Discrete – Time Markov Chain and Poisson Processes
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Lecture: Accessibility and Communication of States

Welcome to the lecture number eight of the course Discrete-Time Markov Chains and Poisson Processes. So, recall that in the last couple of lectures, we have talked about stochastic process, then Markov chain, and then we have seen some properties of Markov chain and one of them was Chapman Kolmogorov equations and then in the last lecture, we have seen how we can use Chapman Kolmogorov equations to find out probabilities in different scenarios. Of course, in those cases, we have to first write our problem in terms of Markov chain, Markov chain's term and then using the Chapman Kolmogorov equations, we can find out what are the probabilities of different events to occur. Now, today, we are going to discuss with some important concepts of Markov chain called accessibility and communication. So, we will discuss those issues today, we will address those things today. And we will see couple of examples how to illustrate this accessibility and communication concept.

So, what is accessibility? Accessibility is a very, very standard term that accessible basically mean that I can go from one place to another place. So, maybe that Delhi is accessible from Noida, that basically means that I can go from Noida to Delhi or Delhi is accessible from Kolkata that basically means that I can access Delhi from Kolkata and similarly Kolkata is accessible from Guwahati, means that I can go to Kolkata from Guwahati. So, this is basically the essence of accessibility, that is true in case of Markov chain also. So, let us see the definition first, then we will discuss other things. So, definition tells that state j is said to be accessible from i. So, suppose I have a state i, I have a state j, now state j is said to be accessible from i, if there exists some $n \geq 0$, if there exist some $n \geq 0$, such that $p_{ij}^{(n)} > 0$. Just recall that what was $p_{ij}^{(n)}$, $p_{ij}^{(n)}$ was $P(X_n = j|X_0 = i)$. So, I can go from i, starting from i, I can go to the state j in n steps that is the probability, that is the $p_{ij}^{(n)}$ and the definition says that, that quantity is strictly greater than 0 if this quantity is strictly greater than 0, then we say that I can go from i to j. And so j is accessible from i under this scenario. Note that I have given one particular thing here, just recall that we have defined $p_{ij}^{(n)}$ for $n = 1, 2, \ldots$. $p_{ij}^{(0)}$ was not defined earlier. So, but because I am taking $n = 0$ also here, $n = 0$ is included here.

So, I am just defining what we mean by $p_{ij}^{(0)}$. So, $p_{ij}^{(0)}$ we have defined a δ_{ij} which means that it is 1, if $i = j$ and it is 0 otherwise. So, that means, I mean the intuition is that, I can go from i to i in 0 step always because I am already in i. So, of course, I can, if I do not take any more step I will be in i. So, that is why this quantity is 1 and $p_{ij}^{(0)}$ is 0 for all $j \neq i$. That the convention that we are going to take in this case, so accessibility means that I have some n note that, some n, if I get one $n \geq 0$, such that $p_{ij}^{(n)} > 0$, $p_{ij}^{(n)} > 0$, I am done, I am done, I can tell that j is accessible from i, I can tell that j is accessible from i in this case. One notation now onwards we are going to use is that $P_i(\cdot) = P(\cdot|X_0 = i)$.

So, instead of writing this $X_0 = 0$ here, I will just write i here that means that given $X_0 = 0$, I am trying to find out the probability. Just for example, for example, that instead of writing that $P(X_n = j|X_0 = i)$, just we have written here, I will, now onwards I will write this, this probability as P_i that means starting from i, I go to j in n step, I will be, I will be in the state j after n step. So, this particular notation I am going to use now onwards, this is nothing, but, to shorten the notation instead of writing this given $X_0 = i$, I will just write i here that basically means that under the condition $X_0 = i$, I am trying to see the probability of some event, I am trying to see the probability of some event. In this case in this example, I am trying to find out $P(X_n = j|X_0 = i)$ that I will write $P_i(X_n = j)$. So, with that notation and with this definition of accessibility, let us proceed, note that this definition tells that $p_{ij}^{(n)} > 0$. So, what does this mean? Does this mean that I can go from i to j and this actually given by next theorem.

