Introduction to Queueing Theory

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Lecture - 10

Poisson Process and its Properties, Generalizations

Hi and hello, everyone; what we have been seeing was this Poisson process. Recall that the definition that we have seen in the previous lecture.

- A counting process $\{N(t), t \geq 0\}$ is said to be a (homogeneous) Poisson process (PP) with rate (or intensity) $\lambda > 0$ if
  1. $N(0) = 0$,
  2. it has independent increments, and
  3. the number of events in any interval of length $t > 0$ has $Poi(\lambda t)$ distribution.
- The (homogeneous) Poisson process has stationary increments. The definition above fixes all finite dimensional distributions of the stochastic process.
- Fix any $T > 0$. Define a new process $N_T(\cdot)$ by $N_T(t) = N(T + t) - N(T)$. Then $\{N_T(t)\}$ is again a Poisson process with rate λ. Thus a Poisson process probabilistically restarts itself at any point of time (Markov property).
- **Alternative Definition**: A counting process $\{N(t), t \geq 0\}$ is said to be a (homogeneous) Poisson process (PP) with rate (or intensity) $\lambda > 0$ if
  1. $N(0) = 0$,
  2. it has independent increments, and
  3. we have the orderliness property:
     $P\{1 \text{ event between } t \text{ and } t + \Delta t\} = \lambda\Delta t + o(\Delta t)$, and
     $P\{2 \text{ or more events between } t \text{ and } t + \Delta t\} = o(\Delta t)$,
     where $o(\Delta t)$ denotes a quantity that becomes negligible when compared to Δt as $\Delta t \to 0$, i.e, $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.

Exercise.
Using the alternative definition and starting from the basic principles, obtain the differential-difference equations satisfied by $p_n(t) = P\{N(t) = n\}$ *as*
$p_0'(t) = -\lambda p_0(t)$
$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t), \; n \geq 1.$
Solve these and show that $N(t) \sim Poi(\lambda t)$ *distribution.*

Now, let us look at some more properties of this Poisson process. First, let us look at the connection between this Poisson process and the exponential distribution that we have seen earlier. So, what is that? It says the following.

- If $\{N(t), t \geq 0\}$ is a Poisson process with rate λ, then the times between successive events (called as inter-event times) are independent and exponentially distributed with parameter λ.
- Let $T_1, T_2, \ldots$ be the inter-event times, where T_n is the time elapsed between $(n-1)$th and nth event.
- We have $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$ which proves that $T_1 \sim Exp(\lambda)$.
- For any $s > 0$ and $t > 0$,
  $P\{T_2 > t | T_1 = s\} = P\{\text{no events in } (s, s+t] | T_1 = s\} = P\{N(t) = 0\} = e^{-\lambda t}$,
  as events in $(s, s+t]$ are not influenced by what happens in $[0, s]$. So T_2 is independent of T_1 and has $Exp(\lambda)$ distribution.
- Similarly, we can establish that T_3 is independent of T_1 and T_2 with the same distribution, and so on.

Now, what about the converse? The converse also holds, meaning that if there are certain events, there is an event right and for which the inter-event times are all exponential IID with parameter λ. Then, the events occurring according to a Poisson process, the number of events that are happening is according to a Poisson process with rate λ. And hence, this gives a characterization of a Poisson process as well. So, if the inter-event times are exponential, then the number of events will follow a Poisson process. If there is a Poisson process, then the inter-event times are IID exponential. So, this is characterization, and this connection is this Poisson exponential connection or Poisson exponential model, and sometimes this is also referred to as the Poisson exponential model. So, it means that in a Poisson process, it is always the case that the inter-event times are IID exponential. If the inter-event times are IID exponential, then the number of events happening is according to a Poisson process with rate λ. So, this connection is also important in many ways, and one way is that in the simulation of a Poisson process, say, obviously like, you may have to simulate in some situations a process. Now, how do you simulate that you will generate an exponential random variable, and that would give you the time for the next event of a Poisson process. So, that is how you will use it. And how you can simulate an exponential random variable because that is we are not touching, but of course, that is very much closely related to the topics that we might use and when you actually apply this whole of queuing theory to some realistic situations.

- If S_n denotes the time of the nth event, then S_n has $Gamma(n, \lambda)$ distribution.

