Introduction to Queueing Theory

Prof. N. Selvaraju
Department of Mathematics
Indian Institute of Technology Guwahati, India

Lecture - 45
G/G/1 Queues: Bounds

Hi and hello, everyone. What we are seeing in this last phase of this course is the General Queuing Models, in which what we have seen in the previous lecture was a generic $G/G/1$ queue, and we did its analysis through Lindley's integral equation. And we showed how complex or how difficult it is if you have to do a complete analysis. It is not that it is impossible, but it involves other tools. But, for more general distribution, even for $M/M/1$, if we have to do it via $G/G/1$, what was the difficulty we exhibited. In general, analyzing such queues involves complex analysis as well as transform inverse. And since these procedures are complicated many cases, we need numerical methods to do the analysis; as an alternatively, one can consider developing bounds and at least approximations to the $G/G/1$ analysis that we have done in earlier. So, that is what we summarized in the previous lecture with the analysis of $G/G/1$.

Now, what we are going to do now is to take up this idea of developing bounds for the performance measures that you can develop, which in turn will help us to get approximation methods or approximation procedures or processes or whatever you are going to call it. But, that we are not going to touch, but we will just be concerned about the bounds. At least we will give us an idea about how one can develop bounds directly so that it is still of some use. So, that is what is the idea here. As you just said that for many queuing systems, direct analytical methods are not available, and hence bounds and approximations may be developed for such systems. Bounds are very useful in many cases. Even in not here, anywhere in any mathematical analysis that you do, bounds are always helps us to get a very generic result. You do not need to put too many assumptions on the model to get a bound that is general nature. So, because they provide always the best-case and worst-case scenarios, lower and upper bounds will give you those kinds of scenarios. Depending upon how you view it, it is one is the best case, the other is the worst-case scenario that is possible in such a situation. So, say, for example, in our case, with an upper bound in mind, suppose if you say the number in the system is this is the upper bound, right. Then accordingly, one can decide on the number of servers so that the congestion does not exceed a decided threshold. Suppose your objective is to keep the number of customers below a certain level equivalently in terms of the congestion probabilities. Typically, this will be specified in terms of those congestion probabilities. Then it will be below a certain threshold; then, one can think. The upper bound can give you the worst-case scenario, and based upon that; then you can decide the number of servers. So, then you are sure that it is never going to exceed based upon that assumptions.

And also that that is a way one would look. And also, approximations can be developed based on the bounds. For example, one very simple approximation that one can obtain is, taking the upper bound and lower bound, and you take a simple average of them. It is an approximation. There are various ways of how one can utilize these bounds to develop approximations. As we said that we are not going to look at the approximation but at least the bounds; we can derive it here. We consider here the upper and lower bounds for the mean delay in, mean line delay, or mean delay in queue for a typical customer in the steady-state $G/G/1$ queue as a function of first and second moments of it is interarrival and service times. Because it is $G/G/1$ queue, so what you know is the mean and variance of this interarrival time as well as service times. Now, based upon that, one can develop bounds; that is what we are going to look at it. Further, like we will also derive, we will not derive that we will, just state it the lower bound when full forms of the interarrival and service time distributions are known. But, somehow, this is not helpful to get the complete analysis. So, you want to use it at least in to try to derive the bounds. So, now, this as we set like can be used to derive approximation. So, that is what is the idea. So, we are going to look at the bounds for the mean line delay of steady-state $G/G/1$ queue is what is our interest, which means W q we are trying to get the bounds for that which in turn can give bounds for the others. That is a way in one would look at it. Before we step into obtaining bounds, there are certain basic relationships that exist for single-server queues, which help us to obtain the bounds. So, we look at several basic relationships. We are not looking at all in some sense; whatever we require to develop the bounds, only those quantities we are looking at it.

So, we are assuming a stationary $G/G/1$ queue with $\rho < 1$, which means a stable queuing system. Now, the relationship involves the interarrival times, service times, idle period, waiting times, and inter-departure times. So, inter departure times possibly we will not do anything, but at least it is also possible to obtain that.

- Recall the iterative equation for the delays is given by

$$W_q^{(n+1)} = \max(0, W_q^{(n)} + U^{(n)}),$$

  where $U^{(n)} = S^{(n)} - T^{(n)}$, $S^{(n)}$ is the service time of the n^{th} customer and $T^{(n)}$ is interarrival time between the n^{th} and $(n+1)^{st}$ customers.

