Introduction to Queueing Theory

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Lecture - 08
Long-Term Behaviour of Markov Chains

Hi and hello, everyone. What we have been seeing so far is the "discrete-time Markov chain" and certain properties ok. We have seen some properties basically what we were interested in like you know, in one-step what happens or in some finite number of steps what happens, but more often; what we are interested when studying this Markov chain is what is called as the long term behaviour or the long-run behaviour of this Markov chain. So, what is the long-run behaviour of this Markov chain, which means that as time after a large amount of time into the future like what is going to be its behaviour, or when the system is in operation for a long time and in some stability is existing or not you want to analyze. And what is its behaviour in such a situation is what would be the general interest in any analysis that you do with the Markov chain theory. One way to characterize that is to look at $\lim_{n\to\infty} P^n$;you know P is the transition probabilities of one-step what is happening, P^n gives you the n-step transition probabilities. Now, n as $n \to \infty$; what is happening to this. So, this is what you want to look at it. Now, if you look at, say, for example: :

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} \begin{bmatrix} 1/5 & 4/5 \\ 2/3 & 1/3 \end{bmatrix}^n = \begin{bmatrix} 5/11 & 6/11 \\ 5/11 & 6/11 \end{bmatrix}.$$

So, this is a very nice behaviour that you are seeing with respect to this Markov chain; Markov chain, we mean like we are characterizing by its TPM. So, initial distribution, its effect also is gone, and you are, in some sense, there is some stability in the system. So, this is a very nice behaviour, but one has to understand that this kind of behaviour does not hold for all Markov chains. So, you want to see for what kind of Markov chains or under what conditions, or under what characteristic the Markov chain has such a behaviour. So, this is the typical behaviour that one would expect to have in a particular system when things are in a stable situation or in equilibrium, in other words. So, the long-term behaviour is related to, say, three concepts:

- Limiting distributions
- Stationary distributions
- Ergodicity

Now, what is a limiting distribution? So, you call a vector $\{\pi_i\}_{i\in S}$ is called the limiting distribution for a MC with transition probability matrix $P = ((p_{ij}))$ if $\pi_i = \lim_{n\to\infty} p_{ji}^{(n)},\ i, j \in S$ (provided the limits exist) and $\sum_{i\in S} \pi_i = 1$.

It may be possible that this limit exists, but not the limiting distribution, which means $\pi_i = \lim_{n\to\infty} p_{ji}^{(n)}$ might exist, but they may not sum to 1 for all $i \in S$. So, in that case, you do not call that a limiting distribution; the moment you use the word distribution, we mean that the sum is 1, limiting because you are looking at the limiting behaviour. So, in that case, you do not call that as limiting distribution; the moment you use the word distribution, we mean that $\sum_{i\in S}\pi_i = 1$, limiting because you are looking at the limiting behaviour. So, it is possible that the limiting probabilities would exist, but not the limiting distribution. So, by this way of defining what do we mean by a limiting distribution, whenever a limiting distribution exists, then it does not depend on the initial state. So, we can write $\pi_i = \lim_{n\to\infty} \pi_i^{(n)},\ i \in S.$. When you take the n-step state probabilities rather than the transition probabilities, and if I take its limit as $n \to \infty$, what I would be getting as the same π's. Because I call limiting distribution only if $\pi_i = \lim_{n\to\infty} p_{ji}^{(n)}$ is true when it does not depend on the initial state, so, that would be then equivalent to, looking at $\pi_i = \lim_{n\to\infty} \pi_i^{(n)}$. So, whether just you look at state probabilities or look at the transition probabilities, you are ending up with this π_i, and what is the interpretation in this sense that you can see that π_i is the probability of being in state i a long time from now. A long time into the future, if you want to ask at an arbitrary point of time, what is the probability that you know I will be seeing state i the probability is π_i. Now the question is because this is what you want to see like in a long time into the future like, what, where would be your Markov chain would be, what is the probability that I could find the Markov chain in a particular state is what is the limiting distribution gives you. Now, the question is, when does a Markov chain have a limiting distribution? In our case, limiting distribution means that it does not depend on the initial distribution.