So, this theorem basically gives us three conditions and these three conditions are equivalent. This theorem gives us three condition and theorem tells that these three conditions are equivalent. So, let us see what are the conditions? So for distinct state i and j, so, here I am assuming that i and j are not same, i and j are two different states. So, the following statements are equivalent. So, what are the statements? First says that j is accessible from i. Second statement says that probability that conditional probability that starting from i, I will be ever in j that probability is strictly greater than 0. So, conditioning on the starting from i the conditional probability that ever be in j will be strictly greater than 0 and the third condition tells that there exists some states i_0 has to be i and i_n has to be j and in between I have $i_1, i_2 to i_{n-1}$ in between these $n - 1$ states can be anything. So, I have some state i_0, i_1, i_2, i_{n-1} and i_n says that if I take the product of these probabilities. These probabilities are that from i, I go to i_1 then from i_1 to i_2 then i_2 to i_3, so on and so forth, finally, i_{n-1} to i_n and i_n is j, so i_{n-1} to j.

If I take the product of these probabilities that has to be strictly greater than 0 for some states i_0 to i where i_0 has to be i, i_n has to be j and in between state can be anything and for some $n \geq 0$. So, these three statements are equivalent if $i \neq j$, if the state i and state j are not same then they are equivalent. So, let us now understand each of them separately and try to prove them. So, first j is accessible from i we know by definition that means there exists a $n \geq 0$ such that $p_{ij}^{(n)} > 0$, that is the definition. Now, what does this physical meaning of the second statement? The second statement says that, because 1 and 2 are equivalent, the second statement says that, if j is accessible from i, that means, starting from i, starting from the state i, I can go to state j, I can go to state j, in ever be in state j, so that means in some step I will be in state j, ever be in state j that means that in some stage I will be in state j. So, it says that, well if there exists $n \geq 0$ such that $p_{ij}^{(n)} > 0$, then the conditional $P($ Ever be in $j|$ starting from $i) > 0$ and notice that starting from i that is equivalent to write that $X_0 = 0$, sorry $X_0 = i$ that is writing same as $X_0 = i$. So, this is equivalently I can tell that $P(\text{Ever be in } j|X_0 = i) > 0$, starting from i that means $X_0 = i$ and what this third condition tells that? Third condition basically tells that well I have i here, I have j here, if j is accessible from i, if j is accessible from i, that means, I have some state in between such that I have a path through the states to go from i to j. I have a path through the states to go from i to j. And there is a path that means that there is a positive of probability that I can go through this route from i to j. I can go through this path from i to j that has positive probability that means, there is a path. So, let us try to now, now prove these things. So, we discussed the physical meaning of this thing and now, let us prove these things. So, proof, first, what I am going to do is that I am trying to prove the equivalence between the statement 1 and statement 2 that means, j is accessible from i and probability ever be in j given starting from i is strictly greater than 0. First we will try to prove the equivalence of these 2 statements. So, the proof goes like this. So, we know that because j is accessible from i, so there exists some n for which $p_{ij}^{(n)} > 0$. So, I take that particular n here for which n, $p_{ij}^{(n)} > 0$ that particular n I have taken here. So, I have written $p_{ij}^{(n)}$ and that is by definition starting from i going to j in n step, I will be in j in n step starting from i, this is basically this probability. Now, you just consider this probability also. This probability again starting from i that any of the $X_n = j$. Any of the $X_n = j$ because I am taking union and union means that collection of everything, so any of $X_n = j$. So, clearly this particular n is a special case of any of the n that means that $X_n = j$

is a subset of union of $\cup_{n=0}^{\infty}\{X_n = j\}$. It is a subset of this quantity because here all the n's are included here I have a particular n. So, that means if I take the probability, the probability when I give the probability here, the probability will be increasing. The probability of this one is less than or equal to probability of this one and that is what I have written in this statement. So, we have that for a particular n, any particular n, $p_{ij}^{(n)} \leq P_i(\cup_{n=0}^{\infty}\{X_n = j\})$.

