Convex Optimization

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Lecture No. # 12

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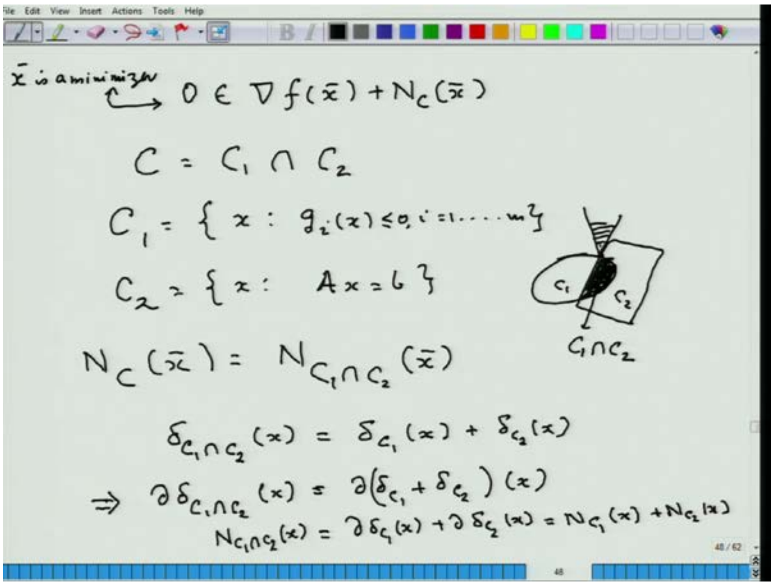

Good evening viewers, and once again we are on the journey of learning convex optimization. See if you remember what we did in the previous lecture was the following; is that we looked at a, we are trying to look into an optimality condition of this form. And we were also looking into a very general optimization problem, where you have both inequality constants as well as equality constants.

Now, we have shown the standard optimality condition can now be used expressing the sum rule used to the sum rule of sub differentials can be expressed in this form. So, once you can express it in this form, question is to compute this one and to compute this one. So, we had earlier computed this one, where you see there is a use of the separation theorem.

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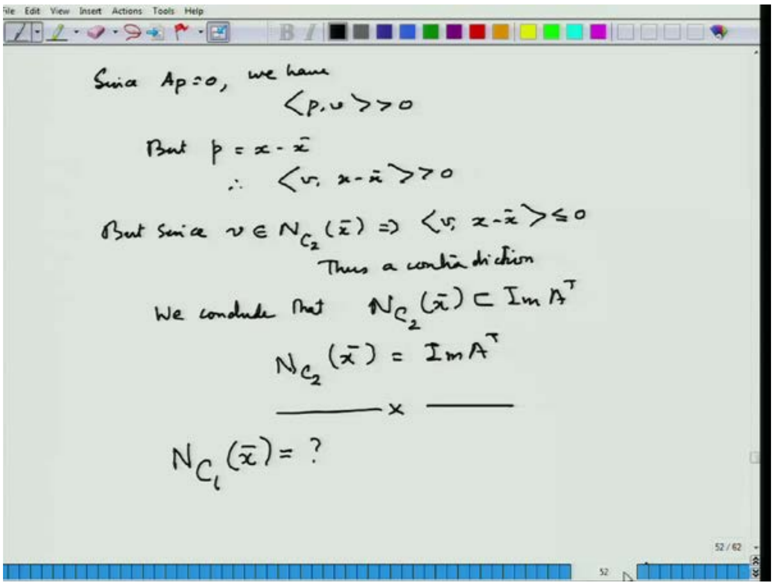

And then we are now going to compute the case for C 1 that is when now, when I have inequality constants what would happen. So, in this case we will need.

(No audio from 01:29 to 01:53) Here, we will need the status condition that is, there exist x hat such that g i of x hat is strictly less than 0 for all i.