And, if there is one exists how to determine it, of course, you can say that you know you raise compute this and do, but is there an alternative way where you know, you can compute this π_i in a much simpler way because this is not going to be an easier way of doing it; obtain these n-step transition probabilities and then take its limit is not going to be an easier way of course, though this is one of the ways you may be able to do for smaller ones, but in a more complex situation this may not be possible. So, what you do like, is there a way or how to determine it is what is the question that you want. Now, prior to that, what we call as a stationary distribution again, there is π_i; this may be the same as that one or otherwise. So, $\{\pi_i\}_{i\in S}$, and you will know why the same notation π_i we are using it for both a little later.

A vector $\{\pi_i\}_{i\in S}$ is called a stationary distribution (or invariant distribution) for a MC with transition probability matrix $P = ((p_{ij}))$ if $\pi_i \geq 0$ for all $i \in S$, $\sum_{i\in S}\pi_i = 1$ and $\sum_{j\in S}\pi_j p_{ji} = \pi_i$ for all $i \in S$.

So, this relationship can easily be derived using your Chapman-Kolmogorov equation by taking its limit as $n \to \infty$. You take n steps, and you break it up into what would happen in between $n-1$ step and the 1 step, and then you take the limit on both sides, take the limit inside the summation you will get to $\sum_{j\in S}\pi_j p_{ji} = \pi_i$. So, one can understand how this limiting and this stationary are related by that process.

Nevertheless, like what is our definition of the stationary distribution. It is a probability distribution that satisfies this $\sum_{j\in S}\pi_j p_{ji} = \pi_i$, and $\sum_{j\in S}\pi_j p_{ji} = \pi_i$, for all i you can put in matrix form as $\pi P = \pi$ is very important. So, you can think of this as $\pi P = 1\pi$ is that is the thing. So, this π is basically the left eigenvector corresponding to the eigenvalue 1, and since it is a distribution, $\sum_{i\in S}\pi = 1$ is what is written as $\pi e = 1$ where e is the vector of all 1's. So, a stationary distribution is a probability distribution such that it satisfies $\pi P = \pi$. Now, whenever a Markov chain, if it starts at this $P(X_0 = i) = \pi_i$ if the initial distribution of this Markov chain is equal to $\{\pi_i\}_{i\in S}$ for the i^{th} state, then you can show easily that $P(X_n = i)$ would also be exactly equal to π_i for all n and for all $i \in S$ and such a scenario we will say that Markov chain is stationary or this is the stationary version of Markov chain. Now, you know like why we call this a stationary distribution because if you start the process at this initial if you start, the process at this distribution means that initial distribution; if you take it to be a stationary distribution, then the state probabilities $P(X_n = i) = \pi_i$ from then onwards would also be given by at every step would be given by exactly by the stationary distribution. So, and hence the Markov chain is stationary.

So, if S is finite, then a stationary distribution would always exist. In general, a stationary distribution may not exist, and even if it exists, it may not be unique; say, for example, you can look at the simple random walk case what happens with respect to a stationary distribution. So, why do we care if our Markov chain is stationary or not. If it were stationary and we know what the distribution of each of these X_n was, then we would know a lot because we would know what is the meaning of this long-run proportion of time that the Markov chain was in any state. So, this π_i basically then gives you the long-run proportion of time that the Markov chain spends in that particular state is what is given by this π_i. So, how does this interpretation come we will see in a moment, but that is what is the interpretation whenever this exists. The stationary distribution exists, one can see. Hence solving for π is an important part of the Markov chain analysis, and we can also relate this to the limiting distribution; we will see that in a moment. So, there is a reason why you want to look at this or compute the stationary probabilities, which is very simple; you have this P, you solve this system of equations $\pi P = \pi$ and $\pi e = 1$. Now, suppose assume that if this P is finite, then $\pi P = \pi$ and $\pi e = 1$ is a system of n plus 1 equations in n, one of them is redundant you can throw away any of these equations from this $\pi P = \pi$ any one of them you can remove, and you can replace with $\pi e = 1$, and you can solve this to get this π that is a typical way of doing it. One can get that to get the stationary distribution.