Now, this S_n, so this is if you denote this the time of the n^{th} event by S_n. So, basically, S_1 will be equal to T_1, S_2 will be $T_1 + T_2$, S_3 will be $T_1 + T_2 + T_3$, and so on. S_n will be the sum of the T's from 1 to N, the inter-event if you sum, that is the time for the n^{th} event to happen, and you know the sum of exponential is gamma. So, the time of the n^{th} event would follow now a gamma distribution, or this is what you can also sometimes refer to as the waiting time distribution, waiting time for some particular number of events to happen. So, waiting time for 10 arrivals, waiting time for 20 arrivals, 20 failures in some production system, and so on. So, these are all would be then gamma distribution is what then you would see. Because of exponential, the sum of exponential IID exponential is gamma. So, that will give you the waiting time distribution in a Poisson process follows a gamma distribution. So, now let us look at a simple example and see what kind of questions can arise in such situations and how they can be answered through a Poisson process type of model. So, here basically, we are looking at the Poisson process model.

Example.

Consider the car insurance claims reported to insurer. Assume that, the average rate of occurrence of claims is 10 per day. Also, assume that this rate is constant throughout the year and at different times of the day. Further assume that in a sufficiently short time interval of time, there can be at most one claim. What is the probability that there are less than 2 claims reported on a given day? What is the probability that time until the next reported claim is less than 2 hours?

Solution : The given situation can be modelled as Poisson process. The number of arrivals of car insurance claims to the insurer in time intervals of length t is a Poisson process $\{N(t),\ t \geq 0\}$, with rate λ as 10 per day.

Required probability that there are less than 2 claims reported on a given day

$$\begin{aligned} P(N(t+1) - N(t) < 2) &= P(N(1) < 2) \\ &= P(N(1) = 0) + P(N(1) = 1) \\ &= \frac{e^{-\lambda \times 1}(\lambda \times 1)^0}{0!} + \frac{e^{-\lambda \times 1}(\lambda \times 1)^1}{1!} \\ &= e^{-\lambda}(1 + \lambda) = \mathbf{11e^{-10}} \approx 0.0005. \end{aligned}$$

Required probability that time until the next reported claim is less than 2 hours (2 hours $= \frac{2}{24}$ days) is $P(T < \frac{2}{24})$, where T is inter-arrival time that follows an exponential distribution with parameter λ. Thus, $P(T < \frac{2}{24}) = 1 - e^{-\lambda \frac{2}{24}} = \mathbf{1 - e^{-\frac{5}{6}}} \approx 0.5654$.

So, there is more than a 50 percent chance that in the next 2 hours, you are going to get 1 claim. So, it is a 50 percent chance is there; that is what it says here. So, like this, in different situations, of course, you can look at the different books when you are dealing with the Poisson process for various kinds of examples models. Basically, the underlying assumption should be such that the Poisson model can fit in there. So, that is what is important. Once you are ensured that the underlying assumption would fit, when things happen in a random fashion, you can always take a Poisson process model as a first cut model. So, next, what we are looking at is basically some more properties of this Poisson process; the first one is what we call the "Poisson thinning process or splitting or decomposition process". What is that?

- Consider a Poisson process $\{N(t)\}$ having rate λ. Suppose that each event is classified as a type I event with probability p or a type II event with probability $1 - p$, independently of all other events. Let $N_1(t)$ and $N_2(t)$, respectively, denote the number of type I and type II events occurring in $[0, t]$. Note that $N(t) = N_1(t) + N_2(t)$.

  Then, $\{N_1(t)\}$ and $\{N_2(t)\}$ are both Poisson processes with rates λp and $\lambda(1 - p)$, respectively. Furthermore, the two processes are independent.
  You can look for the proof elsewhere; it is not a very complex one. But, we can just give a sketch so that you understand the use of the total probability law that we repeatedly said. You apply everywhere, and this is what a situation with.

(Sketch:)

$$\begin{aligned} P\{N_1(t)=n\} &= \sum_{r=0}^{\infty} P\{N(t)=n+r\}P\{N_1(t)=n|N(t)=n+r\} \\ &= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!}\binom{n+r}{n}p^n(1-p)^r \\ &= e^{-\lambda t}\sum_{r=0}^{\infty}\frac{(\lambda pt)^n(\lambda(1-p)t)^r}{n!r!} \\ &= e^{-\lambda t}\frac{(\lambda pt)^n}{n!}e^{\lambda(1-p)t} = e^{-\lambda pt}\frac{(\lambda pt)^n}{n!} \end{aligned}$$

So, this is the expression that you are going to get, and we show that $N_1(t)$ is a Poisson process with rate λp or $N_1(t)$ is a Poisson distributed random variable for the fixed t with parameter λpt. So, similarly, you can do $N_2(t)$, and then you can show that these are independent and so on. This is what it actually means?