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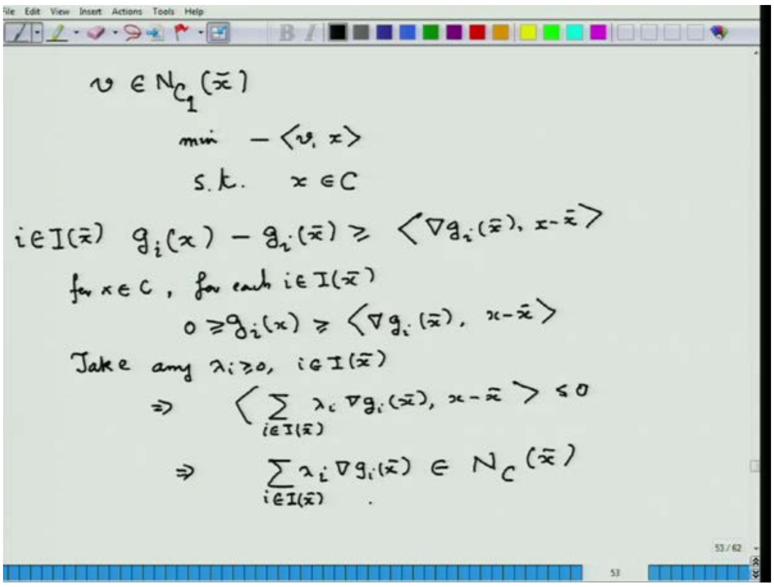

So now, how do I compute it? So, any v which is in this set sorry C 1. Essentially solves the following minimization problem, minimize minus of v, x such that x is element of C. But come on, we have we are essentially we want to find an optimality condition. So, we will so that, we cannot use an optimization problem itself to figure out, how to solve the, how to compute the normal cone.

So, it is very important, that we do it; without the help of optimization problems. So, let us see how can we do it, in a more straight forward fashion. Now, let us see how I would think, if I am possibly getting it as an open problem. You know it is very important that you think, as if this problem has been given to you, and you are trying to solve it. And that is the only way, one can imbibe; the spirit of doing research. And so, here were how do I think about it.

I have constraints and all of these are convex. Now, consider those i's which are in I x bar, that is in the active index set; that is the set of all is for which g i x bar is 0. Then convexity would give me, for each i this fact; this is something which we already know, it will give you this fact. And then, because i is in I x bar this is 0. And, so far x in C what I would have is that here for each i g i x is of course, less than equal to 0 and this is 0.

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Now, take any lambda i greater than equal to 0; i belonging to I x bar. So, that would imply, that I have multiply all of this equations by lambda all this inequalities by lambda i, then add them up; then I will have…

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Now, does this ring a bell. Of course, it does because x was arbitrary element in C. So, what I get is exactly something like a definition of the normal cone. So, what does it shows that, this thing this immediately implies, that summation lambda i…

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Is an element of the normal cone to C at x cone. So, I know; that any element when I where lambda is greater than equal to 0 is of this form, must belong to normal cone. So, the question that arises. Is that for this particular formation of C C C1, that is set of all x as g i is less than equal to 0, does every element v is of this form; that is I am asking the following question

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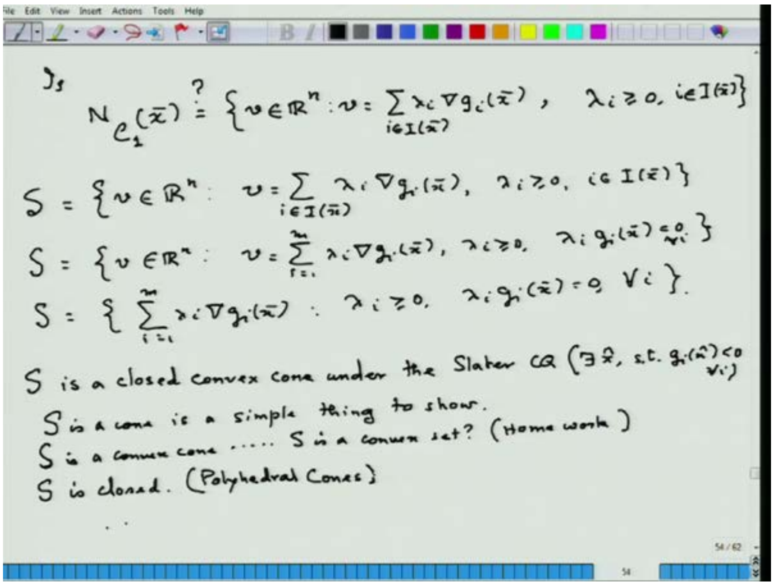

