pay 200. So, X is this. So, again, the unit is rupees, so here everything you need is rupees.
So, X is equal to $50N_1$, $100N_2$ and $200N_3$ that is because lightweight vehicles pay 50, medium weight vehicles pay 100 and heavyweight vehicles pay 200 and $N_1(1)$ is basically the number of lightweight vehicles that arrive in a given hour. $N_2(1)$ is the number of medium-weight vehicles that arrive in a given hour. And $N_3(1)$ is the number of heavyweight vehicles that arrived in a given hour. So, the question is, you need to find the expectation and variance of X.
So, now it is simple. So, for the expectation of X, so it should be 50 times what is so you know, if it is a so here, your $N_1(.)$ has read this and t is 1. So basically, N_1 is Poisson $2\frac{3}{10}$ which is $\frac{3}{5}$. So that is $N_1(t)$. N_2 is Poisson $\frac{3}{4}$ again and N_1 is Poisson so, you get $\frac{4}{5}$. So, these are the rates and since the time is 1 hour so, basically λt is just λ. So, we know that for a Poisson random variable the mean is equal to its rate or equal to the parameter. So, it will be $(50 \times \frac{3}{5} + 100 \times \frac{3}{5} + 200 \times \frac{4}{5})$ and finally, you get it to be 250. So, this calculation you can easily do.
Now, what will be the variance of X? So again, the variance of X should be a variance of this quantity, this is where we will also use the fact that N_1, N_2 and N_3 are independent because then if you have variance of sum of random variables it just becomes sum of the variance provided the random variables are independent.
So, that we are using here again we know that if you have a random variable that has Poisson lambda distribution, its variance is also lambda. So, what you get is the variance of X will be so now for this so it will be $((50)^2 \times var(N_1(1)) + (100)^2 \times var(N_2(1)) + (200)^2 \times var(N_3(1)))$ which you get to be equal to this.
So, the variance of X is given by this. So again, once you write down what X is, then finding

expectation and variance is very simple, you use just the formula. So, for expectation, it is even more simpler. For variance, you need to use this additional fact that N_1, N_2 and N_3 are independent and using that you will get that variance of sum is sum of the variances. So, that answers both questions. So, you see again the main concept that we used here is of Poisson thinning.

Yeah. So, now if the unit is so you are being asked about the amount collected in a given hour so here t is basically 60. So, what you need is it should be $N_1(60)$, $N_2(60)$ and $N_3(60)$. So, basically, the mean will be this times so, $N_1(60)$ will have mean $\frac{3}{5}$, so, it will be λt. So, everything will get multiplied by 60. So, this expectation of X is this times 60 and the variance because again the variance will be λt so, it will be this $\lambda 60$.

So, hope you have understood why the solution which I gave was initially slightly wrong because the unit given here is per minute and you are being asked for something about a given hour, so, 1 hour is 60 minutes. So, it should be not $N_1(1)$ but $N_1(60)$, $N_2(60)$ and $N_3(60)$ and then, so, what the net effect is everything just gets so in the expectation you get a times multiplying factor 60. Similarly, in for variance also you get a multiplying factor 60 that is because, now all these will be so, here you know the λ t is 60. So, if you have a random variable which is Poisson λt, then its mean is λt as well as its variance.

So, the correction is everything should get multiplied by 60 because here the unit is minutes and you are being asked about hour fine and 1 hour is 60 minutes. So, that is why I keep saying that the unit is important. So, here, it is good that I did this mistake so, that you will not do this mistake when you do it, you solve problems on your own because here I made a mistake with the units because the original rate was given in terms of minute and you are being asked about hour. So, the time is basically 60 units fine.

Now, moving to the next problem. So, customers come to dine at a restaurant according to a Poisson process with rate 10 per hour. Suppose that an arriving customer is female with probability $\frac{2}{5}$. Now, if so customers come to dine at a restaurant according to a Poisson process with rate 10 per hour. So, here the unit is per hour and if you take an arriving customer, then that customer is female with probability $\frac{4}{10}$. So, it is so the customer is non-female with probability $\frac{6}{10}$. So, if the restaurant is open for 8 hours a day and you are told 70 customers come to the restaurant came to the restaurant yesterday, then find the probability that 30 female customers came to the restaurant yesterday. So, this is you are being asked about some conditional probability.

So, what is the information that you are given? You are given that yesterday 70 customers came to the restaurant for dining. And each day, the restaurant is open for 8 hours a day and the rate of customer arrival of customers is 10 per hour. And also, you know that a customer is female with probability $\frac{4}{10}$ with probability $\frac{2}{5}$.

