Discrete-Time Markov Chains and Poisson Processes

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Lecture 5

Definition of Markov Chain and Transition Probabilities

Welcome to the 5$^{\text{th}}$ lecture of the course, Discrete-Time Markov Chain and Poisson Processes. In the previous lecture, we have talked about what is stochastic process and then we have pointed out that the stochastic process is basically a collection of random variables and the index set is basically some time set where basically index set T is either $\{0, 1, 2, \cdots\}$ or T is the interval $[0, \infty)$. And then we have talked about what is the state space of a stochastic process?. What is the initial distribution of a stochastic process?. And we have seen several examples related to stochastic process.

In this lecture, we are going to talk about the Markov Chain. What is a Markov Chain?. Markov Chain is basically a particular kind of stochastic process. You say that in case of the stochastic process, we can have different kind of dependency. Like maybe some stochastic process that today's value maybe depends on all the previous values.

For example maybe I talk about the rainfall of today, the rainfall of today may depend on the rainfall of yesterday, day before yesterday, three days back four days back five days back and so on, so, forth. So, that means that the today's rainfall may depend on a long dependency on the previous values or it may happen that today's rainfall only depends on yesterday's rainfall and day before yesterday's rainfall or it may happen that today's rainfall depends on only on yesterday's rainfall.

So, in the case of the stochastic process, we can have different kinds of dependencies. So, today's value may depend on yesterday's value, may depend on yesterday's values and day before yesterday's values, may depend on yesterday's values, day before yesterday's values, three days below the line, four days back the line, on the line and so, on so, forth. All such dependencies can come into the picture and Markov Chain is basically talk about one such dependency and this is possibly the simplest dependencies.

Of course, this is not an independence case, but this is the simplest dependencies that we can talk about that is basically Markov Chain and so, Markov Chain is a stochastic process $\{X_n\}_{n\geq 0}$ and so, notice that when I am writing in this manner, I am assuming in this case, my T is basically $\{0, 1, 2, \cdots\}$.

So, T is at most countable set or infinitely countable set and a stochastic process $\{X_n\}_{n\geq 0}$ is said to be a Markov Chain in short I will write MC, if

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = P(X_{n+1} = j | X_n = i)$$

for all states $i_0, i_1, \cdots, i_{n-1}, i$ and j and all $n \geq 0$, this condition hold true and let us see what is this condition, the condition says that given $X_n = i$, $X_{n-1} = i_{n-1}$ so on so forth $X_0 = i_0$, the conditional probability that $X_{n+1} = j$ is same as given $X_n = i$, the conditional probability of $X_{n+1} = j$. And notice that this has to be true for all states $i_0, i_1, \cdots, i_{n-1}, i$ and j and all $n \geq 0$.

Remark 0.1. *For a MC, the conditional distribution of X_{n-1} (future) given the past $X_0, X_1, \cdots, X_{n-1}$ and present X_n depends only on the present and not on past $X_0, X_1, \cdots, X_{n-1}$.*

So, this is the mathematical form, let us now try to understand the intuitive idea behind this mathematical form. Notice what happened in this case that this part has been removed here. The rest of the part are same. So, this part I have removed from here and then I write these two things are the same. So, what does this mean?. This basically mean that is given in this remark that suppose the n^{th} time is the current time or the present time.

So, I have $n-1$ here, $n-2$ here, so on so forth finally, 0 here. So, these are the past and then $n+1$ here which is future, so, what does this imply?. This implies that the conditional distribution of future this is I am talking about at that time $n+1$, so this is the future. So, conditional distribution at future given present that means, today and past, given present and past does not depend on past and only depend on present.

So, for Markov Chain the conditional distribution of X_{n+1} which is future given the past $X_0, X_1, \cdots, X_{n-1}$ and the present X_n depends only on the present X_n and does not depend on the past, $X_0, \cdots, X_{n-1}$. So, that is basically the idea of Markov Chain. This is one of the simplest ways of imposing dependence structure and the Markov Chain structure says that, if I know the present, the probability distribution of the future can be told that means, I do not need any information about the past to tell the probability distribution of the future.