Is, the set of all v in R n for which v is expressed as…

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So, any v that you express like this, where you can vary the lambda is, my query is is this actual expression. I know that this thing is a subset of this thing, but is the opposite true that is the question. Now, the first steps are following. So, which will now do one by one; let me call this set S as, these are the S.

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Now, this set v can also be expressed in the following way, following equivalent way.

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Obviously, for all i. So, it means that either I can write it like this or I can free up this thing, and write the sum form one to m with the criteria. This sort of complimentary slackness criteria, whichever we want or moreover short hand is of this; that instead of putting this v you directly write elements of this form.

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Now, you know very well that this normal cone is a convex cone, and which is by definition which is that is true, is a close convex cone actually. Now, the question is whether this is also a closed convex cone. So, first step is to we will show that S, these are closed convex cone, under the Slater constraint qualification, which we have said earlier that, there exist x hat such that g i x hat is strictly less than 0 for all i. So, trying to figure out this question; now we will do it step by step. Now, when we want to do it step by step. Let us look into this fact, that it is a cone. S is a cone is a very simple proof.

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And what about its been convex. So, S is a convex set, would be given S in a homework. So, take two elements of this form, and show that there is convex combination is also in S. This is the convex cones, that is now you have to show that S is a convex set. Now, this should be homework, how to show that S is closed. This brings us to the question of polyhedral cones.

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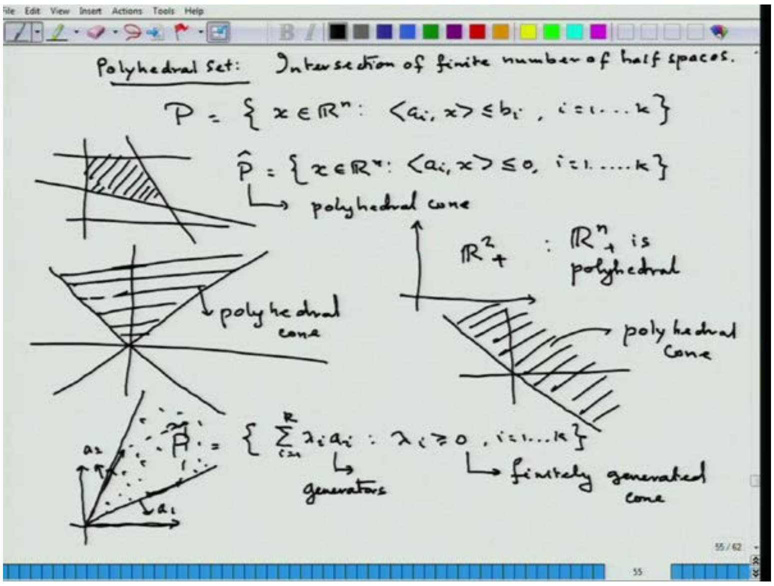

So, let me remind you, what is the polyhedral set? So, polyhedral set is a convex set, which can be represented as an intersection of finite number half spaces. So, this can be represented as an intersection.

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Now, (( )) polyhedral set usually would look like this; the polyhedral set P.

(No audio from 12:40 to 12:53)

So, it is something like this. You have half space here, you have half space here, you have half space here, it could be a set like this. This is a polyhedral set. Now, if I put all b i is equal to 0; that is if I had this P hat.

