Motzkin Decomposition

Prof. Jaun Enrique Martinez - Leqaz
Department of Economic Theory
Indian Institute of Technology, Kanpur

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Special Lecture – 02

The lecture today is going to be on Motzkin decomposable sets of functions. The notion of Motzkin decomposable set was very recently introduced, although their name is after Motzkin, well known mathematician from their first half of twentieth century. The notion of Motzkin decomposable set is inspired by a result it is very well known in linear programming theory, which says that every convex polyhedral can be expressed as the sum of a polytope in a polyhedral a complex cone namely, the polytope generated by extreme coins of the set, and the convex cone generated by the extreme rise of the set.

So, this convex polyhedral are expressed above as sums of a compact convex set the polytope plus a closed convex cone, and the question that arises in a natural way is which are though sets, which enjoy these decomposability property we know that complex polyhedral satisfy this properties, but there is a larger class of sets. And they enjoy many nice properties which complex polyhedral enjoy, the talk is going to be based on three papers.

These are the only three papers as far as I know in which, the notion of Motzkin decomposition is used the three papers are course by Miguel Angel Goberna from the university of Alicante Spain and Maxim Todorov from university de las America spoiler Mexico there is. So, so another course in the first paper (( )) Gonzalez were the time of making this paper was a Maxim Ivanov Todorov, and in the last paper there is another courses of resolution from impact Rio de Janeiro, brazil. The three papers in the journal of mathematical analysis in application, the first two of them were published in 2010, and the last paper is already on line, but not is not the appeared in print yet.

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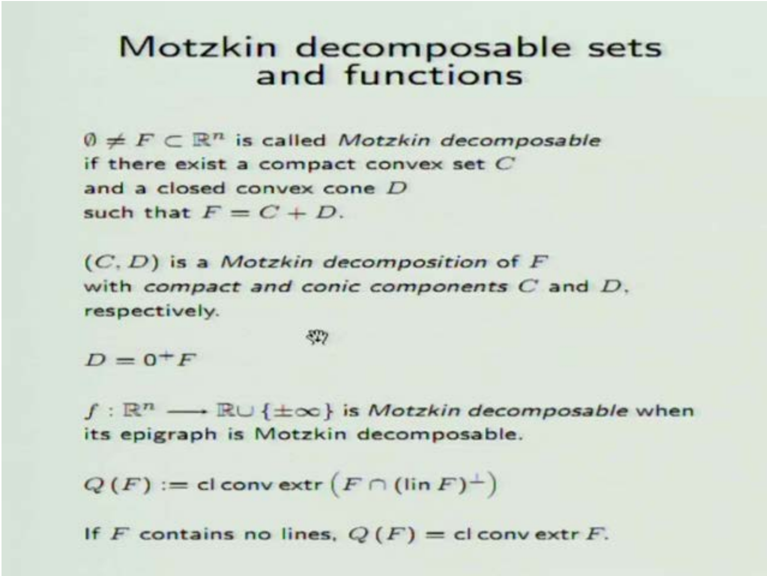

Well. So, let us start with the definition of a Motzkin decomposable set, we are working with subsets of the Euclidean space R n. We consider an empty set F and we call it Motzkin decomposable, if there exist a compact convex set C and a closed convex cone D such that, C plus D is equal to F given that we are adding two convex sets their sum F will be convexed, and since we are adding a compact set with a closed set the sum F will be closed. Therefore Motzkin decomposable sets are close compact sets, but the compact does not hold through it is very easy to see and we are actually, going to see it later there are many close convex sets which cannot be expressed as a sum of this type.

When we thought have such a decomposition, we call the pair C D a Motzkin decomposition of the set and the set C will be a compact component and the set D will be the conic component. I say the conic component because it is uniquely determined by the set and like a compact component, there may be many compact components after all infinitely many compact components for a given set, but there is only one possible conic component which is the recession cone of F namely, the set of all directions along which if you follow one is thirteen from may not be thirty point of the set you never leave the set.

This is the large class of close compact sets. So, I said at the beginning it include all complex polyhedral, but it also includes all compact convex sets because we can take the equal to the single cone of zero, it also includes all closed convex cone because we can take C equal to the single cone of zero.

Then we saw large class where the epicanthic they could class of close convex sets we can. Now, define an essential valued function to be Motzkin decomposable, when it raise a Motzkin decomposable epigraph. For some characterizations this set plays an important role, this is the closed convex of the extreme points of the intersection of the set with their (( )) sub space to the linearity space of F. The linearity space of F is the set of all directions corresponding to lines, which are fully contained in the set or in other words is the intersection of the recession cone with the opposite and we need to consider, this intersection because we want to talk about extreme points and it is well known that, the close complex set has at least one extreme point if and only if it has no lines.

That is to say if and only if the linearity space of F reduce it to 0. So, in this particular case when there are no lines this linearity space reduces to zero, the orthogonal is the whole space. So, the intersection here is. So, prefers the intersection gives F. So, in that case Q of F will simply be the closed convex hull of the set of the extreme points of F, but even if there are no extreme points we are considering extreme points of a smaller set thanks to this trick, we can call this intersection the lines free version of F.

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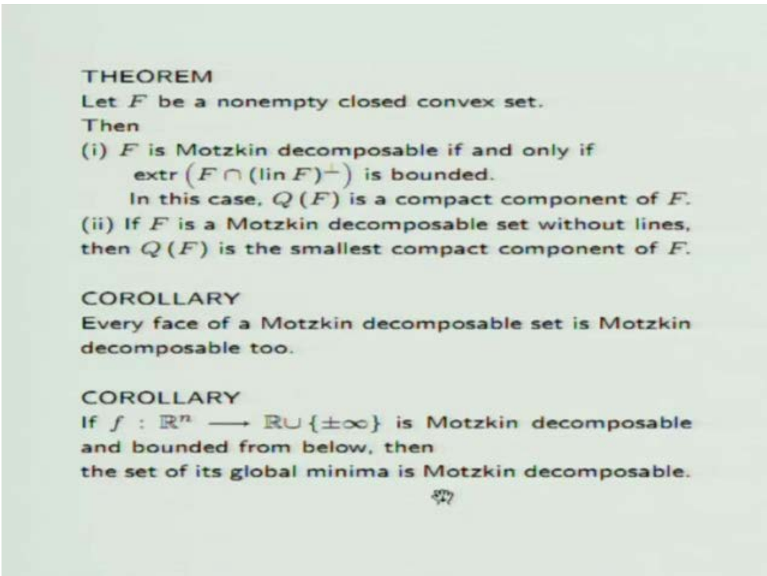

Let us here first characterization of Motzkin decomposability for an empty closed compact set. A set F being non empty closed and compact is Motzkin decomposable if and only if, they set of extreme points of the lines free version of F is bounded and in this case, the set Q of F which we have seen before namely, close compact hull of these set of extreme points is a compact component of F, but as we said before there may be many compact components, but in the case when the set has no lines these compact component is the smallest possible cone component is unique.

More than one compact component, but the cone component is unique.

Yeah it is unique, but compact component they are many.

But in the case, there are no extreme points among the many this is the smallest possible. If there are lines there is no a smallest possible if and there is no minimal one with respect to inclusion ok. Well circular of this theorem, we have that every phase from Motzkin decomposable set, it is Motzkin decomposable too this comes from two facts first that the linearity space of a phase is the same as the linearity space of the whole set and second that one can easily check, the lines free version of a phase is a phase of the lines free version of the set and then the extreme points of the lines free version of the phase will be extreme points of the lines free version of the set.