- Now, let

$$X^{(n)} = -\min(0, W_q^{(n)} + U^{(n)}).$$

  This is the time between the departure of the n^{th} customer and the start of service of the $(n+1)^{st}$ customer. Then, from the above,

$$W_q^{(n+1)} - X^{(n)} = W_q^{(n)} + U^{(n)},$$

This is one of the important equations in our scheme of things. So, basically, the sum of these two or difference. So, I am taking this minus other said or whatever if you take it. So, $W_q^{(n+1)} - X^{(n)}$, would be, $W_q^{(n)} + U^{(n)}$. So, suppose now since this involves minimum or maximum of the same quantities, which are 0 and $W_q^{(n)} + U^{(n)}$. Now, if $W_q^{(n)} + U^{(n)}$ is positive, then $X^{(n)}$ value is going to be 0, and $W_q^{(n+1)}$ is given by $W_q^{(n)} + U^{(n)}$. That is what you will get here. Now, if this is negative, then the $-X^{(n)}$ is going to be $W_q^{(n)} + U^{(n)}$, and $W_q^{(n+1)}$ is going to take value 0. So, you can easily see from this maximum and minimum of this function by looking at the cases when $W_q^{(n)} + U^{(n)}$ is going to be strictly positive, $W_q^{(n+1)}$ is going to be equal to $W_q^{(n)} + U^{(n)}$, and $X^{(n)}$ is going to be 0. And if $W_q^{(n)} + U^{(n)}$ is negative, then this quantity $-X^{(n)}$ is going to be $W_q^{(n)} + U^{(n)}$, and $W_q^{(n)}$ is going to be 0. So, you can easily see. So, in both the cases, if you take the sum or difference whatever you want to consider, sum, I would say $W_q^{n+1} + (-X^{(n)})$ that is what you are calling a sum; otherwise, you take it as a difference. Whatever be; $W_q^{(n+1)} - X^{(n)} = W_q^{(n)} + U^{(n)}$ in both the cases. That is what you are seeing it here.

- For a stationary queue, $E[W_q^{(n+1)}] = E[W_q^{(n)}]$. Taking expectations of $W_q^{(n+1)} - X^{(n)} = W_q^{(n)} + U^{(n)}$,

$$E[X] = -E[U] = \frac{1}{\lambda} - \frac{1}{\mu}.$$

Since X is the time between a customer departure and the next start of service (in steady-state), we also have

$$E[X] = P\{\text{system found empty by an arrival}\} \cdot E[\text{length of idle period}] = a_0 E[I],$$

where $\{a_n\}$ are the arrival point probabilities and I denotes the length of an idle period. This gives

$$E[I] = \frac{E[X]}{a_0} = -\frac{E[U]}{a_0} = \frac{1/\lambda - 1/\mu}{a_0}.$$

So, this is what is the expected idle time is what then you are seeing it here. So, this is the first relationship that we are obtaining it here. The expected idle time in terms of the mean of interarrival and service time distribution and in terms of the arrival point probabilities.

- We now obtain a formula for the expected wait for a stable $G/G/1$ queue in terms of the first and second moments of U and I. Squaring both sides of $W_q^{(n+1)} - X^{(n)} = W_q^{(n)} + U^{(n)}$, we have

$$(W_q^{(n+1)})^2 - 2W_q^{(n+1)}X^{(n)} + (X^{(n)})^2 = (W_q^{(n)})^2 + 2W_q^{(n)}U^{(n)} + (U^{(n)})^2.$$

Note that $W_q^{(n+1)}X^{(n)} = 0$, and $W_q^{(n)}$ and $U^{(n)}$ are independent. Taking expectations of the above equation, and using these facts and $E[(W_q^{(n+1)})^2] = E[(W_q^{(n)})^2]$ for stationarity, gives

$$E[X^2] = 2W_q E[U] + E[U^2]$$

or

$$W_q = \frac{E[X^2] - E[U^2]}{2E[U]}$$

- Now, $E[X^2] = P\{\text{system found empty by an arrival}\} \cdot E[(\text{length of idle period})^2]$. Hence,

$$W_q = \frac{a_0 E[I^2] - E[U^2]}{2E[U]}.$$

- Using $E[U] = -a_0 E[I]$, we have

$$\boxed{W_q = -\frac{E[I^2]}{2E[I]} - \frac{E[U^2]}{2E[U]}.}$$

So, this is what is an expression, a relationship between the mean line delay in a $G/G/1$ queue, which you are expressing in terms of the random variables I and U; I is the idle period $U = S - T$, that is the difference between the service time and the inter-arrival time, is what then U its moments in terms of first and second moments of that you are writing this.