(No audio from 13:14 to 13:26)

This K can vary away straighter than means, that it is some fix number K. So, this is called a polyhedral cone; does all of such half spaces have 0 as one of the element. Now, I will take this one, and I take this one. Now, I just bother about this part, bother about this part. So, I just bother about something above this, above this whatever. So, I can get a polyhedral set, this is a polyhedral cone; this is the polyhedral cone.

So, example of polyhedral cone: important cones are to get the thing R 2 plus, it is this, and this. So, this half space, and this half space. So, R 2 plus an example of polyhedral cone, and thus R n plus is polyhedral. You can make out your own examples of polyhedral cones. Suppose you take, you know to just draw this line passing through the origin; and take this part, this is also a polyhedral cone.

Now, what is important is, that you should know this following fact; that polyhedral cone can be represented in a nice fashion. Now, there was another (( )) definition of a cone which is called a finitely generated cone, that is you have certain generators, that is you have some fixed vectors say a i, say 1 to K. And create all elements of this form. So, suppose I have taken two dimension, I take say two vectors. There is no x condition of linear independence, dependence or anything on this vector; I just few given vectors. So, I say suppose I take, these two vectors in R 2.

Now, any point satisfying this I multiply with a non-negative number; like this, this and add them up; it will come here. And so, any point here so, any any point of this form would be within these two boundaries. So, so given these two vectors which I can call a 1 and a 2, this is the finitely generated cone, generated by the generators a 1 and a 2. So, this a i's are called generators of the cone. And this cone is called finitely generated cone.

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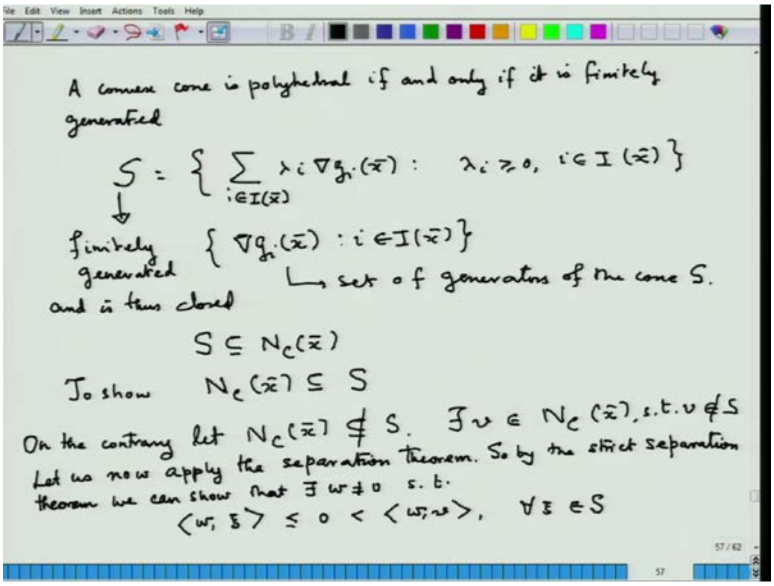

One of the most powerful results in convex analysis or convex geometry is the following, it says that a polyhedral cone or rather a polyhedral cone is finitely generated, and finitely generated cone is polyhedral. In another way of telling is is this, which is more where you write in the sense that, it is stating almost in if and only if terms. A convex cone is polyhedral, if and only if it is finitely generated generated.

(No audio from 17:50 to 18:05) Now, if you look at the set S, which is given in this way. (No audio from 18:20 to 18:32) This set is finitely generated set, where the set of generator are this gradient vectors. This is the set of generators, generators of the cone S.

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Now, if you go back, and look at the definition of polyhedral cone, this one; it is clear that ad polyhedral cone is closed. And so here, because this set is every finitely generated set is polyhedral, every finitely generated set is closed. So, this set S is finitely generated, through these generators, and is thus closed. So the three requirements, S is the cone is done which you can show, it is a convex cone which are homework and S is closed, because of this polyhedrality. The three things, we have done has been now completed. Now, our question is to now show this, whether this is equal to this. Now, how do we do this.