Now, let us tie these two things limiting distribution and stationary distribution based on the properties that the Markov chain may exhibit. So, this is the main result in that connection you can see. So, of course, this particular slide has lots of content; please, try to go through it leisurely. So, for an irreducible Markov chain, a stationary distribution π exists if and only if all states are positive recurrent. So, if you have an irreducible Markov chain, then the existence of stationary distribution π is equivalent to all the states being positive recurrent. So, in this case, the stationary distribution is also unique, and it is given by $\pi_i = 1/M_{ii}$, where M_{ii} is the mean recurrence time to state i. Further, if the chain is aperiodic, then the limiting distribution limiting probability distribution exists, and this equals the stationary distribution, which is also, again, a unique thing that you have here. So, this is what is the main theorem in connecting these two and connecting the concept of stationary distribution, limiting distribution to the properties of the Markov chain; why we classified or looked at the properties of the states of the Markov chain; the reason is basically in order to characterize them and connect them to such kind of quantities. So, if you see here, if it is an irreducible Markov chain, that is what you know we are taking it here. A stationary distribution exists if all states are positive recurrent if somehow you show that the state is irreducible. So, you take any one state, and if you can show that it is a positive recurrent, then you know for sure that a stationary distribution exists, unique, and it is given by $\pi_i = 1/M_{ii}$; what is a stationary distribution it is a solution to that equation $\pi = \pi P$ and $\pi e = 1$, $\pi \geq 0$, it is a probability distribution. So, that π_i would be the same as the π_i, which is connected to the mean recurrence time to state i because whenever you have positive recurrent, M_{ii} is finite; that is what you know positive recurrent means. So, then $1/M_{ii}$ will be something that is strictly greater than 0, and this will be the solution to that. So, that means that pi_i now can be obtained in two different ways. If you know the mean recurrence time, you can get π_i's, or you can solve this π_i from that stationary equation which is called $\pi_i = \pi P$. Further, in such a situation, if the chain is also aperiodic, then the limiting probability distribution exists, and it is equal to the stationary distribution. So, now, we answer the question, if there is a limiting probability distribution, how do we get it. So, the limiting probability distribution exists under aperiodic condition, and how do we get it; we get it as a solution to the stationary equation. So, if you find the stationary distribution, which in this particular case is unique, so, that is the limiting probability distribution as well.

So, these are all the one and the same; if you have an irreducible, aperiodic, positive recurrent Markov chain and the limiting distribution is the same as the stationary distribution and which is a solution to the stationary equation, and it is a unique solution, everything is coming from here. So, this is the ideal situation that one would want to have here. Now we can also see we cannot make a transient or null recurrent Markov chain stationary; that means that if there is no stationary distribution, then the Markov chain is either transient or recurrent and in such case, this $\pi_i = 0$ for all $i \in S$; that means if there is no stationary distribution, you will see examples where the different variations are there. So, if the chain is positive recurrent, then only you have a unique stationary distribution which is also the same as the limiting distribution. If the chain is not positive recurrent or if it is either transient or null recurrent, then there is no stationary distribution; that is what you know you will have here. Now, if the Markov chain is reducible, what will happen? So, the irreducibility component if it is removed, if it is reducible, then the stationary distribution may not be unique; that is what will happen, and no conditions on the period for the Markov chain for the existence and or uniqueness of the stationary distribution. You see, here, this first part of the theorem talks about the stationary distribution where we are not talking about the periodicity of the states. It is irreducible, so all the states have the same period, but we are not talking about periodicity for the existence and uniqueness of stationary distribution; you do not need the periodicity concepts. But it is not true with the limiting probabilities for limiting probabilities to exist; the periodicity is relevant you will see in an example. So, the limiting distribution of a Markov chain is also a stationary distribution; limiting distribution, whenever it exists, is a stationary distribution, and in a reverse way, the existence of a stationary distribution does not imply the existence of a limiting distribution. You might see an example where this would be the case later. And if it can also be shown, as you know given here in this cases, that if the limiting distribution exists, then it is the only stationary distribution because limiting distribution limit you know it it has to exist means it is a unique value. So that means, if that is the stationary distribution, then the stationary distribution is also unique, and that is the only stationary distribution that you will have here. So, this is tying this.