Now, this statement and this statement are exactly same because i is given here, so I am starting from i and union is n so that means, I will be j in some step. Union of $X_n = j$, union is over $n = 0$ to n that means that I will be in j in some step, so ever be in j. So, I will be ever be in j that is exactly this one. So, these two events are exactly same, one is written in words and another is written in mathematical form. So, we have these two probabilities has to be same, we have written the same thing. Now, come to the next one, next one is again the same thing written in a different form it is nothing but P_i, again this starting from I have written i here and then $X_n = j$ for some $n \geq 0$. So, these three statements, these three things are writing of the same thing in different notation. So, I have just written it here to tell you that the same thing can be returned in several ways. And in many scenarios, one is helpful rather than the other that depends on what kind of problem we are trying to solve. So, these three things are same. Now, come to this expression why this inequality is there? This is because it is coming from this one, this is coming from this one. We know that $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ that we know. So, based on that now, I can easily write this this one from here, because this one is less than or equals to if I take $n = j$ as a I said $n = j$ as I said. So, this is same as this is less than equals to $\sum_{n=0}^{\infty} P_i\{X_n = j\}$ and this is $p_{ij}^{(n)}$. So, we can write that $p_{ij}^{(n)} \leq P_i(\cup_{n=0}^{\infty}\{X_n = j\}) \leq \sum_{n=0}^{\infty} p_{ij}^{(n)}$. So, first steps to because I have taken a particular n here where we have all the n, so $\{X_n = j\} \subset \cup_{n=0}^{\infty}\{X_n = j\}$. That is why the first inequality is true and this inequality is true because of this basic inequality in the probability. These inequalities also true. Now, after writing this thing, the equivalence of 1 and 2 can be proved very easily. So, suppose now, that the first condition is true, suppose now that the first condition is true that j is accessible from i. So, that means they are existing in says that $p_{ij}^{(n)} > 0$. So, that means, if I take that particular n here, so this quantity has to be strictly greater than 0. If this quantity strictly greater than 0, now, this ever be in j starting from i that quantity is even bigger than $p_{ij}^{(n)}$. So, this quantity also has to be strictly greater than 0. So, if I assume 1 is true, we can show that 2, you has to be true.

Now, when I try to prove the prove that if 2 is 2 and then 1 is 2, so suppose 2 is true, we will try to show 1 is true. Now, 2 is true, that means this probability is strictly greater than 0, this probability is strictly greater than 0, and notice that summation $p_{ij}^{(n)}$, even bigger than this particular thing. So, this summation $p_{ij}^{(n)}$. So, this shows that $\sum_{n=0}^{\infty} p_{ij}^{(n)} > 0$. Now, that shows that this implies that at least one n is there they are exist a $n \geq 0$ such that $p_{ij}^{(n)} > 0$, why because if there does not exist any n such that $p_{ij}^{(n)} > 0$, then this sum has to be 0. Then the sum has to be 0 because if I add 0 off, whatever time does not matter, it will remain 0. That is why that sum is strictly greater than 0 that says that at least one of the entry has to be strictly greater than 0 otherwise sum will be equal to 0. So, that proves the equivalence between 1 and 2. So, the once I have this particular expression, the proving of the equivalence between 1 and 2 are very, very simple.

Now, we will look into the proof of the equivalence of 1 and 3. So, now, we will look into 1 and look into 3 and we will try to prove the equivalence of 1 and 3. To prove this 1, first we have to understand this particular expression. So, what is this particular expression just recall that $P^{(2)}$ can be written as P^2. So, what does this mean? This is nothing but product of the matrix $P \cdot P$. Similarly, we have also discussed that $P^{(n)}$ is nothing but P^n. So, I need to multiply P to the power P in P for n times I have to multiply P for n times that is the matter. So, now, we know that when we take the multiplication of 2 matrices that mean that I have to take the rows of the first mattress and then the column of the second matrices and I have to take the multiplication of the row times the column row of the first matrix and the column of the second matrix. So, if you keep on doing this, then P^n turns out to be some combination of like this, some combination like this. And that tells that $p_{ij}^{(n)}$ can be written as summation $i_1, i_2, \ldots, i_{n-1} \in S$ and then I have this expression. This is probability moving starting from i going to i_1 starting probability starting from i_1 to go into i_2 and so on so forth. Finally, probability that starting from n by $n-1$ going to state j. So, if I take the sum of these, I will get that, so this one is coming from the definition of matrix multiplication and just you write the matrix multiplication for 2, you can also understand that how this is coming. So, this is actually coming from that and once I have this expression now, proving that 1 and 3 are very, very simple, how? Because suppose first, suppose that 1 is true, so what we try to prove? Try to prove that try to show that 3 is true that is my, that is our job. And now it is easy, because see 1 is 2 that means j is accessible from i. So, what does this mean? This means, there exist n such that this quantity is strictly greater than 0, there exist n such that this quantity is strictly greater than 0, this quantity is strictly greater than 0. So, that means this sum is strictly greater than 0. And this sum is greater than 0 that means that at least one of the entry inside the sum has to be strictly greater than 0. So, that proves the third part. So, that means that that shows that summation $p_{i,i_1}, p_{i_1,i_2}, \ldots, p_{i_{n-1},j} > 0$ and this implies that at least one $p_{i,i_1}, p_{i_1,i_2}, \ldots, p_{i_{n-1},j}$ has to strictly greater than 0, at least one of them has to strictly greater than 0. Otherwise this sum will be 0, if all the entries here are 0, then the sum will be 0, but we know that the sum is strictly greater than 0. Using this we can prove that 1 implies 3. Now, we have to prove that 3 implies 1.