So, in order to this, we have to observe this fact that; what we have shown is that S is the subset of, and now what we have to show is that N c to show N c x bar is a subset of S. So, the idea is to start with the contradiction. So, this sort of arguments that I will be arguing now, is quite common in convex analysis and convex optimization. So, that is imperative that, you listen very very carefully to the arguments that I will go through. Even those one not from mathematical background, I will tell them to be very careful and try to argue, look at the argument carefully.

So, what you learn - what you are learning is modern optimization. And here analysis plays of very very fundamental role, because at the end, a lot of problems in analysis was developed; lot of issues in analysis or ideas in analysis were developed to take care of problems in optimization, and thus analysis plays a fundamental role in understanding optimization problem.

So, now let me say that; on the contrary let this not be true. That is I am telling you let me assume that, what we have what we have guessing is not true. Now, if this is not true, then what would happen. So, there exists of v in N c x bar means, every element in v is not element here; if it so, then it will be a subset. So, there must be v in N c x bar such that v does not belong to S. We have proved that S is close convex cone, and hence a closed convex set and thus, it is important for us to go back, and look at the separation theorems that we have learnt; and now, we are going to apply the separation theorems.

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So, application of the separation theorem, will lead to the following; and let us see how this is done. So, what we are going to show that, v is not in S. So, by the strict separation principle or strict separation theorem.

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We can show that, there exist w not equal to 0; such that, w of chi, chi is a element in this set is less than equal to 0 is strictly greater than.

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Of course, this S is depending on x bar, which I am not mentioning, because you know we have just taken a fixed x bar and we are working. So, S of course, depends on x bar its very may be is more perfect to put S of x bar. So, I am just. So, this is what you have from the separation theorem; now, see let us let us see where we go.

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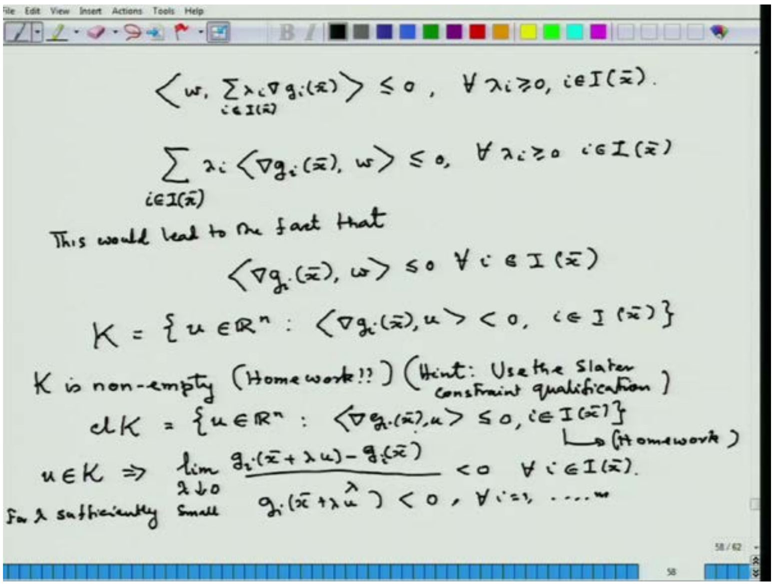

Now, which means my w of chi which is…

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This is less than equal to 0, for all lambda i greater than equal to 0, and i belonging to I x bar. This is what you have straight straight away; what does this show me, this show shows me the following.

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So for, the fixed set of indices, for whatever lambda is greater than or equal to 0. You take this some must always be negative. So, this would immediately allow us to conclude, this would lead to the fact.

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You have really not used Slater condition anywhere, but we will soon use Slater condition, and that is exactly what I am going to show you. Now, let me consider this set K, which is a set of all u in R n.

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Now, you can show that K is non empty. Now, this will be a home work, to show that K is non empty; and the hint for doing this homework is you please use the Slater condition.