So, the future only depend given present and past, the future only depend on present not on past. That is basically the basic idea of Markov Chain and this definition, this basic idea of Markov Chain you will see, will be used again and again throughout this course. So, again I just repeat the same thing because it is a very very important thing that for Markov Chain, we write up with a mathematical form in this manner.

And keep in mind that these basically mean that that we write that probability $X_{n+1} = j$ given $X_n = i$, $X_{n-1} = i-1$ so, on so, forth finally, $X_0 = i$ is same as probability that $X_{n+1} = j$ given $X_n = i$ and this condition has to be true for all states $i_0, i_1, \cdots, i_{n-1}, \cdots, i, j$.

So, whatever the state you take here it has to be true and it has to be true for any time $n \geq 0$ and that means that the conditional distribution of future given past and present depends on the on present not on past.

With that let us proceed. So, now, let us see one particular case of Markov Chain. Markov Chain is a particular case of Stochastic process. Time homogeneous Markov Chain is a particular case of Markov Chain. So, what is time homogeneous Markov Chain?. Time homogeneous Markov Chain is defined as follows that, it is a Markov Chain and a Markov Chain is said to be a time homogeneous Markov Chain, if this condition holds true for all $n \geq 1$, what is this condition?.

This condition basically says that, if I have time $n-1$ here, n here, $n+1$ here, then it is a Markov Chain. So, the distribution of n given the present and past depends only on the present. So, that means for the time homogeneous Markov Chain, what happens is that this probability, which is the probability of this point, given the value of this state at this time point is same as probability of the state at the $(n+1)^{\text{th}}$ time point given the previous one the state at n^{th} point.

So, that basically means that this probability is independent of n. So, if I know that today's values, the probability of tomorrow's value is the same as if I know, maybe 5^{th} February's value, I try to find out what is the probability of the 6th February value.

So, it does not matter whether I am going from 5^{th} February to 6^{th} February, or I am going from 9^{th} February to 10^{th} February, or I am going from 15^{th} December to 16^{th} December. If this i and j, I write in the same way, the probability will not depend on n and that is why you see that there is no n notation here. So, that is why the rest of the discussion we will use this notation p_{ij} that will going to tell us that probability of moving from i to j in one step of time.

So, this notation as I mentioned, we will going to use everywhere and this basically mean that p_{ij} is basically nothing but $X_1 = j$ given $X_0 = i$. So, in one step does not matter whether from 0 to 1 or $n-1$ to n or n to $n+1$ whatever it is does not matter, if p_{ij} does not depend on from which time I am jumping to the next time only thing it is depend on the states i and j and so, this is called the one step transition probability for a time homogeneous Markov Chain.

Time homogeneous that basically mean that this probability does not depend on n, that is basically the final word. So, let us now see this point one by one the first point in this course, we are only going to talk about time homogeneous Markov Chain. A very very special case we are going to talk about here, which is the time homogeneous Markov Chain.

So, the most generalist stochastic process inside that I have Markov Chain, inside that I have time homogeneous Markov Chain, and in this course, we are only going to talk about time homogeneous Markov Chain. So, see that stochastic process can have continuous time also in the first part of the course, we are not talking about the continuous time we are only going to talk about discrete time stochastic process.

Then we are only going to talk discrete time Markov Chain, then we are only going to talk about time homogeneous discrete time Markov Chain in this course, at least in the first part of the course. And at the last part, when we will talk about Poisson Process, that will be basically continuous time Markov Chain. So, we will see this thing later.

Now, this condition is obvious, because p_{ij} is a probability. So, probability has to be greater than equals to 0 not only that p_{ij} greater than equal to 0 it has to be less than or equal to one also. So, p_{ij} lies between 0 and 1 for all i and j in the state space. This one has to be true the reason behind this, so, what is this?. If you see this one what is this quantity?. $\sum_{j=0}^{\infty} p_{ij}$. In this case what I am taking is that my state space is supposed $\{0, 1, 2, \cdots\}$ in this case.