In the case of Poisson arrivals, the above equation reduces to the PK formula.

Now, like here, we just squared $W_q^{(n+1)} - X^{(n)} = W_q^{(n)} + U^{(n)}$ for expected value of the line delay (W_q). Now if we cube $W_q^{(n+1)} - X^{(n)} = W_q^{(n)} + U^{(n)}$ i.e. take the power of three of $W_q^{(n+1)} - X^{(n)} = W_q^{(n)} + U^{(n)}$ on both sides and do similar substitution and finally you get an expression for variance of the wait in queue. W_q is the mean of the wait in queue, variance of the wait in the queue also you can obtain, anyway we are not interested in that.

So, we are not going there. But it is possible to obtain that way. Now, once we have this relationship, we have derived, we have obtained. Now, one can also obtain some relationship for the inter departure time and so on. If required, you can do it; otherwise, for our scheme of things, that is not there. So, we just skip that. Now, what we will do? We will try to obtain some bounds for this single server queues, using these relationships, whatever we have established.

- We will apply some of these relationships to derive bounds that are valid for all stationary $G/G/1$ queues, with $\rho < 1$.
- We first derive a lower bound on the mean idle time. Recall that $E[I] = \frac{E[X]}{a_0} = -\frac{E[U]}{a_0} = \frac{1/\lambda - 1/\mu}{a_0}$. Since $a_0 \leq 1$, we have $E[I] \geq 1/\lambda - 1/\mu$, where equality is achieved for the $D/D/1$ queue.

An Upper Bound:

- We will use this inequality to derive an upper bound for W_q. By rewriting $W_q = -\frac{E[I^2]}{2E[I]} - \frac{E[U^2]}{2E[U]}$, we have

$$W_q = \frac{-(Var[I] + E^2[I])}{2E[I]} - \frac{E[U^2]}{2E[U]}.$$

Since $Var[I]$ is nonnegative,

$$W_q \leq \frac{-E^2[I]}{2E[I]} - \frac{E[U^2]}{2E[U]} = \frac{1}{2}\left(-E[I] - \frac{E[U^2]}{E[U]}\right) = \frac{1}{2}\left(\frac{E[U]}{a_0} - \frac{E[U^2]}{E[U]}\right).$$

- Since $E[U] = 1/\mu - 1/\lambda < 0$ and $a_0 \leq 1$, which implies that $E[U]/a_0 \leq E[U]$, we see that

$$W_q \leq \frac{1}{2}\left(E[U] - \frac{E[U^2]}{E[U]}\right) = \frac{1}{2}\left(\frac{E^2[U] - E[U^2]}{E[U]}\right) = \frac{1}{2}\left(\frac{-Var[U]}{E[U]}\right) = \frac{1}{2}\left(\frac{Var[S] + Var[T]}{1/\lambda - 1/\mu}\right),$$

which can be rewritten as

$$\boxed{W_q \leq \frac{\lambda(\sigma_A^2 + \sigma_B^2)}{2(1-\rho)}.}$$

So, this is what is an upper bound. So, $W_q \leq \frac{\lambda(\sigma_A^2+\sigma_B^2)}{2(1-\rho)}$ is an upper bound, which we have obtained now with respect to $G/G/1$ queue using the relationship and things we have obtained here. We obtain now an upper bound.

Now, we can see that in a similar fashion, we can also obtain a lower bound.

A Lower Bound:

- Recalling that $W_q = \frac{E[X^2] - E[U^2]}{2E[U]}$, we can bound W_q from below if we can find a lower bound for $E[X^2]$.
- We recognize from $X^{(n)} = -\min(0, W_q^{(n)} + U^{(n)})$ that the random variable X is stochastically smaller than T, since $X^{(n)}$ is either 0 or $T^{(n)} - (W_q^{(n)} + S^{(n)})$.