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So, use the Slater constraint qualification is the required guideline to, do this fact. Now, if this is non empty, which shows that for sorry this should be u. Once if once this x, y is fixed; this is a series of convex in equalities actually. This is a series of convex in equalities, and what I have showed that for this series of convex inequalities, a Slater type condition is actually holding.

So, what you can actually show that closure of K is equal to. So, this would also be homework, and this the hint is same apply this later condition now, because because this is non empty. The Slater condition is satisfied for this particular set K. And this, for this system of inequalities, and if I take this system of inequalities. And the Slater condition is actually satisfied then; the closure of this sort of set, that is is this an interior of this sort of set is this.

So, this is also homework. So, that would I am giving quite a number of homework, because these sort of practice is very important for you to have an understanding of the subject. If you do not practice in this manner, you cannot understand a subject - mathematical subject like this. The only way to understand a mathematical subject is to solve problems or try to fill in the gaps when you are understanding fill in the theory which you feel by working them out, and that is very important.

Now, you if you observe that what does this mean. So, if you have a u in K, this would imply by the very definition of the derivative and use of Taylor's theorem; I will have…

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For all i, I would be having this. This is the meaning of this statement; for all i belonging to I x bar. Now, what does this mean that when lambda is sufficiently small, then for every i for, so what happens? Because the limit is strictly less than 0, that is the strictly negative point; after some value of lambda, this differential quotient will be strictly negative. That is the meaning of convergence of a sequence.

So, a convergence as parse, the (( )) notion of the limit. So, for lambda sufficiently small, I would like to want the students that those, who do not want to follow this proof. And feel it is intimating them specially engineering students in our country. So, please do not follow the proof, accept this story; this is true when Salter holds. So, this fact is true in Slater condition would just accept this fact for the moment.

So, for lambda sufficiently small, so greater than 0 and sufficiently small. What do I have, I have the following; this will be, because I this is 0, g i x bar plus lambda u would be strictly less than 0. Now, because for those I which is not in i x bar, for them g i x bar is strictly less than 0. Then by continuity of those g i is, you can get a lambda sufficiently small, such that g i x bar plus lambda u would be strictly less than 0. So, for lambda sufficiently small, I can show this is true for all i equal to 1 to m.

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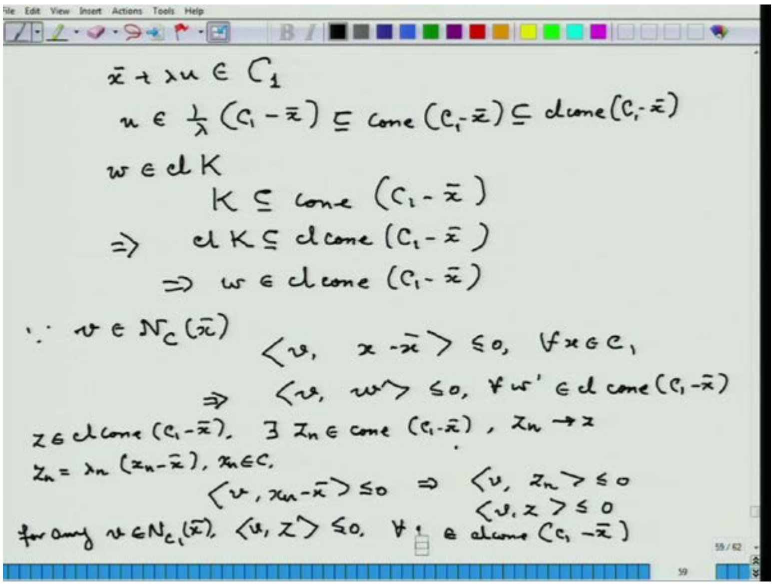

So, once I know this. So, it will immediately tell me that x bar plus lambda u is element of my set C 1; I am basically greater than 0. So, u is an element of one by lambda times the set C one minus x bar. So, this sort of set is of course, subset of the cone or the convex cone in this in this case, generated by set C minus x bar, which is again naturally a subset of the closure of the cone, now what do I show where w belongs to. Now, by our explanation here this w, this w is belonging to the closer of K.