Now, this is nothing but

$$\begin{aligned}\sum_{j=0}^{\infty} p_{ij} &= \sum_{j=0}^{\infty} \text{Probability of moving to } \ j \text{ from } i \text{ in one step} \\ &= \text{Probability of moving to one of the state from } i \text{ in one state}\end{aligned}$$

i.e., probability of sum over j equals to 0 to ∞, probability of moving to j from i in one step. So, finally, if I take the sum that is basically nothing but probability of moving to one of the states from i in one step. Now, you see that means that when I am in a Markov Chain, from i, I have to move one of the state, then that means that probability has to be 1 and that is why basically this sum is it exactly equals to 1.

That is basically the intuition behind this writing. Of course, we can prove it mathematically and the mathematical proof goes as follows that

$$\sum_{j=0}^{\infty} p_{ij} = \sum_{j=0}^{\infty} P(X_1 = j | X_0 = i) = 1.$$

Now, I am taking the summation over all possible values of j and because this is a probability and if I take the summation over all possible values of j then this sum of the probability has to be 1. So, this way also I can write this. Now, move to the next one, next point which says that p_{ij} is called the one step transition probability from state i to state j.

As it is I have already mentioned that from the one step, what is the probability for moving from state i to state j that is the p_{ij} and because we are talking about time homogeneous Markov Chain that does not depend on n. And finally, that P which is the matrix

$$P = (p_{ij})_{i,j\geq 0}$$

is called a one steps transition probability matrix. How this pij looks like?. When we do the example, we will see this one explicitly. (Refer Slide Time: 16:32)

Let us proceed to the next example. This example talks about rainfall. Let us first read the example. It says a suppose that the chance of rain tomorrow depends on the previous weather condition only through whether or not it is raining today and not on past weather conditions. So, if the first line from here to here it says that the rainfall is basically a Markov Chain.

It basically tells that the tomorrow's rainfall only depends on today's rainfall, it does not depend on previous past rainfall. So, it basically says to us that the rainfall is a Markov Chain. Now, let us see that how the probability behaves?. The next line says that suppose that if it is raining today, then it will rain tomorrow with probability α. If it is not raining today, then it will rain tomorrow with probability β of course α and β lies between 0 and 1.

So, if I try to write this stochastic process first what I have to do I have to define that the rain and not rain in real line. I have to map rain and non-rain in real line and that we can do in this manner that suppose 0 denote the state that it is not raining and 1 denotes the state it is raining. So, 0 means not raining and 1 means raining.

Now, in this case, so, I have 2 states here. So, the state space in this case clearly 0 and 1 because either there will be rain or there will not be rain. And X_n is the state of the n^{th} day. So, the X_n can take value 0 if X_n takes value 0 that means that the on the n^{th} day there is no rain and if X_n takes value 1 that means n^{th} day there is rain. So, state space is $[0, 1]$; these things are clear.

So, let us now talk about transition probabilities and in this case to find out the transition probabilities we will have four probabilities, p_{00} that means in one step what is the probability of moving 0 to 0, p_{01} then $p_{1,0}$ and $p_{1,1}$. So, these probabilities I need to find out and once I can find out these probabilities, I can easily write the transition probability matrix. So, let us compute these things one by one.

The first one which is basically probability that $p_{0,0} = P(X_1 = 0|X_0 = 0)$. So, what is the probability that in one step I can move from 0 to 0 and recall that what does 0 mean?. 0 mean that not raining. So, it basically means that I have to find out what is the conditional probability that tomorrow it will be rain given that today it is raining and that probability is given here that if it is not raining today, then it will rain tomorrow with probability β.

So, if it is not raining today, I have to find out what is the probability that it is not raining tomorrow. So, that basically mean that this quantity is same as $1 - \beta$, because the conditional probability of raining tomorrow is β. So, conditional probability of not raining tomorrow will be $1-\beta$.