  Now, you see here $X^{(n)}$ and $T^{(n)}$ you compare. So, $X^{(n)}$ would be either 0 or $T^{(n)} - (W_q^{(n)} + S^{(n)})$ which is basically something you have subtracted from this particular T. So, that one can make us to recognize that this is the distribution; if I look at it now, there will be more mass towards the left side of the distribution of X as opposed to the distribution of T. For the same value of T, T will take exactly this value for the same value of T. But, this X will take either 0 or $T^{(n)} - (W_q^{(n)} + S^{(n)})$ because this cannot go beyond that less than that, or even if it is the case, that will still be true. But it is a nonnegative quantity, so we are looking at this quantity. So, this will be something we have subtracted. So, the same mass or same probability would be distributed somewhere towards left of the point at which this $T^{(n)}$ takes the value. That is the understanding of the stochastically smaller when you compare two random variables.

■ Definition: X is stochastically smaller than T if $P\{X \leq x\} \geq P\{T \leq x\}$ for all x, and this is written as $X \leq_{st} T$.

So this is also called as stochastic ordering or stochastic comparisons in that literature. So, when we say simply stochastically smaller means, we mean this. What does that mean is $P\{X \leq x\} \geq P\{T \leq x\}$ for all x. So, if I draw a simple case, if I will have to look at, suppose if I look at the one distribution function, distribution function, suppose if I look at one distribution function is like this, then the other distribution function would be something like this it will go. So, this will not exceed here. So, that is what it would mean. So, here what you are seeing is this quantity; the blue line is what basically this quantity. For all x, when I pick it up, this blue line lies above this purple line. So, that is what you would mean when you say X is stochastically smaller than T in this particular case. This is what precisely we mean when we say that X is one variable is stochastically smaller than the other variable. So, in this particular case, this is what you are seeing. So, basically, you are observing that X is stochastically smaller than T. What that means is that because of this behavior, now the higher moments for this would be smaller than the higher moments of this quantity. That is the implication.

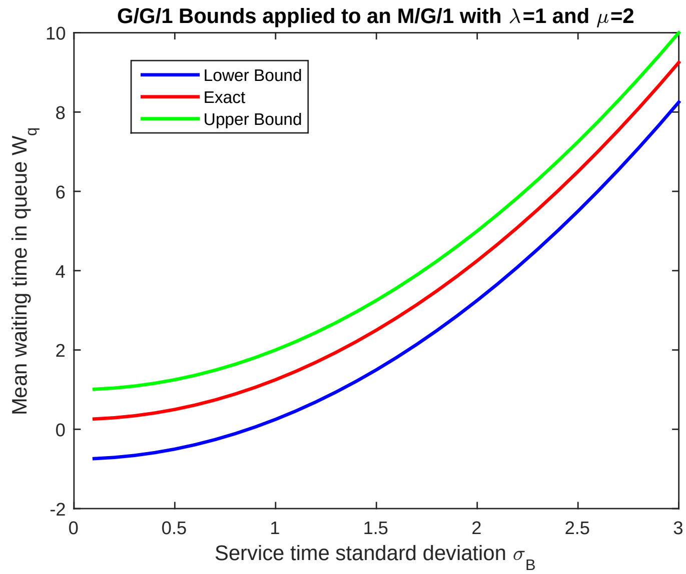
- It then follows that $E[X^2] \leq E[T^2]$. Thus from the equation $W_q = \dfrac{E[X^2] - E[U^2]}{2E[U]}$, we have

$$\begin{aligned} W_q &\geq \frac{E[T^2] - E[U^2]}{2E[U]} \\ &= \frac{Var[T] + E^2[T] - Var[U] - E^2[U]}{2E[U]} \\ &= \frac{Var[T] + E^2[T] - Var[T] - Var[S] - E^2[S-T]}{2E[U]} \\ &= \frac{1/\lambda^2 - \sigma_B^2 - 1/\lambda^2 - 1/\mu^2 + 2/(\mu\lambda)}{2(1/\mu - 1/\lambda)} \end{aligned}$$

Finally, this simplifies to

$$\boxed{W_q \geq \frac{\lambda^2\sigma_B^2 + \rho(\rho-2)}{2\lambda(1-\rho)}.}$$

This lower bound is positive iff $\sigma_B^2 > \dfrac{(2-\rho)}{\lambda\mu}$, and is thus not always of value (but quite useful in many situations).