If the set is bounded then this subset will be bound with two hence we have the corollary as an immediate conclusion. Another immediate conclusion is this last corollary here that we have a Motzkin decomposable function, which is bounded from below then the set of its global minima is Motzkin decomposable and notice that Motzkin decomposable implies non emptiness, every Motzkin decomposable set is non empty. So, here in particular we are saying that a whenever a Motzkin decomposable function is bounded from below, it attains the infimum it attains minimum this property is well known for polyhedral from the expansion.

It is something like a (( )) theorem.

Yeah for quadratic

But I mean, this is abided generalization of a what is known in linear programming, I forgot to say that a polyhedral complex function is Motzkin decomposable because a polyhedral complex function, which is the maximum of a finite collection of affine functions. Yes, an epigraph which is a complex polyhedral and therefore, it is Motzkin decomposable. So, this class of functions Motzkin decomposable contains a particular a polyhedral complex function, they also contain sub linear functions.

Yes

Because sub linear functions are have an epigraph, which is closed convex cone. So, again this we shall follows from the main theorem because one can easily identify the set of global minima, we have phase of the polyhedral of the excuse me of the epigraph of the function.

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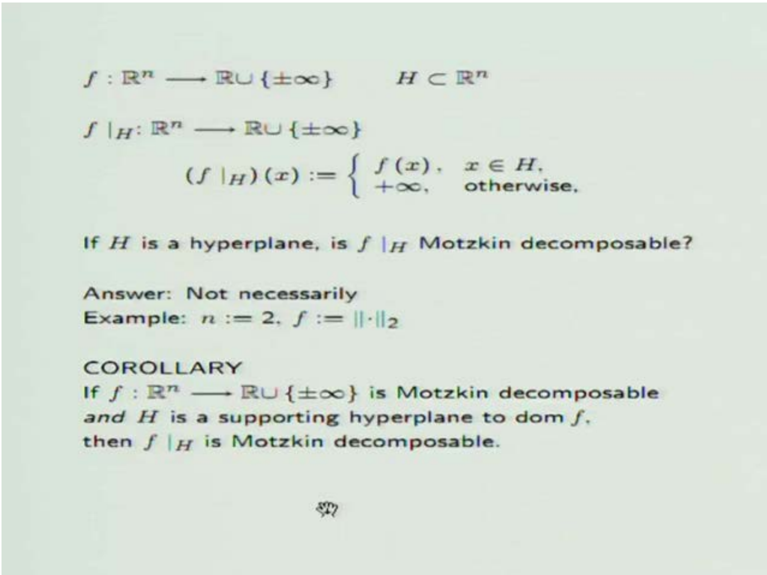

There are some questions, we can make you understand the structure of Motzkin decomposable function for instance consider their restriction of ones and fraction to a high plane is it necessarily Motzkin decomposable, well first of all we need to make the question more precise since, we are dealing all the time with functions. We find on the hull of R n, when we make the restriction we hyper plane, we are no longer working with the function having full domain. So, we have to extend the restriction to the complement of in hyper plane and with which by the value plus infinity.

So, the restriction is actually the function which is obtained from the original function by replacing their values, the function takes outside H by plus infinity. So, is such a restriction Motzkin decomposable.

Sir for second I will go to the bathroom the lights are too much. So, I cannot take the eyes I am feeling very it is too much of light can the light be dimmed a bit I am feeling too much disturbed light (( ))

Decoration whether they are restricted come out in Motzkin decomposable function to a hyper plane is necessarily, Motzkin decomposable and the answer is now, we have an easy example in two dimensions a function of two variables the Euclidean norm. The Euclidean norm is Motzkin decomposable because it is epigraph is the classical ice cream cone as long as it is across convex cone its Motzkin decomposable.

Nevertheless, if you take as the hyper plane a vertical hyper plane not containing the origin not containing the vertex of the cone then, what would be the restriction of this function to this hyper plane. The epigraph of this restriction would be the area enclosed by the a branch of a hyperbola, which is not Motzkin decomposable because the set of extreme point is the whole boundary, which is unbounded therefore, it cannot be according to the theorem, we saw below cannot be Motzkin decomposable.

But if the hyper plane is as to point in hyper plane to the domain of the function then, the restriction is Motzkin decomposable, the reason is that in that case the epigraph of the restriction would be a phase of the epigraph of the full function and then it would be Motzkin decomposable.

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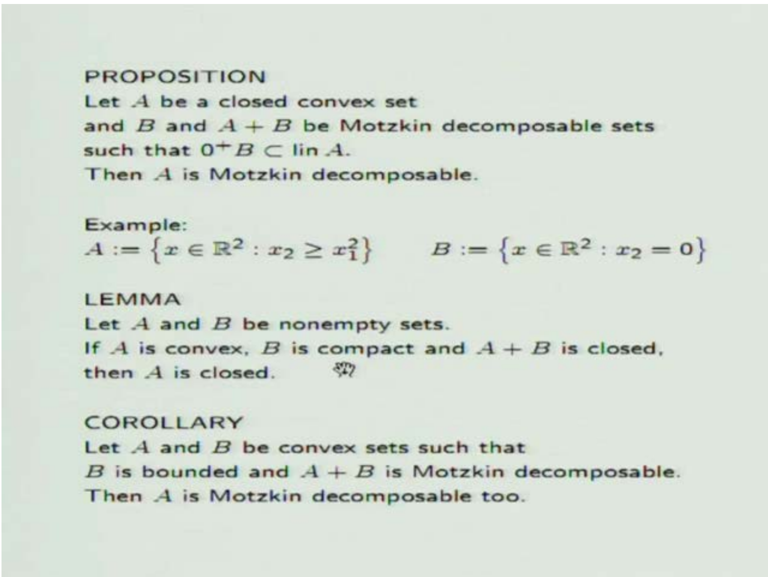

There are several questions with another set related to operations with Motzkin decomposable sets here is one suppose, we have two closed convex sets A and B and one of them say B is Motzkin decomposable and when we make the sum the sum is Motzkin decomposable too can we reduce from this fact, that the other set a is Motzkin decomposable too? The answer is now and so, we will show by an easy example soon, but the answer is yes if we impose the additional assumption that the recession cone of the set B is contained in the linearity space of a to see that, without these assumption their property does not hold look at this example here.

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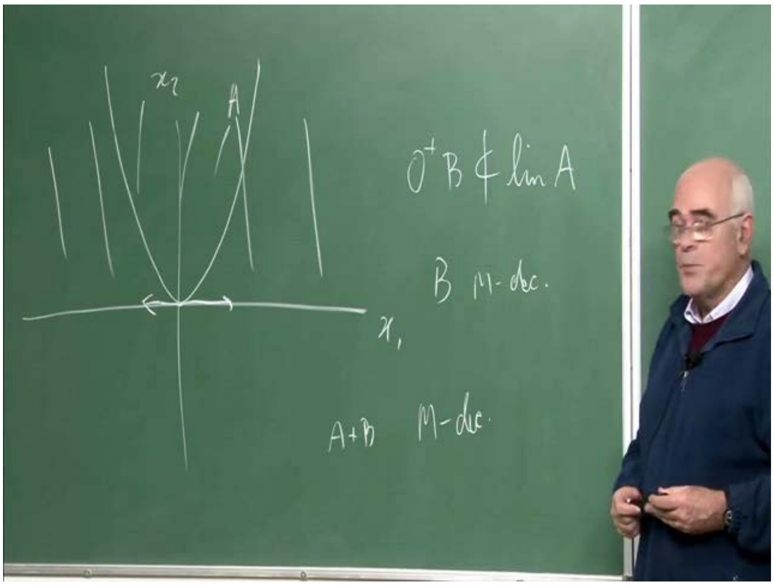

The set A is let us say vertical parabola, the area includes by a vertical parabola. We can make a picture this will be a then the set B is the horizontal axis and what is A plus B, A plus B would be the full upper.