So, this theorem now, like the third notion, is connected with the long-run behaviour, which is what we said is **ergodicity**. Now we say a state is said to be ergodic if it is aperiodic and positive recurrent, then we say this is an ergodic state, and if all states of a Markov chain are ergodic, then the Markov chain itself is said to be ergodic, and in such a case as this theorem, shows here this has a unique limiting stationary distribution, it has a limiting distribution which is unique of course then, that is the only stationary distribution, and that is what we call it as an ergodic. So, for that reason, this kind of theorem that all variations of this are called ergodic theorem, which gives the condition for ergodicity of the Markov chain, but one thing I would point out here is that the way we have said about the ergodicity of a Markov chain or ergodicity of state is in this way as far as we are concerned, but there is a slightly less restrictive notion of ergodicity, and that is actually the complete or full notion of ergodicity, but for our purpose, we will confine ourselves to be in this form. So, the aperiodic positive recurrent state is ergodic. In general, this aperiodicity need not be insisted on. Still, you can define a notion of ergodicity, and that is what one does in a complete study of the Markov chain. However, for our purpose, we mean an ergodic state means, in this case, aperiodic and positive recurrent, and in that case, we will call this as ergodicity because our aim is basically to obtain this unique limiting stationary distribution. So, we want to study the long-run behaviour of the Markov chain, which is characterized by the limiting probabilities. Now the limiting probabilities are obtained through a solution of the stationary distribution and which will have a unique solution. So, we are looking for conditions under which that can happen; these are the conditions: Irreducible Markov chain, positive recurrent, aperiodic then you will have whatever the nice thing that you are looking for, which is the unique limiting stationary distribution. So, this is the ergodic theorem as far as we are concerned in this scenario. Now, let us look at a certain example and how you will use this theorem in this example, and that is the way we are going to use this theorem. This is the main theorem that will be used throughout.

Example 1.

Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. If you see a truck pass by on the road, on average how many vehicles pass before you see another truck?

Let $\{X_n\}$ be a MC with $S = \{0, 1\}$ (0-truck, 1-car) and with $P = \begin{bmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{bmatrix}$. The unique stationary distribution is $\pi_0 = 4/19$ and $\pi_1 = 15/19$.

If you see a truck pass by then the average number of vehicles that pass by before you see another truck corresponds to the mean recurrence time to state 0, given that you are currently in state 0. By the above, the mean recurrence time to state 0 is $M_{00} = 1/\pi_0 = 19/4$, which is roughly 5 vehicles.

So, you can answer such questions using such modelling and analysis; that is what one does.

Example 2.

Now let us take another Markov chain which is given by

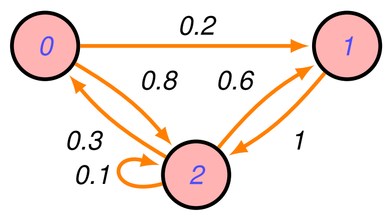
$$P = \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0 & 0 & 1 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}$$

Irreducible, positive recurrent, aperiodic.

$\boldsymbol{\pi} P = \boldsymbol{\pi}, \boldsymbol{\pi}\, \boldsymbol{e} = 1$

$\Rightarrow$ *unique* $\boldsymbol{\pi} = (0.153, 0.337, 0.510)$

Limiting distribution equals $\boldsymbol{\pi}$.



Example 3.

Now look at this example.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Reducible MC

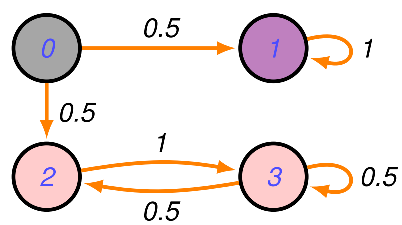
3 communication classes $\{0\}$, $\{1\}$ and $\{2, 3\}$

State 0 is transient, states $1, 2, 3$ are positive recurrent

$\boldsymbol{\pi} P = \boldsymbol{\pi}, \boldsymbol{\pi}\, \boldsymbol{e} = 1$

$\Rightarrow$ *Solution exists but not unique*

$\boldsymbol{\pi} = \alpha(0, 1, 0, 0) + (1 - \alpha)(0, 0, 1/3, 2/3), 0 \le \alpha \le 1$

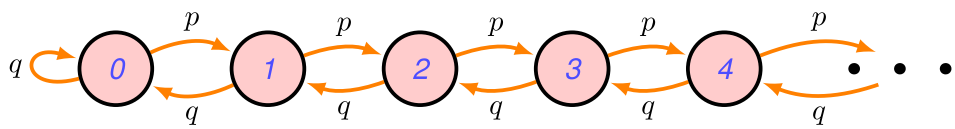


No limiting distribution; limiting probabilities $\lim_{n\to\infty} p_{ij}^{(n)}$ exists, but depends on the starting state as observed from

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0 & 1/2 & 1/6 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}$$

Look at this example this typical this is a typical example that is going to be helpful for us because this is the way we are dealing with this is how we are going to use this ergodic theorem in our analysis.