Similarly I can look into this probability. It is $p_{0,1} = P(X_1 = 1|X_0 = 0)$. So, what is the probability that in one step that Markov Chain goes from 0 to 1. Now, 0 means not raining, 1 means raining. So, that basically mean that I need to find out what is the conditional probability that today it is not raining, what is the probability that tomorrow it is raining?.

And that probability directly given here to be β, so, I can write this one. Now, I have to find out $p_{1,0} = P(X_1 = 0|X_0 = 1)$. So, what is the conditional probability that tomorrow it is not raining, given that today it is raining?. And here it is given that it is raining today then, it is it will rain tomorrow with probability α.

So, it will rain tomorrow with probability α. Not rain tomorrow will be probability $1-\alpha$ and similarly, this quantity is basically probability of $X_1 = 1$ given $X_0 = 1$ which is equals to α, which basically means that I have to find out the conditional probability that tomorrow it is raining given that today it is raining and that probability directly given here to be α.

Now, we know the transition probability matrix if I write the states then the matrix is basically $p_{0,0}, p_{0,1}$ What is the probability moving from 0 to 1?. And then what is the probability moving from 1 to 0 in one step and what is the probability from moving to 1 to 1 in one step and you see that this matrix is exactly this one because $p_{0,0} = 1-\beta$ which is written here $p_{0,1}$ is β which is written here $p_{1,0} = 1-\alpha$.

So, I have written here $p_{1,1} = \alpha$. So, in this way, we can be able to find out the one step transition probabilities and once I have one step transition probabilities, I can write the one step transition probability matrix very easily.

Next example is again about rain, but this example is a little different than the previous one. In the previous example, what we assume in the first line?. That we have assumed basically, the tomorrow's rain depends only on today's rain, not on the past raining. In this case, we are assuming little bit differently that, the chance of rain tomorrow depends on the previous weather condition through last two days not only through last one day, but through last two days.

So, whether it will be raining tomorrow or not that is determined by the probability of whether it will be rain tomorrow or not, that is determined by the by the fact that whether there is any rain today or tomorrow. Today and yesterday. So, whether the chance of having rain tomorrow depends on the past through whether there is rain today or yesterday.

So, now based on that, I have given with some probabilities, the probabilities are that we have four conditions that it will rain for past two days. If it is given what is the probability of rain tomorrow that is 0.7. Rain for today there is rain yesterday there was no rain, then what is the probability of raining tomorrow?. Yesterday it was rain. Today there is no rain what is the probability of rain tomorrow and similarly, there is no rain in past two days.

That means that today there is no rain, yesterday there is no rain, what is the rain tomorrow?. So, these four probabilities are given to me. Now, can we convert this one to a Markov Chain?. That is the question can we convert this real-life problem to a Markov Chain?. If I converted it to a Markov Chain then maybe I can discuss different kind of probabilities that are meaningful to me using the tools of Markov Chain, which we are going to learn in this course.

First notice that if I see the $X_n =$ State of the n^{th} day, like the previous example, then this is not a Markov Chain, because here are the dependency is through 2 days think. So, definitely if I can do in this way, it will not be a Markov Chain. So, that is why I have written here that if I take X_n is the state of the n^{th} then X_n is not a Markov Chain, because it depends on the past. Given the present and past the future depends on present as well as past not only under present. So, this is not a Markov Chain.

So, can I write someway it is so, that is the Markov Chain?. And the answer is that yes, how?. Just we need to define the states a little bit cleverly that is it. If I can define the state cleverly I am done. If I define the state in this manner, that it is 0, if rain both today and yesterday, so, what I am doing, instead of this taking the state of one day only I am taking the state of both the days, today and yesterday.

I have two days maybe that is the today and that is the yesterday. So, I have four possibilities here that rain, rain, no rain, no rain, rain, no rain and no rain, rain. These four possibilities are there. Now, I define each of them as one state. So, the first 0 if rain both the days that is basically the state 0. 1 if rain today and no rain yesterday. So, rain today no rain yesterday, this is defined as state 1.