So, to do this, so we want to prove now is that we want to prove now is that suppose 1 is that suppose 3 is true, suppose 3 is true. Suppose 3 is true, try to show that 1 is true. That is what we try to do. Now, this is true that means that there exists at least one set of $i_1, i_2, \ldots, i_{n-1}$ such that this quantity is greater than 0. So, 1 is true that means that there exists at least one set of $i_1, i_2, \ldots, i_{n-1}$ such that this quantity is greater than 0, that the third part statement says. Now, if this quantity is greater than 0, that means at least one of the entries are greater than 0. So, summation has to be greater than 0, because this product will be always non negative, because this is the product of some probabilities and probabilities that are non-negative. So product of the probabilities are also non negative there. So, if one of the product is strictly greater than 0, then the sum has to be strictly greater than 0. And this shows that there exists some $n \geq 0$, such that this quantity strictly greater than 0. So, that shows that if 3 is true, 1 is also true. So, that shows that the equivalence between 1 and 3. So, previously we approved the equivalence between 1 and 2. Now, we have proved the equivalence between 1 and 3. As so that shows that all these 3 conditions, 3 statements are equivalent condition. So, that means showing that j is accessible from i is same as showing that second condition that ever in j $P($ Ever be in $j|$ starting from $i) > 0$. And which is again equivalent to show that there exists a path from i to j. And this path in this case is through $i_1, i_2, \ldots, i_{n-1}$. Through $i_1, i_2, \ldots, i_{n-1}$, I can go from i to j. That is the idea.