Thing accept this line.

No no using this line.

(( ))

Whole whole half space.

Is a full.

Upper half space.

Upper half space…

Then you have b is Motzkin decomposable.

Because it is a close complex cone it is a sub space, same with A plus B is a half plane. So, it is also Motzkin decomposable nevertheless, our original set A the parabola is not Motzkin decomposable for the simply soon, as we saw before an example the set of extreme points is unbounded is the whole boundary of course, in this case.

Here the case b is not bounded.

Yeah, but does not matter the condition the result as you can see here is that we are assuming that both B and A plus B are Motzkin decomposable.

B and a plus b.

B is the horizontal axis.

A plus B is the upper half plane.

Such that the recession cone.

But we do not have.

Is linear

Inclusion of course,

This is not contained in linearity space of A.

Because the recession cone of the this is the cone of B.

And the linearity space of A is reduce it to zero A contains no lines. So, the inclusion does not follow this shows that these extra assumption in the preposition in the super flows, but in particular if the set B is bounded is compact then the recession count reduce it to zero, it will be containing the linearity space of A. So, in such a case the property holds this is what is said in this corollary at the bottom of the page in which on top of this assumptions getting some refinement, we are not assuming that A and B are closed again B are just complex and these complex, but we do not say closed or compact.

Then in spite of this we could assumptions we get the same conclusion, if A plus B its Motzkin decomposable then A must be Motzkin decomposable. This is based on this lemma, which is a very easy statement in in complexity, but which the proof of which is now, completely obvious I could not say it is an exercise, but it is not a standard proofing complexity. It says if we have two non empty sets such that, A is complex just complex B is compact just compact and A plus B is closed then a must be closed just one option based on this result does not hold through without complexity of set.

For instance consider any set which is then, but different hold set R n then at the unit ball or a ball of arbitrarily small close ball, which is compact then A plus B is the closed space. Then it is closed nevertheless, A was not closed was (( )) subset also with complexity, this we shall only holds to in finite dimension, in an infinite dimensional space it does not hold through. Consider for instance are reflexive one at space with the with a with topology, then you can repeat the argument I have used. Now, by choosing Athens hyper plane which is complex then, you are a closed ball which is compact and under the with topology, then you get the whole space you repeat.

The argument I have used before, but even recently one week ago professor Nicholas (( )) gave me a very nice example showing that even in an arbitrary one at space not necessarily, reflexive and using the norm topology not the week topology, this we shall does not hold I mean, we can take this we shall as correct recession of finite dimensionality this result holds to if and only if the space is finite dimensional.

Well, I am going to present several characterization we have seen already is sum of a Motzkin decomposability, one of them will be based on the notion of the conic representation of one on empty closed convex set.

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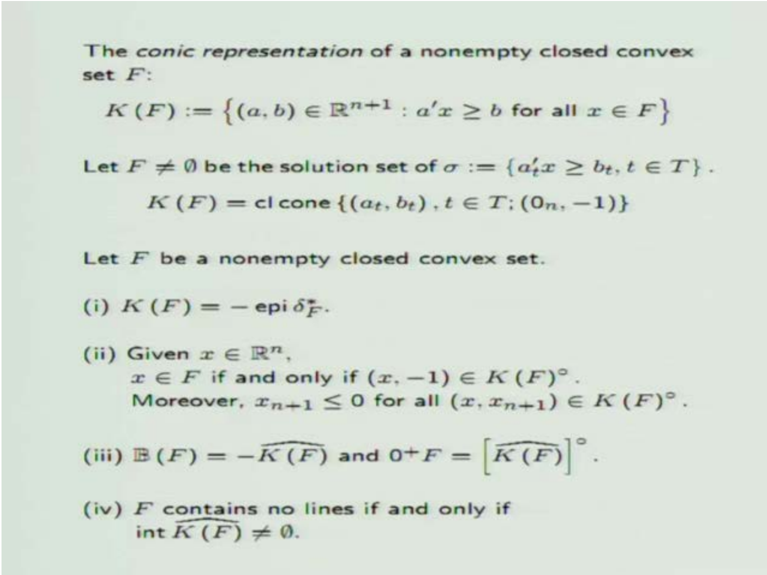

The conic representation K of F is roughly spoken, the set of all linear I n a qualities which are satisfied by all the elements in F. So, far we consider all coefficient vectors A, B in R n plus one corresponding to an equalities, which are satisfied by all vectors in F or points in F and then the set of all these coefficients make the conic representation of F clearly this is closed convex cone.

Semi definite programming semi infinite programming.

This is a notion which either know, if it was introduced, but at least is extensively used in a linear infinity programming. Yeah, this is very much used in in that field then for instance it is well known. In fact, this is a sub addition of Farkas lemma that, if our closed convex set is the solution set of a given and linear inequality system with possibly infinitely many inequalities, we are not saying anything here about the cardinality of the then the conic representation is the closed convex cone generated by all the inequalities.

Let say together with this particular vector all components are zero accept the last one which is minus one, the close convex hull of all this all this vectors, the reason for which these particular vector appears is that there is always an inequality in a quality, which is having consequence of a given systems of inequalities, not this another way of looking at this K of F.

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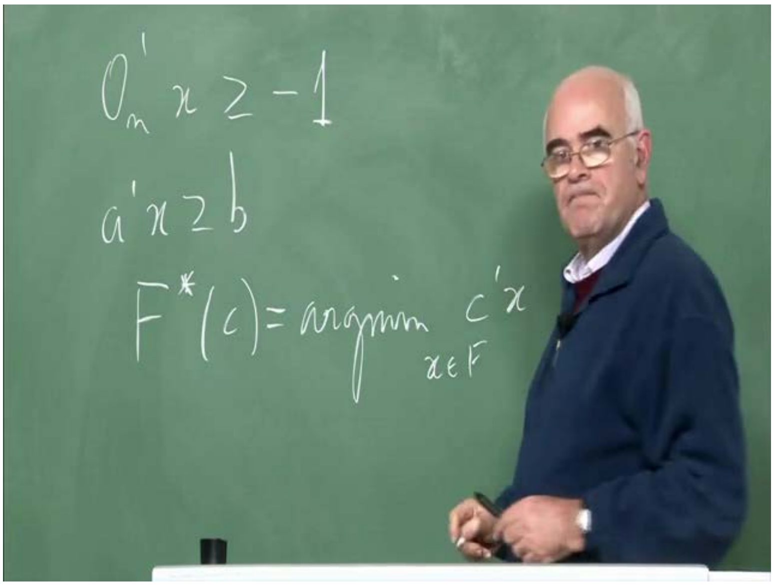

In this case is the set of all consequence linear consequences of this linear system and there is also a consequence which is which really holds namely, you have an inequality zero n prime x. We are saying greater than equal to minus one this is this clearly holds proof for every x and this is the reason why this vector appears here or in other word did you have an equality a prime x greater than or equal to B, you can always decrease the right hand side will be a valid in equality. The conic representation is. In fact, minus epigraph of the super function of the set, we also have that a point belongs to the set if and only if when, we are the last coordinate minus one then the resulting point belongs to the dual count positive polar of the conic representation of the set and moreover for every vector in this polar set dual cone the less coordinate less than or equal to 0.