So, w belongs to the closure of this set K - C 1 sorry. Now, since w belongs to the closure of K, one has to note something; there is the following that. What is the closure of K. So, I take any u in K, that u for any y take in K, this is what is happening right. So, naturally if u is in the cone of this, and the closure of K. So, what I am finding is that K is the subset of the cone disk; from these two equations, I actually have that K is the subset of cone of C 1 minus x bar.

So, this would imply that closure of K is subset of closure of cone of C 1 minus x bar. So, this would imply that w is an element of the closure of the cone C 1 minus x bar. Now, since w is an element of this, and v is the element of this; takes any v. So, take, but my v this v was in the normal cone, but it was not in the set S. So, since v is in this set v times x minus x bar is less than equal to 0 for all x in C. But, this implies that v times w is also less than equal to C or w dash, for all w dash in the closure of the cone generated by C 1 minus x bar. It is natural, because you multiply by some lambda. So, that will be in the cone of C 1 minus x bar, and you take the closure that is you construct take an element element of the closure.

So, elements elements of the form lambda N x n minus x bar would be going to that point. So then, once you pass on to the limit, then you will get this expression; that is ok. Let me take, let me see what happens. So, take z element of closure of cone of C 1 minus x bar. So, there exist z n in the cone of C 1 minus x bar; such that, sequence z n is converging to z. Now, z n can be expressed as some lambda n, because in the cone cone of this x n minus x bar right. Now, since C is closed where C is where x n is where element of C, where since C is closed.

Now, if x n converges; now, I am telling z n is converging to z. So, x n must converge; since c, is closed if x n converges it will converge to some element in the set. So, what does this mean, from what I get from here. I get immediately from this expression v of x n minus x bar is less than equal to 0, which would immediately tell me v of lambda n x n minus x bar less than equal to 0. So, v of z n would be less than equal to 0, and if I pass on to the limit z is going to this.

So, v of z is less than equal to 0. So, what I have showed that for any v element of N c x bar v of z is less than equal to 0, for all v element of the closure of the cone generated by C 1 minus sorry C 1 minus x bar. I do not need this. So, here I use the fact that x N is in C, use the definition that v is in the normal cone, then multiply by lambda N to give me this, as I pass on to the limit, I will have this. So, for any z sorry not v any z, in the closure of z which is an element of the closure of the cone of C 1 minus x bar; v z is holding, I mean v z is less than equal to 0.

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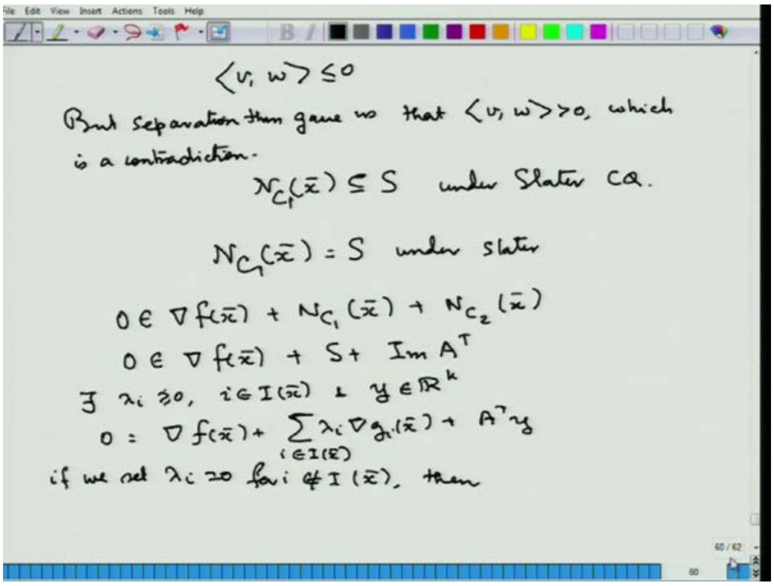

So, v of w; so what I get here, is that v of w is less than equal to 0. Naturally, because w I have shown to be in the closure of cone of C 1 minus x bar, but let me go back a while and see what does the separation theorem give me. The separation theorem gives me that v of w is strictly greater than 0 for that particular v, which we had taken, and thus here we have a contradiction, but we had proved, but separation theorem gave us.