Example 4.



With $0 < p < 1$ and $q = 1 - p$, MC is irreducible and aperiodic.
If MC is positive recurrent, then a unique stationary distribution exists and equals the limiting distribution.
If MC is null recurrent or transient, then there is no stationary distribution (and hence no limiting distribution).
Try solving $\boldsymbol{\pi} P = \boldsymbol{\pi}, \boldsymbol{\pi}\, \boldsymbol{e} = 1$

First part gives $\pi_n = \left(\frac{p}{q}\right)^n \pi_0, n \geq 1$ and the normalization condition implies that

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \sum_{n=0}^{\infty} \left(\frac{p}{q}\right)^n.$$

If $p < q$, then the geometric series converges and $\pi_0 > 0$ (and hence $\pi_n > 0$) and we have a solution to the stationary equations.

▶ *MC is positive recurrent if $p < q$.*

If $p \geq q$, then the geometric series does not converge and $\pi_0 = 0$ (and hence $\pi_n = 0$) and there is no stationary distribution.

▶ *MC is not positive recurrent if $p \geq q$.*

So, and hence in the long run, what you would find the long-run proportion of time then would become zero; that is what if that is the interpretation that you are taking it, again, this $\pi P = \pi$, $\pi e = 1$ solution even though you are getting I mean as a unique solution. For example, in this particular case, when $p < q$, it has two interpretations again: One is the long-run fraction of time that the process spends in that particular state. The other is; what is the probability that a long time from now you would find the process in that particular state, these are the two interpretations. The second one is based upon limiting distribution; the first one is depending upon that it is $\pi_i = 1/M_{ii}$ because of that kind of thing that you have here. So, this is not positive recurrent if $p \geq q$. So, our interest. So, we do not want to see whether this particular thing either is null recurrent, or transient, so we do not distinguish between this for our purposes, but if one wants to analyze further then, you can further look at when this will be null recurrent when this is transient. So, that is it is always possible to do, but our interest is mainly to obtain a stationary distribution, a limiting stationary distribution. So, that would require only positive recurrent. So, we will look at only whether the chain is positive recurrent or not whenever you have irreducibility and aperiodicity is ensured in this case.

So, now, you see here how we are using the theorem; we are not proving that the chain is positive recurrent and hence it will have a unique limiting stationary distribution, and we are finding $\pi P = \pi$. We directly try solving $\pi P = \pi, \pi e = 1$, and whatever condition is required to get this distribution, the solution to the stationary distributions is a unique distribution that is the condition for positive recurrent. So, you can also show now that or otherwise if even if you are not taking this route, if you have to show this positive recurrence of this particular chain, it will be possible only if $p < q$. So, under this condition, only this chain will be positive recurrent. So, this is the condition for stability of this chain, or this is the condition for the positive recurrence of this chain. So, we are trying to use this theorem in order to arrive at this; we will be using it in that way only like when we are dealing with the queuing system typically. We will try to solve this and whatever condition is required to be put on the parameters so that we have a unique stationary distribution case, and that is the condition for the positive recurrence, and under that condition, the system is going to be stable, and that is how we are going to infer.

Example 5.
Now, look at this example what happens here.
$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
MC is irreducible and positive recurrent.
Has a unique stationary distribution $\pi_0 = \pi_1 = 1/2$ *(Ths system spends half of the time in each state).*
What about the limiting distribution?

$$P^n \textit{ does not converge as } P^n = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, n \textit{ even,} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, n \textit{ odd.} \end{cases}$$

No limiting distribution [Note: MC has period 2].

But, if we choose the initial distribution as $\boldsymbol{\pi}^{(0)} = (1/2, 1/2)$, then $\boldsymbol{\pi}^{(n)} = (1/2, 1/2)$ for all n.
i.e., Even though $\lim_{n\to\infty} P^n$ does not exist, it is possible for $\lim_{n\to\infty} \boldsymbol{\pi}^{(n)}$ to exist, but only if the starting state is chosen randomly according to the stationary distribution.