The barrier cone of F which mean, the set of all linear function which are bounded above on F is minus this set by K of F that we mean that projection of K of F one to R n, K of F is in R n plus one we believe the last coordinate, when we get the projection on to R n this is the meaning of this expression over here and we also kept that the recession cone of F is just the dual cone of this projection.

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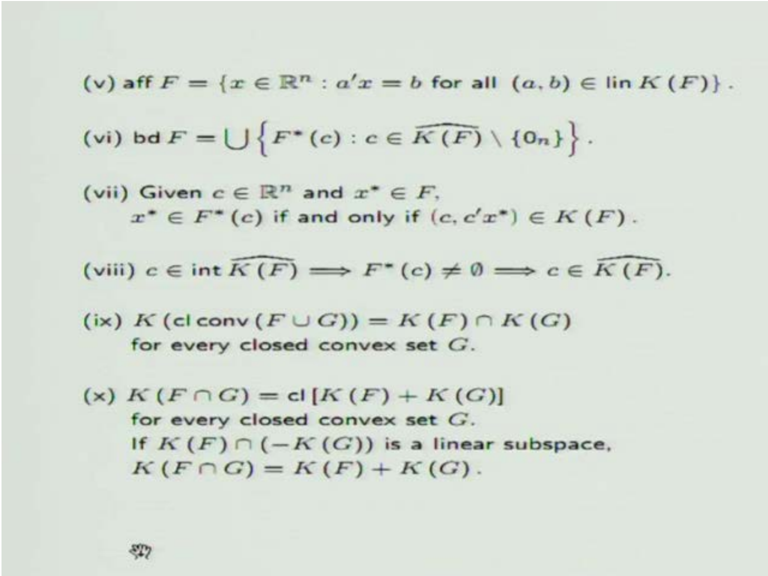

Moreover, F contains now, lines or in other words F contains some extreme point, if and only if the interior of this projection is non empty, the affine set the affine; however, of F is the set of points which shows all this system of equalities. Where the equations I had taken from the linearity space of the conic representation of F also the boundary of F can be expressed in terms of the conic representation, in this way I have to explain here then representation.

Here F star of C denotes argmim of the linear function C argmim over F or C prime x so. In fact, what this property is telling as is that the boundary is argmim of this exposed phases. Moreover, an optimal point for the linear function determined by C is such that, these point belongs to the conic representation of F and vice versa more over for C in the interior of this projection, these optimal solution set is non empty and these does not imply that C is in the interior of the projection, but C must belong to the projection sum is a (( )) for conic representations.

A conic representations of the close convex set of union intersection of the conic representations of the set and for the intersection is the closure of the sum. Moreover, if the two conic representation have this property, that the intersection is a linear subspace the intersection of one with minus the other then this formula simplifies closure becomes super flows and the conic representation of the intersection in that particular case reduces to the sum of the conic representation of this set.

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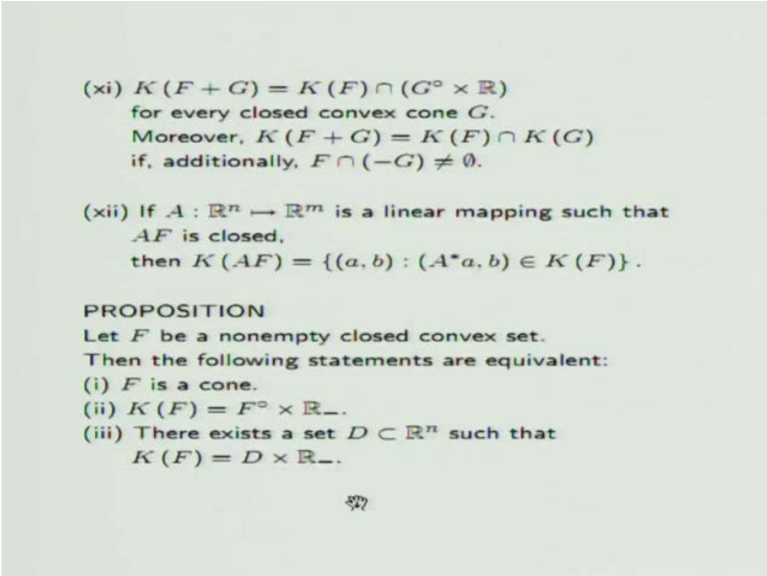

When the set on set G is suppressed from this cone we have this equality which give such an expression for the conic representation of this sum. And in particular we have this intersection when F intersection with minus G is non empty we have also an expression for the conic representation of the linear units of a close convex set under the assumption that this linear units is closed.

One can characterize those complex sets, which are cones by using the conic representations they are, those complex cones for which the conic representation has this structure is a Cartesian product where the where the second component is the non positive real line and if sets name of course, then the first set in the Cartesian product must be the dual cone to F.

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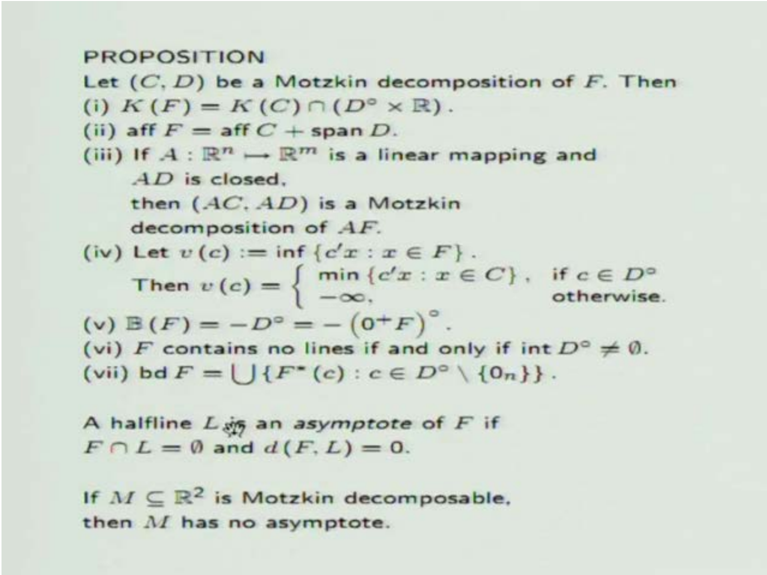

We also have this proposition, which summarizes any properties of complex Motzkin decompositions of a set. Suppose, we have one such decomposition then they conic representation of F is the conic representation of C intersection, this set where D is the conic component. So, there the recession cone also we have that the affine hull of the set is the affine hull of compact component plus the linear space generated by the conic component. if we have a linear mapping such that, the units of the recession cone is closed then the units of the compact component and the conic component make a Motzkin representation of the image of F and that that linear mapping.

First property is important because it deals with solution sets of linear problems. Linear problems subject to non-linear set belongs to F this property says that whenever, a linear function C prime x is bounded below on F that is to say when the infimum is not minus infinity, then it is attained it is a minimum and this happens, when C belongs to the dual cone to D if and only if.

I just want to. So, the forth property is very interesting from the optimization point of view. So, minimizing a convex set over a Motzkin decomposable set.

A linear function over a Motzkin decomposable set.

A linear function over a Motzkin decomposable set and if it is bounded below then the minimum is attained.

See the element of D naught. So, C as the element of D naught we can guarantee that is bounded below.

Yes.

Motzkin decomposable set.

But I have one question that second line says minimum C transpose x x belonging to C why should be x belonging to f.