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So, that is the contradiction. So, we do not. So, what we have assumed is not true, and thus N C x bar is a subset of S under Slater. So, which means N C x bar is equal to S under Slater C and C 1 x bar. So, the optimality condition this.

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Can now be written as...

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So, we already proved that, these major weight transposes; C 2 is nothing but x capital X is equal to P; set of all x as capital X is equal to P. So, this will give me. So, there would exist lambda i, lambda i greater than or equal to 0; where i is in i x bar. And y element of

R, K just I have to go back and check; what was my x equal to v is in R K right. So, for y in R K, we have 0 is equal to grad f x bar plus summation. So, I must have an element v from this, and element from the major weight transpose as thus 0 is equal to grad of x bar plus, that element from S plus that element from major weight transpose.

So, what does an element of x look like, is look like lambda, i grad g i x bar i over i x bar. So, that is exactly what I am writing down.

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So, or we can write if we said, lambda is equal to 0, for i not element of i x bar, then the optimality condition is the following.

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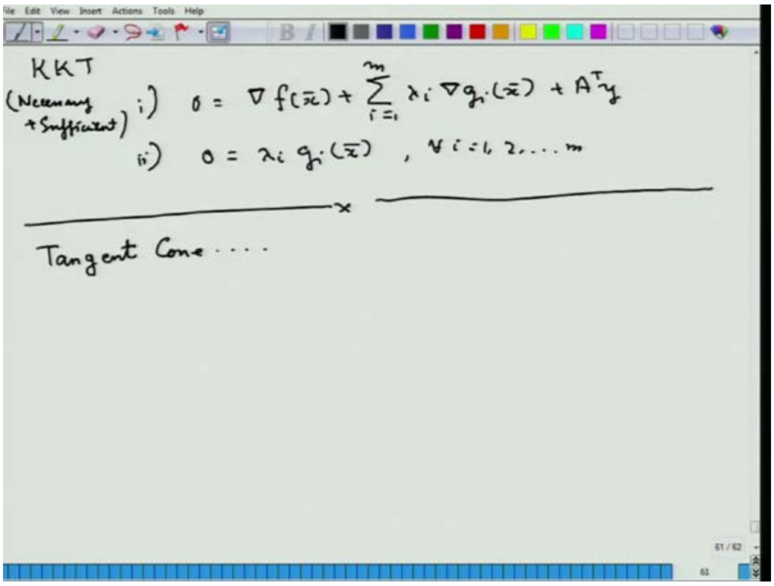

The Karush Khun Tucker condition; necessary and sufficient condition, necessary plus sufficient condition is this.

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So, I can have the complimentary slackness condition. That is how what you what you are trying to say is that whenever g i is strictly less than lambda i must be 0. So, lambda i g i x bar is any way always 0. So, this condition has to be satisfied; which will immediately guarantee that whenever g i x bar is strictly less than 0 lambda is equal to 0, that would allow you to extend this expression, just from i element of i x bar which is not so, nice to look at. I would any slightly uncomfortable, to have it have this summation of at the whole range. That is actually, but whenever lambda is not in i x bar you are putting lambda is 0. So, this is just nice, and compact you are representing this.

But these are some other role, we will see very soon and with this. So, we have now got an Karush Khun Tucker type, necessary and sufficient optimality condition for a convex programming problem. And with this we end to days talk, and we will start describing the notion of the tangent cone from the next lecture. The tangent cone has very silently kept up in the lecture today, but we really did not mention it, but it during its job. We will talk about tangent cone in the next class. So, thank you good bye.