Now, if I take an $M/G/1$ queue with $\lambda = 1/$ and $\mu = 2$, which means $\rho = 1/2$, and if I plot these two quantities and this is what it would look like in a typical way. What you are seeing this green line is the upper bound, and this blue line is the lower bound, and this red is what is the exact value because in this particular case, by PK mean value formula, you can obtain the exact values in this particular case. $M/G/1$ with $\lambda = 1$, which are Poisson process arrival, but only service time, anyway, service time variability is involved in the PK mean value formula. So, we can use to get the exact value. Now, you see how these values are looking like. All of them are quadratic in the variance of service time distribution which is the standard deviation which is σ_B is, what then you are seeing it here. So, this is what you are observing. Now, this should immediately give a clue that suppose if I, even in this particular case, whether I take a simple average of this lower and upper bound, that should give me a good enough approximation for the actual value. What I mean to say suppose this is the actual value. Now, if I take this and this and the average of these two points, I will be somewhere closer to this. So, that might give you a good approximation. Of course, one has to study that for $\rho = 1/2$ for which values of ρ this and like that, but one can develop approximations which might hold in certain situations and under certain conditions that would be a quite good approximation. Then one can measure how good is the approximation and so on. But, visually, at least from this figure, you can easily see that the simple average itself for such ρ's in such situations when this G distribution is so complex that you are not able to do with business with $M/G/1$ or even this M is gone, and then you are looking at $G/G/1$ even then case one can look at this kind of thing. So, one can take a simple approximation; one can think.

At least it gives an idea that one can think along those lines to get this. Bounds help in that way. It not just gives the worst case and best case scenario with respect to the quantities of your interest, but it can also help you to get going with construction of or the creation of appropriate approximation techniques or certain heuristics if you are further using this in some optimization and so on, it will be of quite help when you are looking at the bound. And this is what you see with this simple lower and upper bound. But, lower bound, we have certain reservations, but there could be another lower bound that can also be obtained.

- Let it now be assumed that the interarrival and service distributions are both known and given by $A(t)$ and $B(t)$, respectively.

But, since you might ask, suppose if this is the case, then I can use more information or if suppose if A is supposed to be Poisson suppose if you assume that then I can use $M/G/1$ techniques and so on, one can, or even with some simple $A(t)$, $B(t)$ one can do a similar Lindley's approach also one can do. But, if it becomes too complex to do that, so then you are left with no choice but to look for certain approximations, and that is what one can look at it. Now, if I know the distribution, whereas, in the previous ones, we did not use that, we used only the mean quantities and variance quantities of; even if you look at

$$\boxed{W_q \leq \frac{\lambda(\sigma_A^2 + \sigma_B^2)}{2(1-\rho)}.}$$

σ_A^2, σ_B^2, and mean of this A and B is what we have used to write to derive the lower and upper bounds in this particular case. But, here we have we have the complete distribution; then, whether one can get a better lower bound, yes, possibly.

- Then another lower bound on the stationary line wait may be found to be

$$W_q \geq r_0,$$

where r_0 is the unique nonnegative root when $\rho < 1$ of

$$f(z) = z - \int_{-z}^{\infty} [1 - U(t)]dt = 0,$$

where $U(t)$ is the CDF of $U = S - T$.

- We will not prove the above assertion here (you may refer elsewhere, for example, to the text for a proof).

But, what is important is; what is this r_0? r_0 is the solution of this equation or the root of $z - \int_{-z}^{\infty}[1-U(t)]dt = 0$ which can be shown to be a unique nonnegative root. One can take $f(0)$, which will be less than 0, and f of some large value it will be positive, so that means it must cross the axis at some point in time, one can think, and then you can look at greater than 0, less than 0, two different point. So, the in-between it must cross once, at least. So, one can show all those, but this is what is the bound. So, what is the bound here? $W_q \geq r_0$. Here r_0 will be nonnegative root. So, it will never go below 0. So, this works out to be a better bound in many cases, as you will see here.