No rain today and rain yesterday that is defined to be 2 and no rain on both the days today and yesterday, that is defined as states 3. So, if I define the state in this manner, now, we can say this is a Markov Chain now, if I say Y_n is the state defined as above so, Y_n equals to 0 that basically mean that rain on n^{th} and $(n-1)^{\text{th}}$ day.

Y_n equals to 0 basically mean that rain on n^{th} and $(n-1)^{\text{th}}$ day because state 0 basically means that rained in consecutive days n^{th} and $(n-1)^{\text{th}}$ day, today and yesterday. So, now, if I say Y_n is the state based on this state space, then the collection of $\{Y_n\}_{n\geq 0}$, is basically a Markov Chain and in this case, we can see this is the transition probability matrix.

In the transition probability matrix, I will not discuss all the terms here, but few terms let us discuss how we can find out this term. Let us try start with the first term. So, in this case again let me write the states here. The first one is $0, 0$. So, at first I need to find out what is the $p_{0,0}$ which is by definition probability that $Y_1 = 0$ given $Y_0 = 0$ so, what does this mean?.

This basically means that 0 means that rain both days. So, this means that consecutive two days I have rain, so, if or I write in terms of n maybe $n+1$ and in that is the general one, so, I have n so, what does this mean that?. If $Y_n = 0$ what does this mean?, this means that the I have rain on n^{th} day, I have rain on $(n-1)^{\text{th}}$ day. Because the state 0 basically the rain today and yesterday; today is n, yesterday is $n-1$ so rain in both the days.

And what does $n+1$ mean?, $n+1$ equals to 0 means so maybe $n+1$ is here. So, $n+1$ equals to 0 mean this is basically $Y_n = 0$, $n+1$ equals to 0 mean that $(n+1)^{\text{th}}$ day I have rain and n^{th} day I have also rain. This is $n+1 = 0$ is it?, So, basically again let us repeat it. See that state 0 basically mean that I have rained in two consecutive day today and yesterday.

So, $Y_n = 0$ basically means I have rain on n^{th} day, I have rain on $n-$ first day. Now, $n+1 = 0$ means I have rain on $(n+1)^{\text{th}}$ day, I have rain on $n+1-1 = n$, so, n^{th} day. So, $n+1 = 0$ mean rain on this day and this day and $n = 0$ means that rain on a n^{th} and $(n-1)^{\text{th}}$ day. So, that means that

$$
\begin{aligned}
p_{0,0} &= P(Y_{n+1} = 0 | Y_n = 0) \\
&= P(\text{Rain on } (n+1)^{\text{first day}} | \text{Rain on } n^{\text{th}} \text{ and } (n-1)^{\text{first day}}).
\end{aligned}
$$

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So, now I can go back and can I can see that the probability of rain tomorrow given there is rain in past two days is 0.17 so, that is why I have written these probabilities 0.17 and that is that is written here. Let us talk about atleast one more; let us talk about what is the probability of this one; it is basically $p_{0,1}$. What does 0, 1 mean?.

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p_{0,1} = P(Y_{n+1} = 0 | Y_n = 0) = 0.
$$

$Y_{n+1} = 1$ given $Y_n = 0$. Again, $Y_n = 0$ basically means that rain on n^{th} day and $(n-1)^{\text{th}}$ day and $Y_{n+1} = 1$ that basically mean that Y_1, state 1 means that today rain and yesterday no rain. So, that means I have rain $(n+1)^{\text{th}}$ day and no rain n^{th} day.

So, $Y_{n+1} = 1$ means that rain today and no rain yesterday that means that $(n+1)^{\text{th}}$ day I have rain and n^{th} day I have no rain. So, that basically means that there is actually a contradiction that this even basically it is given even mean that on n^{th} day I have rain and then I am trying to find out the probability of some event which is basically no rain on n^{th} day. So that cannot occur together. So, that is why that probability has to be 0 which is written here. So, in this way, we have written all the probabilities in this case, and you can just cross check this one, this I leave as an homework for all of you.