So, now how the recurrence becomes an important idea, a concept to be determined in a Markov chain. So, how to determine the presence of recurrence in a Markov chain. There are many results available in the literature one sufficient condition that we will use at some point of time in our analysis is what we have given here, which we are treated as a result which is the following.

Result: An irreducible, aperiodic chain is positive recurrent if there exists a nonnegative solution of the system

$$\sum_{j=0}^{\infty} p_{ij} x_j \le x_i - 1 \ \ (i \neq 0) \quad \text{such that} \quad \sum_{j=0}^{\infty} p_{0j} x_j < \infty.$$

Like, there are different varieties of conditions, various conditions depending upon, various nature of the systems and equations, and so on, which can be given in order to show the positive recurrence of the chain. This result we are stating because this is what we will be using it later on when we are dealing with at least one or two queuing models at some point of time, right, I mean little bit the semi Markovian set up or in beyond and around that type.

Now, $\pi = \pi P$ has also an interpretation which is also quite useful as far as we are concerned when we are trying to write down the equations for describing the Markov chains. We said that it could be depicted; Markov chain can be depicted through the transition state diagram, but if you have to do analysis, you have to write down this equation $\pi = \pi P$ so; that means, essentially P matrix you have to specify. So, that equation can be given interpretation. So that one can write down that equation in a nice easier way.

Look at the scenario; what is the meaning of this π_i. So, what is this π_i? This π_i is the long-run proportion of time the process spends in particular state i; since every time period spent in a particular state i corresponds to a transition into state i, we can also interpret π_i as the long-run proportion of transitions that go into state i, this exact same thing you can give for the out of state. Meaning since every time spent in state i correspond to a transition out of state i because you can always; whether your time is spent here in a particular state, whether you are looking at this point or this point like depending upon that, so, the long-run proportion of time the time spent every time period spent in state i corresponds to a transition out of state i. We can also interpret pi as the long-run proportion of transitions that go out of state i. Now, when whenever it is going out since p_{ij} is the probability that this of going from state i, so given that you are in state i, p_{ij} is the probability that you will move into state j.

So, the proportion $\pi_i p_{ij}$ is the long-run proportion of transitions that go from state i to state j; now, if you think of the transition from state i to state j as a unit of the flow of something, the flow from state i to state j, then $\pi_i p_{ij}$ would be the rate of flow from state i to j. If the transition is a flow, then multiplying by this π_i would be the rate of flow from state i to j. Similarly, with this interpretation, we can interpret this π_j as the rate of flow out of state j, and if this is $\pi_i p_{ij}$ is the rate of flow from state i to state j, now if I sum over all i then $\sum_{i \in S} \pi_i p_{ij} =$ "rate of flow into state j". $\pi_j =$ "rate of flow out of state j". So, what you end up is having this equation $\pi = \pi P$ has the interpretation that the "rate of flow into state j" $=$ the "rate of flow out of state j" for every $j \in S$.

So, the meaning when we say that the stationary distribution is that the vector the stationary distribution vector achieves the balance of flow, and hence the system is in equilibrium or the system is in stable condition, the stability of the system. That is why sometimes this is also referred to as equilibrium condition or equilibrium probabilities because this is what ensures that, of course, intuitively, if you think that is what it usually happens like when the system is in equilibrium means that the rate of flow into or suppose if you think about a queueing system that, the rate of flow in and the rate of flow out of the system they need to be at equilibrium in order that the system is in equilibrium. If there is an imbalance here, then the system will move away from one of the equilibrium points. So, that is why $\boldsymbol{\pi} = \boldsymbol{\pi} P$ are called "Balance Equations" or, rather more specifically, "Global Balance Equations," but we might simply call these balance equations. So, all stationary distributions must create global balance because, by the name of it, it must create the global balance equations. We will simply call balance equations because we will be dealing mostly with global balance equations. Why the word global? Because there is another concept called "Local Balance Equations," which is if the stationary probabilities $\boldsymbol{\pi}$ also satisfy $\pi_i p_{ij} = \pi_j p_{ji}$, for every $i, j \in S$, the rate of flow between two states now you are looking at it, not for the whole chain. So, the rate of flow from i to j if it is equal to the rate of flow from j to i, then you see there is a local balance now; these locally two states are in a balanced state; that means that globally also they will be in the balance state because any two states are in balance and hence it would imply. How do you see that you take $\pi_i p_{ij} = \pi_j p_{ji}$, now you sum over i on both sides, what you would get is exactly the global balance equations. So, if you can find a vector $\boldsymbol{\pi}$ that satisfies local balance, then people also satisfy global balance because this implies not the other way around.