- Finally, putting the upper and lower bounds together gives

$$\boxed{\max\left(0, r_0, \frac{\lambda^2\sigma_B^2 + \rho(\rho-2)}{2\lambda(1-\rho)}\right) \leq W_q \leq \frac{\lambda(\sigma_A^2 + \sigma_B^2)}{2(1-\rho)}.}$$

Now, once we have the bounds for W_q, appropriately, one can get bounds for L_q and other quantities and so on.

Illustration with M/M/1 Case

- We now apply the bounds to the $M/M/1$ queue. Recall that we have obtained the CDF of U as

$$U(t) = \begin{cases} \dfrac{\mu e^{\lambda t}}{\lambda + \mu} & (t < 0) \\ 1 - \dfrac{\lambda e^{-\mu t}}{\lambda + \mu} & (t \geq 0). \end{cases}$$

The lower bound r_0 is then found by solving $f(z) = 0$ for its unique nonnegative root; that is,

$$0 = r_0 - \int_{-r_0}^{\infty} [1 - U(t)]dt = r_0 - \int_{r_0}^{0} \left(1 - \frac{\mu e^{\lambda t}}{\lambda + \mu}\right) dt - \int_0^{\infty} \frac{\lambda e^{-\mu t}}{\lambda + \mu} dt$$
$$= r_0 - r_0 + \frac{\mu(1 - e^{-r_0 \lambda})}{\lambda(\lambda + \mu)} - \frac{\lambda}{\mu(\lambda + \mu)} = \frac{\mu^2 - \lambda^2 - \mu^2 e^{-\lambda r_0}}{\lambda\mu(\lambda + \mu)}$$

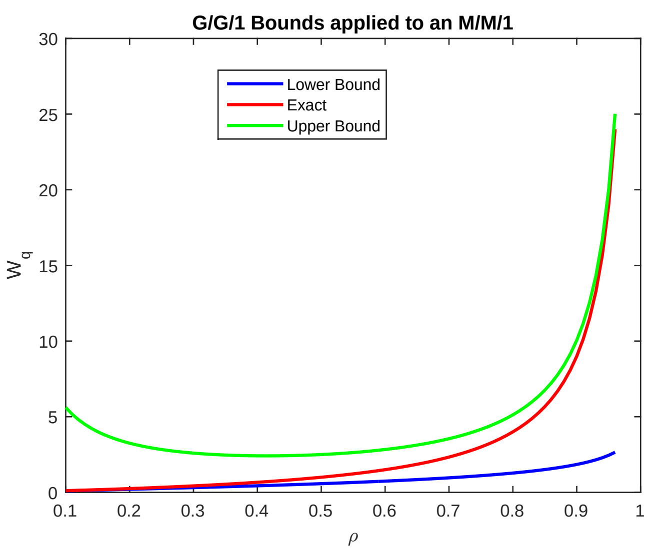
So, $\mu^2 - \lambda^2 = \mu^2 e^{-\lambda r_0}$, or $1 - \rho^2 = e^{-\lambda r_0}$. Hence finally

$$r_0 = -\frac{1}{\lambda} \ln(1 - \rho^2).$$

The upper bound for the $M/M/1$ queue is

$$\frac{\lambda(1/\lambda^2 + 1/\mu^2)}{2(1-\rho)} = \frac{1 + \rho^2}{2\lambda(1-\rho)}.$$

You can compute the other lower bound for this $M/M/1$ case and see what happens there. So, that is for you to explore. It can be very easy because once you assume $M/M/1$, what is mean and variance, so you just put it in the other bound and see what other bound means $\frac{\lambda^2\sigma_B^2 + \rho(\rho-2)}{2\lambda(1-\rho)}$ and see what happens here, with this $(\rho - 2)$ terms and all what happens there. So, this is the upper bound, $-\frac{1}{\lambda}\ln(1-\rho^2)$ is the lower bound, and $\frac{1+\rho^2}{2\lambda(1-\rho)}$ is the upper bound for this $M/M/1$ case. So, if I plot this, what you would observe is this kind of things.



- Both r_0 and the upper bound go to ∞ as ρ goes to 1.
- Indeed, this asymptotic behaviour is true of the upper bound for all $G/G/1$ queues in the sense that the bound always gets asymptotically sharper.
- In turn, the lower bound gets sharper as ρ goes to zero.