Where b C when you are writing min when you are replacing the information with min

No, it is see its equivalent. Thank you, for pointing this out because it is an important observation in principle the function is defined over, if I mean we are minimizing over F.

But this says that there is at least one point in C where the minimum is attained there is no type of here.

No no I understand now every element in C is also element in F because it can add C plus zero because C plus.

Yeah C is always a subset of F.

So, the interesting factor minimum would be attained on C...

Not on the cone only on the compact part the minimum would be attained.

No no the minimum can be attained everywhere, but among the optimal solutions there is earliest one which belongs to C among optimum right if C belongs to D.

Like a linear programming.

Yeah.

Beyond optimal solution there is a vertex.

Yeah, exactly.

That is why I say belonging to the compact component.

Yeah right right yeah.

The fifth property is also interesting, it says that the barrier cone is minus the dual cone to recession cone it is interesting why? Because it is telling us in particular that the barrier cone is closed, which is not always the case in general.

Yeah barrier cone is not close

Yeah, but in actually sets we are closed, but your cone are (( )) they are all hyperbolic complex sets. So, this show something which is after all emanated consequences of the definitions that every Motzkin decomposable set is a hyperbolic complex set. The sixth property is that the set contains no lines, if and only if the interior to its dual empty seventh property is the boundary of F is the union of the exposed phases, determined that element belonging to the dual count to the recession cone except, the origin in two the dimensional case. A Motzkin decomposable set has no asymptotes here, we define asymptotes as a half line which does not intersect the sets, but its distance to the set is equal to zero; that means, you can find points are very close to each other one in the set one in the line, but they do not intersect.

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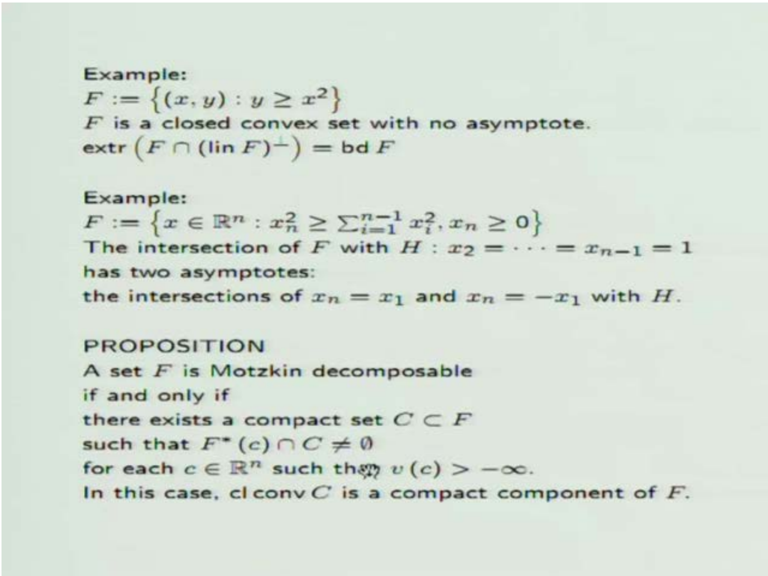

So, one can prove that this holds in R 2, but the complex does not hold through, even in R 2, I mean the further theories no asymptotes does not guarantee that the set is decomposable example take the parabola there is no asymptotes, which closed compact set, but the extreme points is the whole boundary. So, it is unbounded. So, it cannot be Motzkin decomposable.

What about higher dimensions? The statement which goes through namely that a Motzkin decomposable set has no asymptote. The set hold through in higher dimensions, the answer is now, look at this example we are considering a cone this is a cone and if you take the intersection with this hyper plane, it phase to asymptotes and of course, the the asymptotes of a subset as long as they do not intersect the set are asymptotes of this set.

Look look at this example in the dimension, three I have told you have seen these for a different purpose before we were talking about the restriction to a hyper plane again say. So, this count in the case n equal to three is the ice cream cone then, take a vertical hyper plane as before not containing the origin, they intersection is a hyperbola which has asymptotes and the asymptotes of the hyperbola will be asymptotes of that cone this is not very, this is very obvious after listening the explanation, but these I do not make the explanation and I ask you, which that that the set of asymptotes in intuitively at least I would answer no because I mean, it consist of a lines.

Whether it will have a asymptotes

Yeah

That would to now I understand.

So, in the case n equal three or higher than three, it does not hold another we solve this that a set is Motzkin decomposable if and only if there is a set C such that whenever, we have a linear function bounded from below on C then as you have said before there is an optimal solution of the corresponding minimization problem with this linear function, which belongs to C with a compact component. If such a set exist we know that set is Motzkin decomposable and the closed complex hull of this set is a compact component of F.

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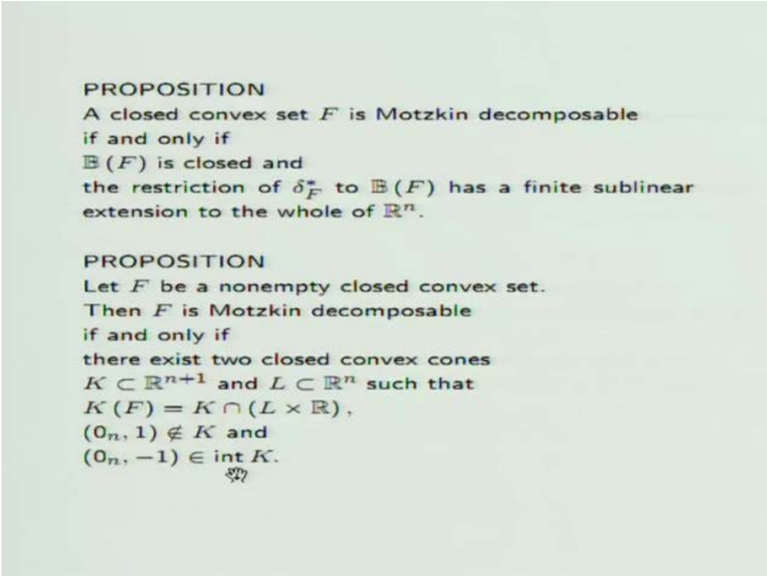

Another, characterization I said that Motzkin decomposable set are hyperbolic means, the barrier count is closed that the complex thus equal to true. We need an extra assumption, we need the restriction of the super function of the set to the barrier count to have a finite sub linear expansion to the hull R n.

I just like to recall just a barrier cone definition barrier cone is the…

A set of linear function, which are bounded above on the set.

Barrier cone

Set of.

Barrier cone

You are moving a hyper plane in this direction you cannot go beyond they are corresponding orthogonal vector pointing, in this direction is general meaning of the cone.

No no there is some relations some polarity relations with some other set.

Yes with the recession cone.

Yeah exactly.

The recession cone is the polar of the earlier cone.

Exactly exactly.

But if you take polar again, you do not get the barrier cone you get the closure of the barrier cone.

Because, they do not have closeness.

Polar polar of the recession cone is the barrier cone

No is they closure of the barrier cone in general.

The closure of the barrier cone.

No no.

The polar of the barrier cone is the recession cone.

This is true.

Here the barrier cone is not closed. So, you cannot tell it

Exactly

Yeah.

When it is closed you call the set hyperbola.

Yeah

Then our Motzkin decomposable sets are hyperbolic, but the hyperbolic complex set need not be Motzkin decomposable, one needs an extra condition which is this one, this to put function here are restriction to the barrier cone, which has a finite value everywhere finite value sub linear extension to the whole of R n.