So, with that, let us proceed. So, let us now see what does this communication mean? So, the definition of the communication comes from the definition of accessibility. And the intuition behind the definition of communication is as follows. See, as I told that maybe the Kolkata is accessible from Guwahati that means that I can go to Kolkata from Guwahati. On the other hand Guwahati accessible from Kolkata that means I can go to Guwahati from Kolkata. Now, because in this particular case that Guwahati and Kolkata both of them are accessible from the other that means that these 2 places communicates between themselves, that means that these 2 places communicate between themselves. So, that is the essence of the communication that if I have 2 states, i and j, if I can go from i to j and j to i, that means if i is accessible from j and j is accessible from i, then we say that the 2 states i and j communicates. So, let us now go by the definition line by line, 2 states i and j are said to be communicated say to communicate, if i and j are accessible from each other So, that means, from i, I can go to j from j, I can go to i. So, if here i, here j. I can go from i to j, I can go from j to i. Note that that may not be one-step but maybe in few time step I will go. Some, time step I can go from i to j and some other time step I can go from j to i. And then using the definition of the accessibility that in mathematical term I can write that there exists $m \geq 0$ and $n \geq 0$ both of them are interior such that $p_{ji}^{(m)} > 0$. So, from j, I can go to i in m step, ffrom j, I can go to i in m step and from i, I can go to j in n step that comes from here, So, there from i, I can go to j in n step, in m step I can go to j to i. So, that means I can go from i to j, I can also go from j to i and that means they communicates. So, some notation this notation now onwards we are going to use j is accessible from i that means I can go from i to j that will denote by this arrow notation and keep in mind this arrow does not mean that I will go in one-step. It mean that in some step I can go from i to j, but whatever the state that depends from problem to problem. In some problem maybe I can go in one-step, may other some problem I can go in multiple steps, just like you can take this example suppose I want to go to Indore from Guwahati, now, there may not be direct flight to Indore there may not be direct flight to Indore but what I can go is that from Guwahati I can go to Delhi and from Delhi I will go to Indore. So, in one-step in this case, I cannot go I have to take 2 flights. So, one-step I may not be able to go but the things are that they are accessible because via Delhi I can go. Similarly, from Indore I can come back to Guwahati via Delhi, so maybe in this case again one-step I cannot go. But via Delhi I can go I have to take 2 steps, but that is okay, I can go that is it, if whether I can go or not that is the important here. Now, if suppose that tomorrow some direct flights started between Guwahati and Indore, so that means from tomorrow I may go directly from Guwahati to Indore, in one-step I can go from Guwahati to Indore So, it is important in this case in case of accessibility and communication, the importance is that whether I can go or not, how many step I need that is a different issue. I will we are not clear about that in this particular case, whether I can go or not that maybe in multiple steps that maybe in one-step is does not matter as long as we are concerned about accessibility and communication. And the communication, we will denote by this notation. So, you can understand from here that here i to j, I have 1 side arrow, but when the communication there I have 2 side arrows that mean from i, I can go to j, from j, I can go to i. So, that is the essence or intuition behind communication and this is the definition of the communication here. So, let us now proceed. So, before proceeding this 1 let us again talk about some practical thing. So, in our family, we have different relationships, like I have my father, my mother, grandmother, sisters, brothers, so on so forth. So, now in some relationships has some special characteristics whereas some other relationship do not have those characteristics. Just for example, talk about the relationship between myself and my father. So, my father suppose, I am Ganguly and my father is Molay Ganguly. So, the relationship from A to M is father, but the relationship from M to A is son. So, that means, in this particular case, M is father to A that does not mean that A is also father to M, that is not true. But suppose, I am Ayon Ganguly and I have a brother suppose the name is a Bhasker, I have a brother name is Bhasker. Now, in this case, this relationship is brother whereas the opposite relationship is also brother. So, you see that there are 2 different kinds of relationships here, one is that that when it is a father relationship, that means M is father to A that does not mean that A is father to M, in fact, A not father to M. But A is brother to B that also implies that B is brother to A. So, we have such thing in practice, then when I am talking about relationships, some relationships are both sides, this relationship holding both sides, in some cases, the same relationship does not hold in both side but some other relationship comes on the opposite side like father and son in this case that if I go from A to M it is father but M to A is son, but A to B is brother, B to A is also brother. So, such kind happened in case of communication also. So, communication, I can think as a relationship, the reason is that in this case, I am talking about the relationship of 2 persons, when I am in family, I am talking about the relationship between 2 persons relationship between myself and my father or myself and my brother, myself and my sister, myself and my mother and so on so forth or the relationship between my father and my mother, all those things we can talk about. So, but the same kind of things are there in case of the communication also. In case of the communication, we have 2 states i and j and as a relation communication is in between, so that communication I can think as a relationship. So, in family I have 2 persons, I have a relationship between 2 persons, in case of the communication you have the Markov chain I have 2 states and I have a relationship between these 2 and the relationship is nothing but the communication that from i, I can go to j and from j, I can go to i. So, now, in case of such cases, I can talk about some property of that particular communication relation. I can talk about some properties of the communication relation. The properties are coming from that of our normal life, normal family. First property is called reflexive and that is that I has a relationship with I. So, myself has a relationship with myself and what is that? That is the self-relationship. So, I has self-relationship with I, I here means myself, I here myself, so I have leadership self-relationships with I and the other side I has the relationship with I that is the self-relationship, that is the first condition, that is the called the reflexivity. That I have communication with I and that is true in case of our communication also I will come back to this one also why this is true in case of the communication also. First thing from our family perspective, then we will come back to our communication thing. Symmetry, symmetry means if i is communicating, if i has a relationship with j, then j has the same relationship with i like brother. So, brother, brother is a symmetric relationship because that A is brother to B, B is also brother to A, so that i is related to j that imply an implied by j is related to i. So, that is there and then now talk about the transitive relationship. Transitive relationship says that if i and k are related then by some relation and k and j is related by the same relation, then i and j are also related by the same relation, i and k are related by one relation, k and j is relation by the same relation, then i and j are related by the same relation. That is true for brother, you see that that is true for brother, but that is not true for father because suppose I have now a son. Maybe I have now a son maybe called the name of my son maybe start with S, then it is also if it is I have not, sorry, I have a son now, I have a son now, his name start with the S. So, his name is, this relationship again is father relationship, but of course, that does not mean that M is father to S. So, this is not symmetric, father relationship is not symmetric, it is not transitive, but this relationship is transitive also, the brother relationship is transitive also because if B has another brother called C, then A and C is also brother. So, A is product to be B, B is product to C, then A is brother to C, so the brother relationship is symmetric transitive, but father relationship is not symmetric, not transitive. So, some relationships have some properties, some relationship may not have that property. And for our case, the main idea is that this communication relationship has these 3 properties and we can very easily show this thing from the definition of communication. First reflexive, why this reflexive thing is true, because we know that pii 0 is strictly is equals to 1, so it is strictly greater than 0. So, there exists some n such that $p_{ii}^{(n)}$, in this case I take $n = 0$ and $p_{ii}^{(0)} = 1$. So, this reflexivity holds true that i is communicating with i and i is communicating with i, on the other hand also. So, this is true. Now, think about the symmetry. Symmetry tells that that that i related to j, i is communicating with j that means, there exists some n such that this quantity is greater than 0, some m such that this quantity is also greater than 0. Now, j is communicating to i that mean because this is greater than 0 and this is greater than 0, this part is also true. So, the approval for 1 and 2 is very trivial, but the approval of the last 1 is not that trivial. Let us discuss now, the approval the transitivity.