Let us proceed to next slide. So, this is a very important Markov Chain which is called simple random walk. So, what is a simple random walk?. The idea is very simple that the state space in this case is the all the integers like $\{0, +1, -1, +2, -2, +3, -3, \cdots\}$, it is going on both the sides.

So, the state space is given by the set of all integers. And then what kind of transition we can do?. From any state if I am here, I can either move this side or this side. So, in one step, if I am here, I can either go to 1 or I can go to -1. In one step from 0, I cannot go to 2 or I cannot go to -2, I cannot go to 3, I cannot go to -3 from 0. So, in one step, I can either go one step towards the left, or one step towards the right.

And they have some probabilities like from 0 going from 0 to 1 in one step that probability is p and 0 to -1 that probability is $1-p$. So, I take a right step with probability p and I take a left step with probability $1-p$ and if a Markov Chain has this structure, this is called a simple random walk. And if the $p = \frac{1}{2}$, that means I can go with $\frac{1}{2}$ probability towards left and $\frac{1}{2}$ probability towards right.

So, the probability of going left and going right at same then that simple random walk is called simple symmetric random walk. So, this is a very special case of an example of Markov Chain and this random walk is very very important Markov Chain, it has been used in a very very even lot of real-life scenario. I will discuss; so, just for the timing take this one.

So, the next one is the gambling model and this model is a very very popular one. So, how does this model goes?. The model goes like this. So, suppose a gambler went to a gambling in a casino and what he is playing is as follows that if he wins again he gets rupee 1, and if he loses a game he actually loses rupees 1.

Now, what is the probability of a win?. Probability of winning is p and probability of losing is $1-p$. So, that means that the gambler will win rupee 1 with probability p and loses rupee 1 with probability $1-p$. But the thing is that the gambler does not have infinite money. So, there are two barrier on both the sides. Barrier is that if gambler reaches money 0 that means, he is broke then he has to stop the play or he decides that he, if he reaches the fortune N, he will stop the game.

So, if he has the rupees N, he will stop the game. So, you can think is maybe N, I can take 1000 rupees, so, if he wins 1000 rupees stop and quit the game and go back home or if he loses the game, I mean if he broke that means he does not have any money he has to leave the game.

This is the scenario. So, now in this case, if I say that X_n is the fortune of the gambler fortune means how much money that gambler has, that is basically the fortune of the gambler after the n^{th} game then this X_n is a Markov Chain and what are the possible values of X_n?. X_n can take any value between in the set $\{0, 1, \cdots, N\}$. At the 0 he has to leave the game, at N he will quit the game and in between if you have in between he will play the next game that way this is going on. So, how the transition probabilities look like?

Note that if he reachs 0, then he will be 0 because if he reach 0, he has to leave the game he cannot have more money by playing so, once he reaches 0 he will be 0 forever, so $p_{0,0}$ has to be 1. See $p_{0,0}$ which is probability that $X_1 = 0$ given $X_0 = 0$ this has to be 1 because once he reachs 0 he will be for always. Similarly, $p_{N,N}$ is also 1 because once it is N quit the game and so, he will be N forever.

So, that is why it will be 1. But in between I have something like that, suppose he has i amount of money. Now, what will happen?. Either see he will win and if he when he has $i+1$ amount of money or he will lose in case he loses, he has $i-1$ amount of money. So, from here either he can go here or he can go here and from here to here that probability is p that is the winning probability and from here to here, the probability is $1-p$ that is the losing probability.

And that is what I have written here that from i, he can go to $i+1$ with probability p and from i, he can go to $i-1$ with probability $1-p$ and what are the possible values of i?. i can take any value from 1 to $n-1$ in between except 0 and n any values from the state space this can take and what are the other p_{ij}?, other p_{ij} has to be 0 because from i in one step you cannot go to $i+2$ or any values in the side or on the right side.

Similarly, you cannot go to $i-2$ or any value to the left of $i-2$. So, that is why basically this is the one step transition probabilities in the case of the gambling model and in this particular model we will come back once again during this particular course. With that I stop in this lecture. Thank you for listening.