What is these finite sub linear extension is they a super function of a compact component. This is very easy to to understand, I have no time to give more explanations, but this is not difficult to obtain. Another characterization of Motzkin decomposability in terms of a conic representations. A set if Motzkin decomposable if and only if there exist two closed complex cones K and L. Such that, the conic component excuse me the conic component representation of the set has this structure complex cone and their first complex cone K does not contain this element, this vertical vector up wards and the interior of the cone contains these vertical vector which is downwards.

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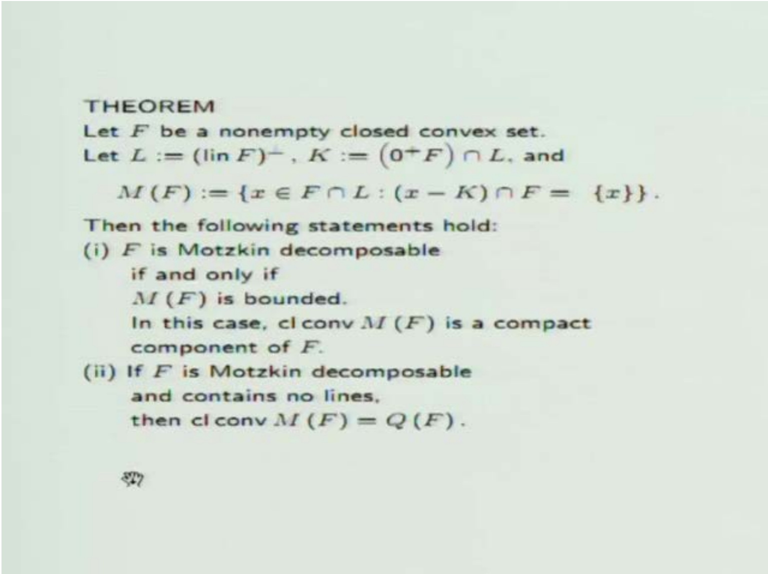

Another set which is important for the restriction of Motzkin decomposable sets, is the set M of F define here we call L they orthogonal linearity's space of F. We call K the intersection of the recession count with L this makes K point. In general the recession point of F contains lines, but when we make the intersection, the intersection does not contain lines any more then. Now, we consider which can be easily seen that is nothing, but the efficient points of the set F with respect to the ordering induced by k.

Then if and only if set is bounded, the set is Motzkin decomposable and in such a case the closed convex hull of this set like a compact of this set and in the case, when our Motzkin decomposable sets contains no lines these close convex hull coincides with the close complex hull of the extreme point the set Q of F is. So, at the beginning of the talk this means, this is the smallest compact component in such a particular page when there is no line.

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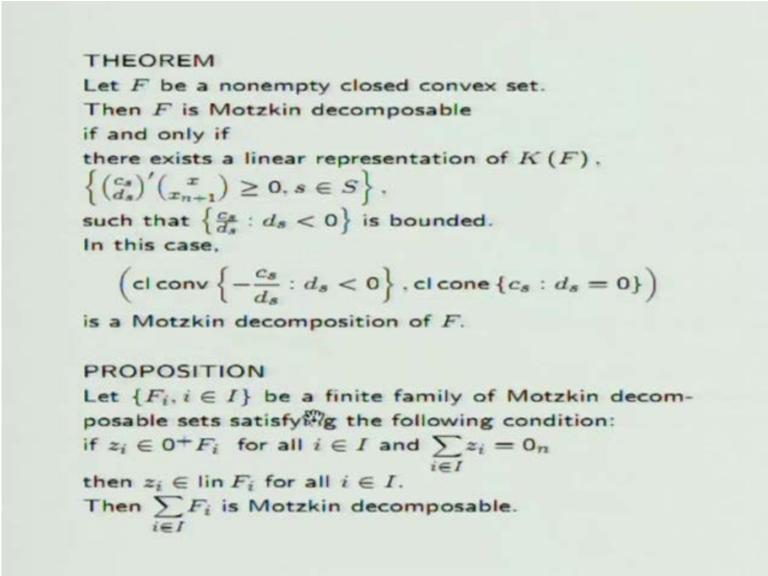

Still another representation or characterization of Motzkin decomposable sets term, this time not the conic representation of linear representations of the conic representation. F is Motzkin decomposable if and only if there is some linear representation of the conic representation of F such that, this set of coefficients normalized coefficients would say is bounded in which case, we have here specific a Motzkin decomposition of this set. Unfortunately, Motzkin decomposable set do not enjoy many nice (( )) we saw before that you we make the intersection, we are going to repeat this later and the intersection may fail to be Motzkin decomposable, here we have the same with the sum the sums.

The problem is that when we are the cones, the cones their cones by definition, but the sum make them to be closed. We have no problem, when adding the compact components, but when adding the conic components there is a problem, but this this appears, if we use these assumption that whenever, we have a sum of elements each one belonging to the recession cone of one of the sets. In this sum is equal to zero then the elements must belong to the linearity space of the sets.

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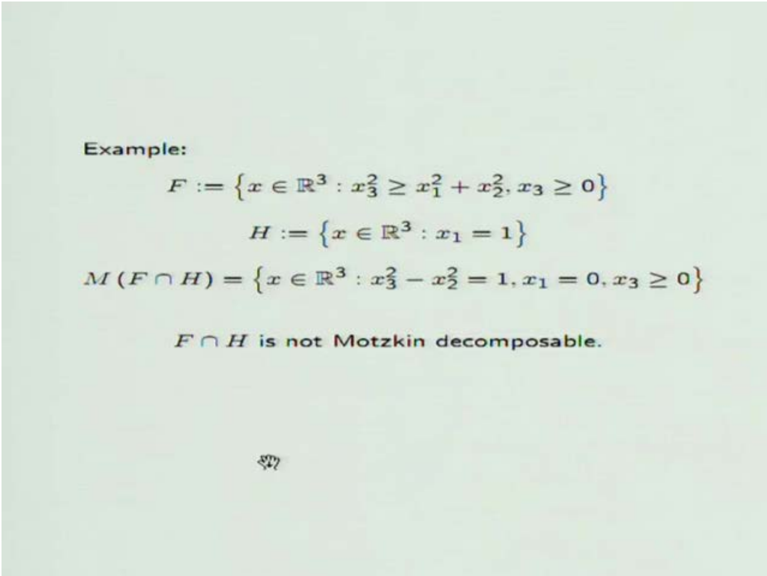

In this case, the sum is Motzkin decomposable and this is the example as I was mentioning before once more, we see the ice cream cone we see here a vertical hyper plane not containing, the origin we make the intersection and the intersection, we have seen already twice that is not Motzkin decomposable. So, this that is the sum pleasant property that the intersection of Motzkin decomposable sets is not necessarily Motzkin decomposable.

In the last part of my presentation, I am going to talk about a special kind of Motzkin decomposable obtained by. So, called truncations to explain the idea very easily is we have closed convex set Motzkin decomposable or not we take a hyper plane and well in in this particular example we have two truncations, which start the intersection with the two half spaces and one of them happens to be compact well it it tells out.