So, the proof of the transitivity goes like that first says what is this saying? It i is communicating with j, so that means, there exists some n such that this quantity is strictly greater than 0 and there exists some m such that this quantity is strictly greater than 0. Then k is communicating to j, so that means, there exists some maybe I call l such that j is strictly greater than 0 and there exists some maybe q such that this quantity is greater than 0, there exists some n such that this quantity is 0 greater than 0 m such that this quantity is greater than 0. This one is true because of is communicating with j and because j is communicating, then k is communicating with j that is why there exists l and q such that both the quantities are strictly greater than 0. Now, think of this i to k. So, first notice that from i to j in $(n+l)$. So, what I have done is that this n, I have taken, this l, I have taken, so using the Chapman Kolmogorov equation, I can write this one as $p_{i,i_1}, p_{i_1,j}$ and this is in n step, this is in l step and I take the summation over $i_1 \in S$.

This is the using Chapman Kolmogorov equation this equality holds true. Now, so that this n because this n_1, when I am taking this belongs to S, so i1 also take the value k. So, I can right this quantity is greater than or equals to p_{ik} in n step p_{kj} in l step. So, I have taken a particular value of i_1, $i_1 \in S$ and k is a member of S in number of steps pairs. So, that is why I have taken a particular value of i_1. So, that means, one term I have kept here because all of them are probabilities. So, this one particular term will be less than or equal to this quantity is greater than or equal to this quantity. Now, both of them are strictly greater than 0 because one here, another is here. So, this quantity is strictly greater than 0. So, that shows that from i to j, I can go in $(n+l)$ steps. So, that means that j is accessible from i. Similarly, now, we can show that i is accessible from j just writing that try to find out that prove p_{ji} in $(m+q)$ step, the same manner I will do. I will do and we can show that this quantity strictly greater than 0 and that shows that i is accessible from j. So, j is accessible from i, i is accessible from j that shows that i and j communicates. So, that is 3 properties that are satisfied by communication relationships. And if these 3 properties are satisfied by some relationship then that relationship actually partitions the state plus like our family C talk about the brother relationship. In case of the brother relationship, you see that my brothers they are 1, class 1, 1 class of my family, one set of my family. If you think of the of all of your uncles including your father, so that is another class of brothers that is another, another, another, another subset of the family. And you notice one very important thing here that all the members in the class why are all my brothers belongs and all the members in the class or my had all my uncles and father belong these 2 sets are disjoint set there is no common member which belongs to both the sets. So, that means that my family is partitioned by this relationship. And the same thing happens in this case also and what we can do? We can tell here is that if I have this state pairs set is given by that. Then the equivalence relation actually partition the state pairs into several groups and hired the member of each group, each class that the all the members they are all the members they are communicating among themselves, all member communicates. All members here communicates. Similarly, all members here communicators, all members here communicates, all member here communicates. But a member from here may not communicate with the member will not communicate with a member of the other class. A member of 1 class will not communicate with a member with another member of the other class, but the thing is that we see that I can go from here to here, that is not a problem, but I cannot come back from here to here, that is the. So, that is the idea of the communication and that says that communication is an equivalence relation. And any equivalence relation actually partitions the set, in this case, the communication in this case that communication relationship partitions our state pairs and it partitioned the state pairs into communicating classes that each parts are called the communicating classes. This is one communicating class, this is another communicating class, this is another communicating class, this is another communicating class and this is another communicating class. We will discuss all these concepts using some examples also. We will see some examples in the next slide.