The other important property is "Superposition" is just the reverse of what might happen, what has happened to the previous scenario.

- Let $\{N_1(t)\}$ and $\{N_2(t)\}$ be two independent Poisson processes with rates λ_1 and λ_2, respectively. Then $\{N(t)\}$, where $N(t) = N_1(t) + N_2(t)$, is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.

(Sketch:)
Note that $E(z^{N_i(t)}) = e^{\lambda_i(z-1)t},\ i = 1, 2.$

Therefore, the PGF of $N(t)$ is, using the independent of $N_1(t)$ and $N_2(t)$,

$$E(z^{N(t)}) = E(z^{N_1(t)})E(z^{N_2(t)}) = e^{(\lambda_1+\lambda_2)(z-1)t}.$$

So, this is the reverse. Again superposition from different streams the arrivals are coming, and if they are all independent, the combined arrival process now would be a Poisson process. So, that is what you are looking at here in this particular scenario. So, this is a superposition property. Another property that is also important in our context and to say that why this particular process is called a completely random process; is because of this fact which we are looking at now. So, what is that where we have?

- Consider a Poisson process $\{N(t)\}$ with rate λ. Given that $N(T) = n$, the n event times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent uniform random variables on $[0, T]$.

Recall: Let $X_1, X_2, \ldots, X_n$ be IID RVs with PDF $f(\cdot)$. If we let $X_{(i)}$ denote the ith smallest of these RVs, then $X_{(1)}, \ldots, X_{(n)}$ are called the order statistics. The joint PDF of $(X_{(1)}, \ldots, X_{(n)})$ is given by

$$f_{(X_{(1)},\ldots,X_{(n)})}(x_1, \ldots, x_n) = \begin{cases} n! \prod_{i=1}^{n} f(x_i) & \text{for} \quad x_1 < x_2 \ldots < x_n \\ 0 & \text{otherwise} . \end{cases}$$

The marginal PDF of $X_{(i)}$ is

$$f_{X_{(i)}}(x) = \frac{n!}{(n-i)!(i-1)!} f(x)(F(x))^{i-1}(1-F(x))^{n-i} .$$

- Using the above, what we have is that

$$f_{(S_1,\ldots,S_n)}(s_1, \ldots, s_n) = \begin{cases} \frac{n!}{T^n}, & \text{for} \quad s_1 < s_2 \ldots < s_n \\ 0 & \text{otherwise} . \end{cases}$$

So, this Poisson process is sometimes said to be a ”completely random” process or a purely random process. This the word or the justification for that comes from the fact that the events are happening in a uniform fashion in a way that is uniformly distributed in time. Say, for example, that at a particular time t, one event has happened. Now, given the fact that, say, at a particular time t_0, one event has happened, what is the probability that it would have happened at any point in the $[0, t_0]$; if you look at it and that is uniformly distributed between $[o, t_0]$, meaning, that it could have happened at any point of time starting from 0 up to t_0 at any point of time; it has an equal probability of occurring. So, that is the uniform nature; it could happen in a random fashion.

- So what the uniform property means is that: Given that n events have occurred on $[0, T]$, the *un-ordered* event times are independent and uniformly distributed (or equivalently, the *ordered* event times follow the order statistics of n independent uniform random variables).
- One important consequence of the uniform property of the Poisson process is that the outcomes of random observations of a SP $\{X(t)\}$ have the same probabilities as if the scans were taken at Poisson-selected points.

So, you have a timeline. Now, you look at this process at some random point of time, and also you look at the process at a Poisson selected point which means that wherever an event happens according to some Poisson process. So, you are looking at, you are marking that points in the timeline, and you are looking at the process only at that time points how the process will look like, and you are looking at arbitrary selected time points of the process. Now, how will these views of the process will be for you when you are looking at these two sets of points. So, this uniform property of the Poisson process makes the observations at whichever time point you are looking at is irrelevant; they have the same property. So, they have the same probabilities as if the scans were taken at a Poisson selected points; for us, this is an important property which is called PASTA, meaning ”Poisson arrivals see time averages.”
► When $\{X(t)\}$ represents the number in a queue, this property is called **”PASTA”** (Poisson Arrivals See Time Averages).