How many truncations were here we have the sets is divided in two parts

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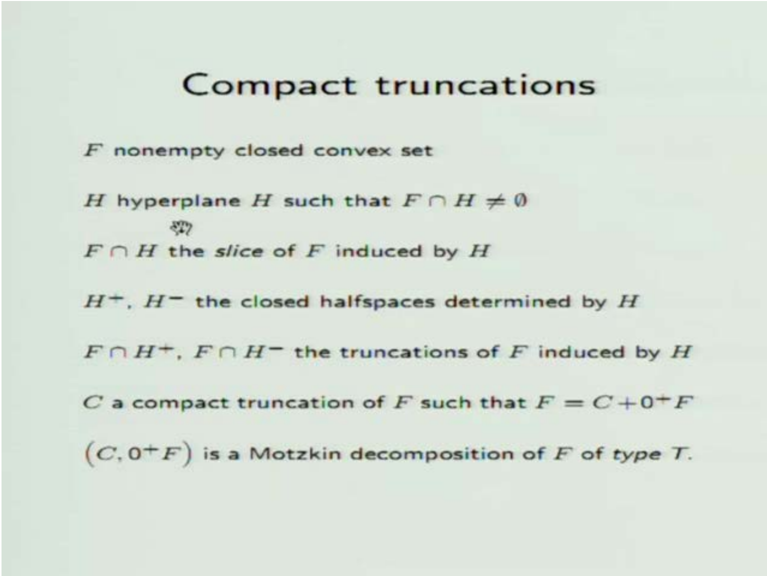

The truncation is look at the definition over here, we take a hyper plane which intersects the same. This intersection we call the slice of the set induced by the hyper plane and we consider the two closed such spaces determined by H is to have have spaces. Determine two intersection, which are called the truncation? If one of these truncations happens to be compact not only compact, but a compact component means that when you have the recession form you get the whole of the set then, we have we say that Motzkin decomposable of type T was the truncation type.

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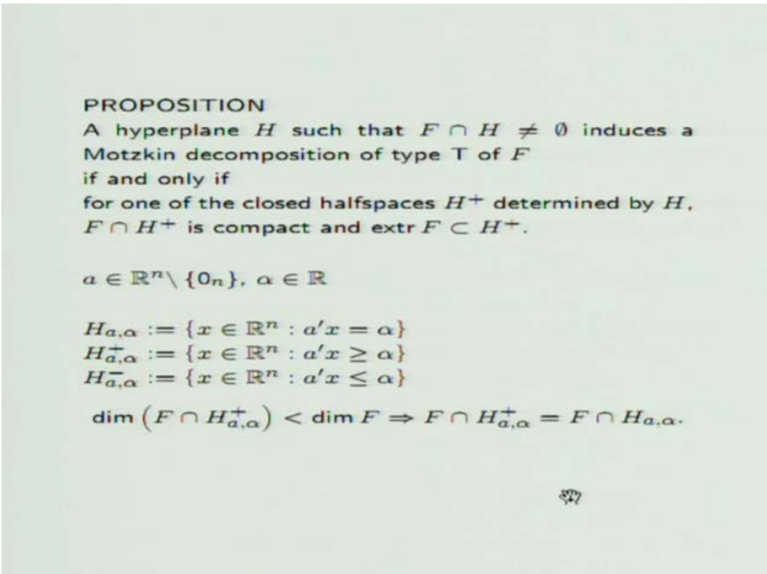

So, these are decompositions of time in a very particular way. Then a first question towards is when thus a hyper plane inducing a non empty slice, in this are Motzkin decomposable of type T, the answer is when one of the closed half spaces determined by H is compact this is; obviously, unnecessary condition, but this compact truncation contains all the extreme points of the set.

Then only you can have

If and only.

Then only you can have that decomposition

Yes.

That is quite intuitively clear otherwise.

Yes exactly and notice that an immediate consequence of this procession is that if the set admits Motzkin decomposable of type, it can contain no lines if it contains one line it is impossible to have a compact condition very easy to prove.

Well, we need to introduce here some notation which will be used like that, but first an observation is that when the a truncation H has smaller dimension, then the set the truncation coincides with the slice or in other words, if the slice is properly contained in the truncation that I mention of the truncation must be exactly the same as as the dimension of the of the phase.

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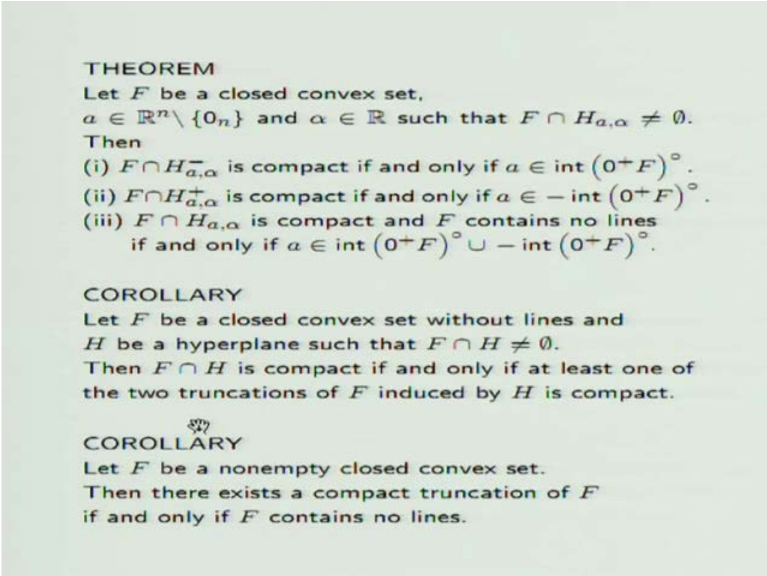

The question now, is when half space induces a compact truncation and here you have the answer, if and only if the vector belongs to the interior to the dual of the recession cone. If they have spaces define with less than or equal to or to minus the set it is defined as here a way around. Then we also have that this slice is compact and F contains no lines, if and only if a belongs to one of this two sets this consequences. If it is one is that if we have a set we have lines, then for a hyper plane inducing an empty slice the slice is compact, if and only if at least one of the two truncations used by H is compact.

We saw before that if there is a compact truncations, there cannot be no lines, but now from this results it follows that the convex there is a compact truncation if and only if there are no lines and we are talking about Motzkin decomposition of type T. I am just talking about compact truncations.

yeah

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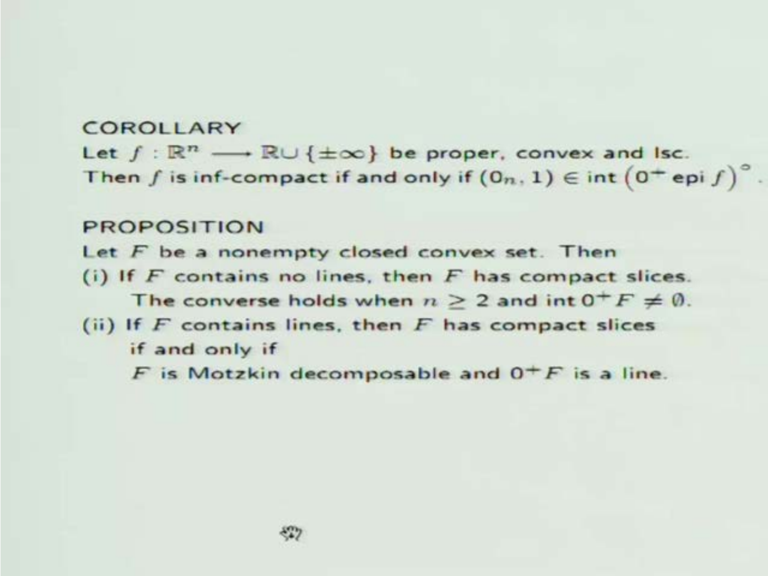

This which is different, in particular when we apply one of the results we have seen before to to functions we get that for lower semi continuous for a complex function in compactness which means, that they lower level sets are compact of course, if and only if this particular vector belongs to the interior to the dual cone of the recession cone of the epigraph of the function.