Before going into that example, let us see some more definitions which are quite meaningful in this case, the first definition is about the closed communicating class. So, a communicating class C is said to be closed if this happens, what does this mean? Notice that this C is a class, C is a communicating class. And i belongs to here and I can go from i to j, i to some j, I can go, look, j is another state I can go from i to j and this implies that $j \in C$, so that j has to be in C. So, j is somewhere here. So, that mean closed. So, if i, I can go from i to j, if i belongs to your communicating class then j must belong to the same communicating class then, C is called in that communicating class is called closed. And that mean that if I have a closed communicating class and if I am already in that class I cannot go out of the class. If I have a communicating class C and if I am already in the class C, then I cannot go outside that closed class that is the idea of the communicating class. Now, talk about the absorbing state, absorbing state or absorbing class. So, suppose i is a state which is a absorbing which is called absorbing state. If I consider this class, this is the singleton set this class, this is a close class. If i as a singleton class, singleton set is a closed class then we say that it is absorbing class, what does this mean? That means, once I mean I cannot move out for my i, I will be i forever, so that class is that kind of classes is called the communicating class, sorry, absorbing class and, absorbing state. So, this kind of state we have seen earlier just recall that the examples we have talked about in the previous class, previous lecture. So, in that lecture, we have talked about some classes that once I am in that class I will be there forever like in the last example, when we talk about the class 4, sorry, class 3 which means that the sequence of 3 consecutive head has already occurred. So, once I am in that class, I cannot go out of this class. So, that class that kind of class is called absorbing class. So, we will see some more examples as requested and finally, irreducible Markov chain that mean that your state pairs S is a closed a communicating class, closed communicating class. So, a Markov chain is said to be irreducible if all state communicate with each other that means, a single communicating close class is there a single communicating class is there. So, if all the states communicate among themselves in a Markov chain, then that Markov chain is called irreducible Markov chain and for that simply says that for the irreducible Markov chain starting from any state I can go to any other state. There is a positive probability to go to any other state starting from any other state. That is the irreducibility mean.

Now, let us see some examples. So, let us start with this example. Before starting this example, see that we have pointed out that if I am given with the initial distribution and the one-step transition probability matrix of some Markov chain, the Markov chains all the probabilistic characteristics of the Markov chains are specified. So, in this case, of course, I have not given the initial distribution, but I have given only the one-step transition probability matrix. And the interesting thing here is that with that of course, do not specify all the probabilistic characteristics of a Markov chain, but, given the one-step transition probability matrix, I can check whether the classes communicates, classes accessibility, closed class, all those things, we can actually see. To do this 1, the easiest way to do is that to draw the graph, what kind of graph let us discuss that. So, in this case I have 3 states 0, 1, 2. Suppose these are the states, now, you see that in one-step there is a positive probability for going from 0 to 0. In one-step, there is a positive probability from going to 0 to 0. So, I just write the path like this from 0, I can go to 0 with positive probability. Similarly, in one-step I can go from 0 to 1. So, with positive probability I can go from 0 to 1. So, I just write like this. So, this way I will add all the edges, so it is kind of a graph we are plotting where we are writing and here you can see that 0, 1 and 2 are nodes and these are the edges and edges means there is a positive probability of going from 1 node to another node. So, that means that well I have the I in this case from 0 to 0 with positive probability I can go so this edge comes from 0 to 1 with positive probability I can go so these edge come. But there will be no direct edge between 0 and 2, because in one-step, I cannot go. Here I am writing everything for one-step thing, Next, from 2, I can from 1, I can go to 0 with positive probability, 1 I can go to 1 with positive probability, 1 I can go to 2 with positive probability. So, this arrow comes here, this loop comes here as well as this arrow comes here. Similarly, from 2, I can go to 2, from 2 I can go to 2 with positive probability in one-step and from 2 I can go to 1 with positive probability in one-step. So, that I can go in one-step with positive probability. So, this graph is nothing but whenever I have a directed edge, that means I can move from 1 to another state in one-step, whenever I do not have, I mean, there is a positive probability of moving from 1, one node to another node in one-step, whenever there is no edge or no directed edge between 2 nodes that means from that particular node, I cannot move to another particular node in one-step. So, in this case, see that I do not have any edge between 0 and 2 that signifies that in one-step, I cannot go from 0, to 2 but there is a directed edge between 0 to 1. So, that means that in one-step, there is a positive probability for going 0 to 1. Now, once I draw this 1, the taking that what are the classes communicates all those things is very, very simple. First, think about state 0 and 1, from 0, I can go to 1, there is no problem in one-step I can go. From 1, also I can go to 0. So, clearly, because in one-step actually I can go. I can go in one-step with positive probability. So, that means that state 0 and state 1 they communicate among themselves. In one-step, I can go from 0 to 1, I can there is a positive probability of going from 0 to 1 in 1 similarly, in one-step, there is a positive probability to go to 0 from 1 and 1 to 0 both sides are possible. Similarly, so that means that 0 and 1 are actually communicates. Similarly, 1 and 2 also communicates because in one-step i there is a positive probability of going to 2 from 1, similarly, there is a positive probability of going to 1 from 2 in one-step. So, that means 1 and 2 also communicate among themselves, whatever 0 and 1? In this case, of course, I cannot go in one-step, but you see that from 0, in one-step there is a positive probability of going to 1 and from 1 there is a positive probability of going to 2. So, that means 2 is accessible from 1, 2 is accessible from 0. 2 is accessible from 0 because from 0, I can go to 1 with positive probability in one-step and from 1, I can go to 2 with positive probability in one-step and that means from 0, I can go to 2. Of course, in this case in one-step I cannot go but in 2 step I may be able to go. Similarly, from 2, I can go to 1 then from 1, I can go to 0. So, these 2 classes also communicates, these 2 states also communicates. So, in this case, I have these 3 states, all the 3 states communicates and so this is 1 class in this case just 1 class exist in this case, 1 communicating class exist in this class and this class has to be closed because this is the whole, the step is this is the whole step was. And because all the steps in this case communicate among themselves, the Markov chain is the irreducible Markov chain, the corresponding Markov chain is a irreducible Markov chain.