So, this is what you would see if you apply this particular bound, and this is why generic observations that one can make. Now, the point is that as you get more and more information about the inter-arrival distribution or the service type distribution, one can get better bounds. I mean, that is how. You; the other bound if you are computed could have been meaningless in this particular case, but at least this blue line here it has given up to this good approximation or good close enough value to the exact value. That is what you are seeing it here.

Let us take this simple example.

Example.
Consider a $G/G/1$ system with empirical distributions.

The interarrival time is 10 minutes with probability $2/5$ and 15 minutes with probability $3/5$.

The service time is 9 minutes with probability $2/3$ and 12 minutes with probability $1/3$.

We then get $\sigma_A^2 = 6$ and $\sigma_B^2 = 2$ (both in square minutes), with $\lambda = 1/13, \mu = 1/10$ and $\rho = 10/13$.

The upper bound is obtained as

$$W_q \leq \frac{(1/13)(8)}{(2)(3/13)} = \frac{4}{3} \textit{ minutes}.$$

To find the lower bound, we have to find the nonnegative root of the nonlinear equation

$$f(r_0) = 0 = r_0 - \int_{-r_0}^{\infty} [1 - U(t)]dt.$$

Example.

The random variable $U = S - T$ is given by

$$U = \begin{cases} -6 \ (= 9 - 15), & \textit{with probability } 2/5 \ (= (2/3)(3/5)) \\ -3, & \textit{with probability } 1/5 \\ -1, & \textit{with probability } 4/15 \\ 2, & \textit{with probability } 2/15 \end{cases}$$

and its CCDF is given by

$$1 - U(t) = \begin{cases} 1, & t < -6 \\ 3/5, & -6 \leq t < -3 \\ 2/5, & -3 \leq t < -1 \\ 2/15, & -1 \leq t < 2 \\ 0, & t \geq 2. \end{cases}$$

Example.

Hence,

$$\int_{-r_0}^{\infty} [1 - U(t)]dt = \int_{-r_0}^{2} [1 - U(t)]dt = \begin{cases} \frac{2}{15} r_0 + \frac{4}{15}, & 0 \leq r_0 \leq 1 \\ \frac{2}{5} r_0, & 1 < r_0 \leq 3 \\ \frac{3}{5} r_0 - \frac{9}{15}, & 3 < r_0 \leq 6 \\ r_0 - 3, & r_0 > 6. \end{cases}$$

Since the upper bound is just over one, we surmise that the lower bound should probably be less than one. That is, it is the solution to $r_0 = \dfrac{2}{15} r_0 + \dfrac{4}{15} \Rightarrow r_0 = \dfrac{4}{13}$, which must be correct, since r_0 is the unique nonnegative solution.

Therefore,

$$\frac{4}{13} = 0.3077 \textit{ minutes } \leq W_q \leq \frac{4}{3} = 1.3333 \textit{ minutes}.$$

It is possible (after some amount of calculations) to obtain the exact value of W_q here and is given by $W_q = 0.37724$.

So, this is what we have for our $G/G/1$ queue. What we have done is that we have considered in general, and we have can try to do how one can analyze. But that is a very complicated theory. Complex means it involves complex analysis and transform inversions, and so on. But, one can obtain certain bounds; in such situations, these bounds are quite useful in many of these practical situations because at the end of the day, no matter what do what you do and mathematical model is itself is an approximation of reality. So, you are putting one more layer of approximation to that to get these bounds, approximations, and so on. To develop an approximation, one can also do a best-case, worst-case scenario analysis in many of the situations where you have to make certain decisions with respect to the because it is a managerial decision that we have to take. So, then you can use these bounds in such situations. It has its own uses. Though sometimes the bounds may be off the mark encompass into exact, but when you are not able to get exact, but at least some idea you can get from where to where it can lie. So, in this particular case whether, what is going to be my W_q, whether it is going to be 2 minutes, 5 minutes, 7 minutes, I do not know, but at least I have some idea now that it lies between 0.31 and 1.33 minutes. Some idea, something is better than nothing. So, that is what you would obtain here. But, as we said, these bounds can be used to develop approximation but which is not in our scope. So, we leave it that. But, one can explore if you want to do that. So, this is all we have for our $G/G/1$ queues. And we stop here. Thank you. Bye.