Now, the question is whether the number in the queue you are looking at, any random point of time, or at the arrival instance. Now, the arrival instance, when you are looking at the number in the queue or any random point you pick, and then you are looking at the number in the queue. Now, how will these two probabilities will differ in general, they need not be the same, but for Poisson arrivals, they will be the same. So, that is what is called the "PASTA" property, which is very relevant. Because at what time you are looking at, the process is also important in a certain context. So, this property gives you that it is irrelevant as far as Poisson arrival is concerned because Poisson arrivals see time averages. Random point of time when you mean it is the time averages over a long period of time. What is the number that you are looking at that is what will give you the time averages. Now, Poisson arrivals will see time averages is what this property is; of course, this is an important property with respect to the Poisson level case, and that is because of the consequence of this uniform property of this Poisson process.

So far, what we have seen is this Poisson process idea and its certain properties; of course, there may be many more properties, but the properties that are most relevant for us we have listed out and their implications and consequences also we have listed out. Then, again there are many generalizations of this Poisson process. As we said, this is a first cut approximation; a very elementary level approximation process is what is done by the Poisson process model. There could be many generalizations; at least one generalization that is very relevant for us is what we call the "**Compound Poisson process**." What is that mean? We are defining it in this manner.

Let $\{N(t)\}$ be a Poisson process and let $\{Y_i\}$ be an IID sequence of strictly positive integer RVs that are independent of $N(t)$. Then, the SP $\{M(t), t \geq 0\}$, where $M(t) = \sum_{i=1}^{N(t)} Y_i$, is said to be a compound Poisson process (CPP). Basically, you have a scenario where you have a Poisson process where the events occur according to a Poisson process, but the number of events that occur at a particular event selected time is not just 1; it could be more than 1. If it is 1, it becomes a simple Poisson process; if it is more than 1, then it becomes a compound Poisson process in general. So it means that the events occur in some sense in groups or in clusters; that is why this is also called as a "cluster Poisson process". Clustered Poisson process means the events occur in clusters, but the inter-event times are exponentially distributed, but the event occurs in clusters in groups, in batches; that is what it would mean. So, that batch size is what effectively is this $\{Y_i\}$; if the batch size is equal to 1, then it becomes a simple Poisson process; if it is for a general one, then it becomes a compound Poisson process.

► If $\{N(t)\}$ is Poisson with rate λ, then $E(M(t)) = \lambda t E(Y_1)$ and $Var(M(t)) = \lambda t E(Y_1^2)$, which we have already obtained in the earlier case when we are dealing with random sum recall, random sum quantities that we have dealt with. So, that random something is one such instance is occurring here is what then you are seeing it here.

Example 1.

Suppose that buses arrive at a sporting event in accordance with a Poisson process, and suppose that the number of fans in each bus are independent and identically distributed. Then $\{M(t), t \geq 0\}$ is a compound Poisson process, where $M(t)$ denotes number of fans who have arrived by time t.

Example 2.

Suppose that customers leave a supermarket in accordance with a Poisson process. If Y_i, the amount spent by the ith customer for $i = 1, 2 \ldots$ are IID, then $\{M(t)\}$ is a compound Poisson process, where $M(t)$ denotes the amount of money spent up to time t.

Many saturation is used in queueing mainly; we were looking at bulk arrivals, bulk service things where you will need this compound Poisson process. You can always think about bulk arrival systems as having compound Poisson process arrivals. The other generalization which we may not deal with in this course, but it is very relevant from a practical point of view, and hence I will just hint it, it is not we will not do much on this is what is called a nonhomogeneous or nonstationary or inhomogeneous Poisson process. We said this $\lambda(t)$ the parameter λ that we have got for a Poisson process is constant with respect to time, which means that the rate of arrivals happens at a constant time at a constant rate at all time points. We just set an example of an insurance claim that comes. Whether it is daytime or nighttime or if you are considering only day as one time unit or you know whatever is the case. You are assuming that throughout the day, it occurs at the same rate and so on. But in many situations, this may not be the case; for example, if you take a hospital scenario. Then, even insurance claims, for example, if you are looking at, say, agricultural claims. In certain seasons only you get more number of claims; in certain seasons, you will get less number of claims. So obviously, this will be time-dependent. Similarly, in the case of a hospital, the arrival rate at morning and lunchtime and evening and late-night, things are not going to be the same rate. Same thing in a traffic junction; if you are looking at the arrival of vehicles like it will not be the constant rate of arrival throughout the day. Then, one can easily do business with that, but that is not going to be the case; obviously, be time-dependent ones. So that means this lambda is not a constant, but it is a function of time. So, this is more realistic than a simple Poisson process, a nonhomogeneous Poisson process.