When there is a set here compact of slices, if it contains no lines it always is compact slices and the compact holds to in dimension two or higher. If the interior of the recession cone is non empty in dimension one this is not true one can also give examples showing that this does not hold to the recession cone is an empty interior. When F contains lines it contain compact slices, in just a very special case when it is Motzkin decomposable and the recession cone is exactly one line, one full line that is no other possibility.

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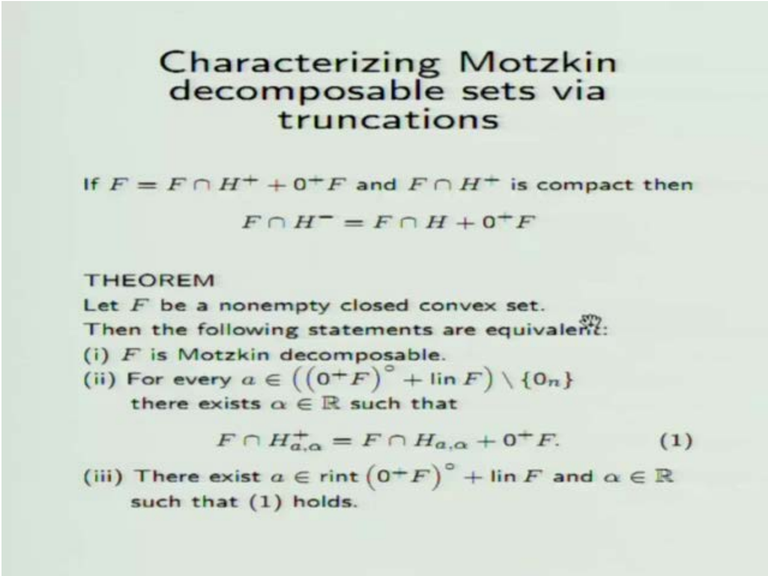

Now, how to characterize Motzkin decomposable sets by means of truncations. First an observation, if we have a compact decomposition of type T, this is what we have here a truncation, which is compact plus the recession cone then, the possibly unbounded they are possibly unbounded truncation is also Motzkin decomposable with the compact component being the slice. After this observation, we make this statement which says that, a closed convex set is a Motzkin decomposable if and only if for every element in this set we have this decomposition.

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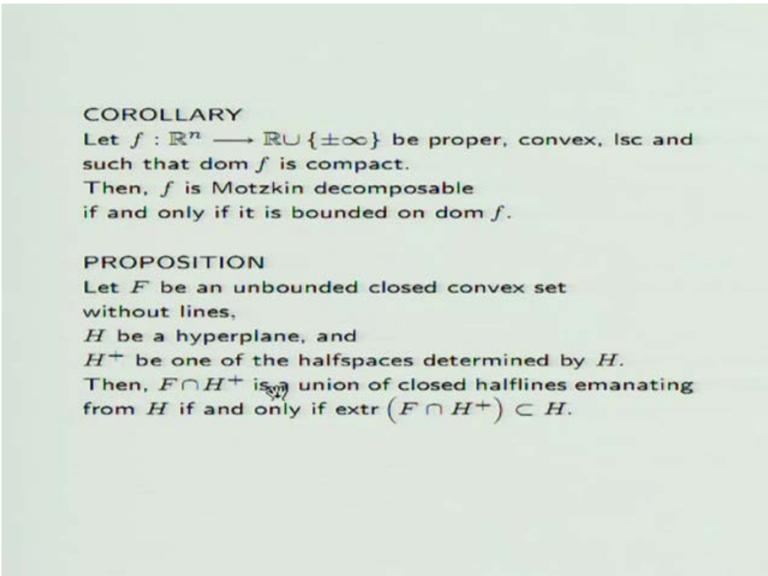

We are not saying this is a compact a Motzkin decomposition of type T, we are saying we have this expression if this expression holds for every A in this set F is can be decomposable and. In fact, it is sufficient to have just one element in this set, which is a smaller number one for F for having a Motzkin decomposability that is to say for having this property for every element in the larger set.

When we translate this for function, we get that if we consider the function compact domain then this function is Motzkin decomposable, if and only if the function is bounded above automatically, the case because we are dealing with the lower semi continuous function on a compact set this is when the domain is compact. We later on consider the general case, where a more general case, but first after precision we will apply for that itself. If we have an unbounded complex set contained in no lines, if H is a hyper plane and H plus is one of the half spaces determined by H then these half space is a union of closed half lines emanating from H, if and only if the set of the extreme point of their truncation is contained in this line.

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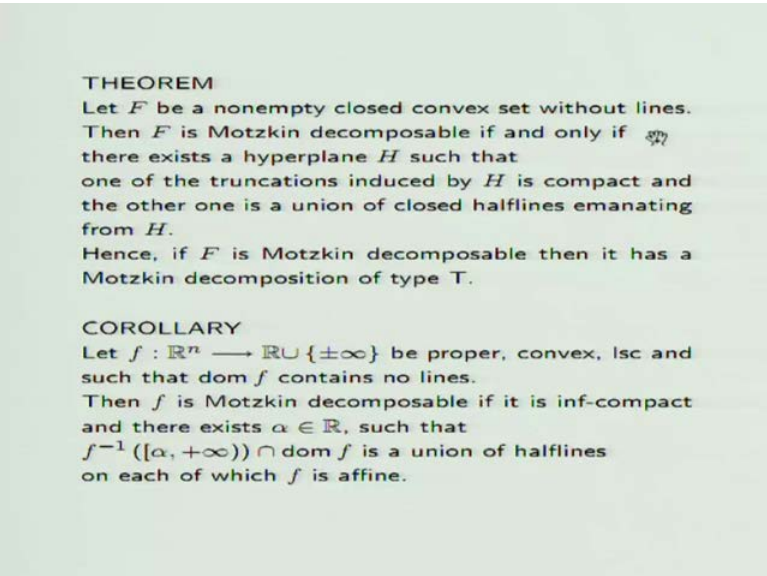

This will be used next to prove this theorem a set is Motzkin decomposable, if and only if there is a hyper plane such that, one of the truncations induced by H is compact, the other one is the union of closed half lines emanating from H and from here it is very easy to prove that, if a set is Motzkin decomposable then it says a Motzkin decomposition precision of type T applying, the previous result to functions we have that if the domain of the function contains no lines then Motzkin decomposability means, two things first that the function is information compact or lower level sets are compact.

But then there is one level alpha such that, on that complement of that level set of that lower level set, we have a union of half lines on each of which the restriction of F is affine then, I will finish by presenting just one result about a smallest Motzkin decompositions of type T. Suppose, for a Motzkin decomposable set there is a smallest compact T component, I mean there is a Motzkin decompositions for which the compact component is the smallest among all possible compact components belonging to a Motzkin decomposition of type T, then this smallest compact component is the smallest in the larger set of all possible compact components of Motzkin decompositions whether, they are or type T or not.

But we have an example in three dimensions, which shows that in some cases there are there is a smallest compact component which is not of type T this means, that for a close compact set without lines, there is a smallest compactly component if and only if it is the smallest component among all, but we know that the smallest compact component among all is Q of F. So, if and only if Q of F is obtained by truncation Q of F. I recall is they close complex hull of the extreme points of the set.

So, if this close complex hull is obtained can be obtained by truncation, then this is we know the smallest compact component then it is the smallest compactly component. So, if and only if this happen one is smallest compact component, but there are examples which a smallest compact component exist, but it is not obtained by truncation then it is not there is no smallest even not a minimal compactly component and this is the end of my presentation .