Not every Markov chain would be locally balanced, but they will be globally balanced. So, the local balance equations, whenever they are existing is much simpler to solve than the global balance equations, but we will mostly consider the global balance equations, and we will work on that rather than the local balance equation because then we do not need to worry whether the particular Markov chain has local balance and to know what the local balance idea to be whether it is it will be there or not like we need to bring in some more concepts. Like for example, it is tied to the concept of what is known in the Markov chain as the reversibility concept. We do not need to look at that. So, we will always work with global balance equations, and from there, automatically, things will follow. So, that is about the interpretation of $\pi = \pi P$, which you will use. Of course, when we do continuous time, it will be more clear at this point of time, we do not need to pay too much for that. But just understand that there is a balancing concept, that is what the stationary distribution would do, and that is why the system is in equilibrium you are looking at. Now, the **memoryless property** we said that the time a Markov chain spends in a particular state is going to be memoryless, and hence it is geometrically distributed is what we are trying to see here. So, if for a Markov chain, if $p_{ii} = 0$, then the time the particular chain spends in a particular state i is equal to 1 because the next moment, it is going to go out of the state. Now for a MC with $p_{ii} > 0$, the number of time units that the system spends in the state i (also known as *sojourn time* or *waiting time* or *residence time*) is geometrically distributed. How you will see you can easily see from the following argument

- Assume that the MC has just entered the state i. It will remain in i at the next step with probability p_{ii} and it will leave this state at the next step with probability $1 - p_{ii}$.
- Independent of what happened in one step, similar property holds in the next step as well.
- Let $\tau_i = \min\{n \geq 1 : X_n \neq i\}$. Then, the distribution of the sojourn time in state i is
  $P_i\{\tau_i = n\} = (1 - p_{ii})p_{ii}^{n-1}, \quad n \geq 1.$

Now, there is one typical simple Markov chain that has some relation to queueing theory, queueing system, or which is a very simpler one which further theory can be developed though we are not going to do that is known as the birth-death chain.

- A birth-death chain is a special of a DTMC with $S = \{0, 1, 2, \dots\}$ and with TPM
  $$P = \begin{bmatrix} 1-b_0 & b_0 & 0 & 0 & \cdots \\ d_1 & 1-b_1-d_1 & b_1 & 0 & \cdots \\ 0 & d_2 & 1-b_2-d_2 & b_2 & \cdots \\ \vdots & & & & \ddots \end{bmatrix}$$
  where $b_i > 0, i \geq 0$ is the probability that a single birth will occur at the next time step, $d_i > 0, i \geq 1$ is the probability that at the next time step a single death will occur, and $1 - b_0, 1 - b_i - d_i, i \geq 1$ is the probability that the state will not change at the next time step.
  ► Transitions to nearest neighbour states only (multiple births and/or deaths not possible).
  ► Very useful model for queues and obtaining of solutions is easier because of the special structure (tridiagonal) of the TPM.

So, this is a very useful model for queues as well because when the queues are increasing or decreasing by one, this is precisely what would be the model for it. And because the special structure which has this what we call tridiagonal structure of this P. So, this can be analyzed nicely and where various birth and death rates or birth probability structures this can be analyzed very nicely. But more we will see with respect to continuous time rather than in discrete-time this birth-death process. So, we will see more details about similar ones in the continuous-time rather than in the discrete-time because most of our models are going to be in continuous time, and hence we will be concentrating more on the continuous-time Markov chain. But for the continuous-time Markov chain to understand that theory, you need to understand the theory of the discrete-time Markov chain, and hence we are doing this whole week we have spent on analyzing the discrete-time Markov chain. So, these are the basic concept that is needed for us to understand certain things in queueing analysis queueing models when you try to analyze that, and that is what we have done this week.

We will see the continuous-time Markov chain things in the following lectures. So, here we end with the analysis, or the ideas of discrete-time Markov chains; here.
Thank you, bye.