Let us proceed to next example, again a one-step transition probability matrix is given and again if you see, we can write, we can draw the diagram, the graph like this and again the interpretation are same that if there is a rejected is I can go, there is a positive probability of going from 1 to another in a one-step. So, I have this now, I can see that 0 and 1 communicates, so clearly 0 and 1 communicates. What about 0 and 2? See that from 0, I cannot go to 2 because there is no path from 0, I cannot go directly to 2, from 0, I cannot through from 0, I can go to 1, but from 1, I cannot go to 2. So, these 2 communicate among themselves. So, these 2 makes 1 class, 1 communicating class. Now, talk about 2, from 2, I can go to 3, fine, but 3, I cannot come back to 2 because 3, from 3, I can only go to 3. So, of course, 3 is accessible from 2, but 2 is not accessible from 2, but 2 is not accessible from 3. So, that means these 2 class do not communication that means that 2 is separately 1 communicating class and 3 is separately another communicating class. So, we have these particular thing 0, 1 is 1 communicating class, 2 is along 1 communicating class, 3 is along another communicating class. 0, 1, this comma will not be there. 0, 1 is closed that means, once you in 0 you cannot come out of this class. Similarly, once you are in 1, you cannot come out of this class. From 0 you can only go to 1, 1 you can only go to 0. So, you will be always here, you will be always here. Similarly, 3 is closed because once you are 3, you are in 3 you can only remain in 3 because only one path, one directed path is there, one directed edge is there. So, you will be in 3. So, this 0, 1 and 3 are closed, plus 2 is not closed. Reason is that from 2, I can go to 1. So, from starting from 2, I can go out of this particular class out of this particular communicating class. So, that is why 2 is not closed. And of course, this is not irreducible because I have 3 communicating classes in this particular Markov chain.

Similarly, let us talk about this it is a little bit bigger, but the things are almost same. You have this particular diagram and now in this case it is easy to see, again we have 3 communicating classes that 0, 1 and 2 these actually make 1 communicating class, 3 alone makes 1 communicating class and these 2 makes 1 communicating class. And in this case 4, 5 is closed, this 1 is closed, the others are not closed. The reason being that from if I start with this 1 suppose I am in 0, from 0, I can go to 1 with positive probability, from 1, I can go to 2 with positive probability and from 2, I can go to 4, I can go to 3 with positive probability and when I go that I am out of this class. So, clearly the first class is not closed. Similarly, this class is also not closed because from 3 there is a positive probability of going to 3 going to 4 that means, I can go out of this communicating class. So, that is why this is not also closed and there is no question of irreducibility because I have multiple communicating classes. Recall that irreducibility meaning that I have only 1 communicating class and in this case I have multiple communicating classes. So, clearly this is not irreducible Markov chain. So, these are some examples. So, today's lecture, just recall we have learned what is accessibility, what is communication and then we have pointed out that communication satisfies 3 or 3 conditions, flexibility, symmetry and transitivity. And that means that the communication can be seen as a relation and that relation partitions the status and each partitions are called communicating classes. And then in through some examples, we have shown that how using diagram using graph, we can easily find out the communicating classes, closed classes of a Markov chain from one-step transition probability matrix. With that I stop. Thank you for listening.