- A NHPP can be thought of as a PP where λ is replaced by a time-dependent function $\lambda(t)$.
  ♦ More realistic than a PP (as the rate of occurrence of events are not the same across times).
- A counting process $\{N(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson process (NHPP) (or inhomogeneous or nonstationary Poisson process) with mean event rate $\lambda(t) > 0$ if
  1. $N(0) = 0$,
  2. it has independent increments, and
  3. we have the orderliness property:
     $P\{1 \text{ event between } t \text{ and } t + \Delta t\} = \lambda(t)\Delta t + o(\Delta t)$, and
     $P\{2 \text{ or more events between } t \text{ and } t + \Delta t\} = o(\Delta t)$.
- NHPP loses the stationary increments property.
- For a NHPP $\{N(t), t \geq 0\}$ with mean event rate $\lambda(t)$, the number of events in a time interval $(s, t]$ is a Poisson RV with mean $m(t) - m(s)$, where $m(t) = \int_0^t \lambda(u)du$.
- The function $m(t)$ is sometimes called the mean value function and it represents the cumulative expected number of events by time t.

  [Note: $\lambda(t)$ represents an expected arrival rate].

Example.

Consider a nonhomogenous Poisson process with $\lambda(t) = \begin{cases} 5 & \text{if} \quad t \in \ (1, 2], (3, 4], ... \\ 3 & \text{if} \quad t \in \ (0, 1], (2, 3], ... \end{cases}$.

Find the probability that the number number of observed occurrences in the time period $(1.25, 3]$ is more than two.

Solution : $N(3) - N(1.25)$ *has a Poisson distribution with mean*
$m(3) - m(1.25) = \int_{1.25}^{3} \lambda(t)dt = \int_{1.25}^{2} 5dt + \int_{2}^{3} 3dt = 6.75.$
Hence, $P\{N(3) - N(1.25) > 2\} = 1 - e^{-6.75}(1 + 6.75 + \frac{(6.75)^2}{2}) = 0.9643.$

Of course, nonhomogeneous in our course, we are not going to deal with it, but the better thing is you are aware of such a situation it can be handled in a quite a bit easier way using the simple properties that we already have our Poisson process you can do. You remember that stationarity properties say stationarity increment property is not there; that is what you have to keep in mind when you are dealing with the nonhomogeneous thing. Before we wind up, you know we just note a few points.

- It is a special case of continuous Markov Chains but the Poisson process is also a special case of a larger class of processes called **renewal processes**. A renewal process arises from a sequence of nonnegative IID random variables denoting times between successive events.
- For a Poisson process, the inter-event times are exponential, but for a renewal process, they follow an arbitrary distribution.

The quantities of interest are, again, the number of renewals that happens by time t, just like the number of events that happen in a Poisson process by time t, which we have found out to be a Poisson distribution so that you can get the quantities immediately. But here, the inter-event times are not following an IID exponential, but it is an IID random variable, and it follows a general distribution that has a support on the nonnegative real numbers that is what you call a renewal process.

Now, what happens is that at the points of the events, the process will behave like a Markov Chain, but otherwise, in general, it will be a renewal process, it is what. So, the only thing now is the memoryless property, and everything will be lost. So, you need to keep track of when the last event has happened because it is not exponential anymore in interevent time. So, the same problem that you are doing with the Poisson process if you remove the exponential and if you have to do this, then you need some more ideas to tackle the problems; that is what it will become we will see when we start the semi-Markovian queues, we will see a little bit about this renewal process before we start the semi-Markovian queues.

But many of the properties of the PP also hold true in the renewal context as well (and we will see about the renewal processes later). So, with this, we will end this our discussion on the Poisson process with these ideas in mind, the simplest case of a continuous Markov Chain; we will now do the continuous-time Markov Chain ideas we will see in the next lecture.
Thank you, bye.