Now, so you are given that yesterday 70 customers came to the restaurant, under this information you are asked what is the chance that or what is the probability that 30 exactly 30 customers were female among the customers who came to the restaurant yesterday. So, what is the probability that 30 female customers came to the restaurant yesterday given that total customers who came to the total number of customers who came to the restaurant

yesterday is 70. So, this is a question about conditional probability.
So, let us see how to solve it. So, let $N(t)$ be the number of customers arriving up to time t. $N_1(t)$ be the number of female customers arriving up to time t and $N_2(t)$ be the number of non-female customers arriving up to time t. So, again, we write down the counting processes. 1 is for the number of customers and which is $N(t)$, $N_1(t)$ is for the number of female customers and $N_2(t)$ is for the number of non-female customers fine.
Now, again, using the concept of Poisson thinning, we get that $N_1(.)$ is a Poisson process we rate 10 times 2 by 5 per hour and $N_2(.)$ is a Poisson process with rate $10\frac{3}{5}$ per hour that is because the total rate is 10 and a customer is female with probability $\frac{2}{5}$. So, non-female with probability $\frac{3}{5}$. So, the rate of female customers will be $10\frac{3}{5}$ and the rate of non-female customers will be $10\frac{3}{5}$ per hour. And again, $N_1(.)$ and $N_2(.)$ are independent. So, that is what we get from the theorem of Poisson thinning. So, again here, the first concept that we use is of Poisson thinning.
So again, you see even the previous problem like vehicles arriving at a toll gate or customers arriving or at a restaurant, all these are very real-life situations. So, you have seen like in while working with Poisson processes, we have solved problems which are you which you see in everyday life very much. So, these are all very real-life problems that we are solving using these concepts of Poisson process.
So, what you were asked? So, you were asked that given that $N(8)$ equal to 70. Why $N(8)$? Because you are told that the restaurant is open for 8 hours a day. So, your time here is eight hours and the rate is given also as per hour. So, the time unit is 8. So, given that $N(8)$ equal to 70 that is because you were told that 70 customers came to the restaurant yesterday. You are asked what is the probability that $N_1(8)$ equal to 30. So, the number of female customers who came to the restaurant is 30.
Now, conditional probability, we write down the formula. So, it is $\frac{\mathbb{P}(N_1(8)=30 \cap N(8)=70)}{N(8)=70}$. Now, if $N_1(8) = 30$ and $N(8) = 70$, then that is simply that $N_1(8) = 30$and $N_2(8) = 40$ right because if there were 70 customers, then if 30 were female, then obviously 40 were non-female divided by probability N8 equal to 70. And now I write down the rates.
So, this is basically probability $N_1(8) = 30$ and $N_2(8) = 40$ but this intersection will become just a product of the probabilities that is because of the fact that $N_1(.)$ and $N_2(.)$ are independent and what is the rate of $N_1(.)$ that is 10. So, it is $10\frac{2}{5}$ which is 4 per hour and $N_2(.)$ is 6 per hour. That is why you have $\frac{e^{-4\times 8}\frac{(4\times 8)^{30}}{30!} e^{-6\times 8}\frac{(6\times 8)^{40}}{40!}}{e^{-6\times 8}\frac{(6\times 8)^{40}}{40!}}$.
Now, if you just do the algebraic manipulation you finally get this answer. So, this gives you the answer the probability that given that 70 people dine yesterday, the probability that 30 of them were female. This is the probability. Now, let me just point out a more general fact from this calculation. So, here it was 30. So, in general, if the problem if you are asked to calculate what is the probability that $N_1(8) = k$ given $N(8) = 70$, then were like again if $N(8) = 70$, so, the number of female customers so k will vary from 0 to 70.
Now, if you just follow the same calculation, you will see it will be just $\binom{70}{k}(\frac{2}{5})^k(\frac{3}{5})^{70-k}$. So, this is this particular thing is for $k = 30$. Now, you can easily recognize this formula so this

is precisely the PMF of binomial. So, given that $N(8) = 70$, the probability that $N_1(8) = k$ that has the same distribution as binomial 70 with success probability $\frac{2}{5}$.

Again, it is very intuitive because 70 customers came to the restaurant yesterday and here you are trying to find out if that customer is female or not. So, female is the success probability. So, among 70 customers, what is the probability that you have 30 successes? So, that is just the setup of binomial. So, 30 is for the given problem but in general, if you have to say the probability that k female customers came so, it is binomial 70, so n is 70, and P is $\frac{2}{5}$.

That is because the customer is female with probability $\frac{3}{5}$. So that is the success probability. So, if you are so, this conditional probability turns out to be or the conditional distribution of $N_1(8)$ given $N(8)$ is binomial. And so, the parameters are so if $N(8) = 70$, so that is 70 and P parameter is basically the probability that the customer is female. So, this is a general fact. So, we have solved this problem. So, we will stop here today. Thank you all.