Curves And Surfaces.

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Module-III.

Surfaces-2: First Fundamental Form.

Lecture-13.

Conformal Mapping.

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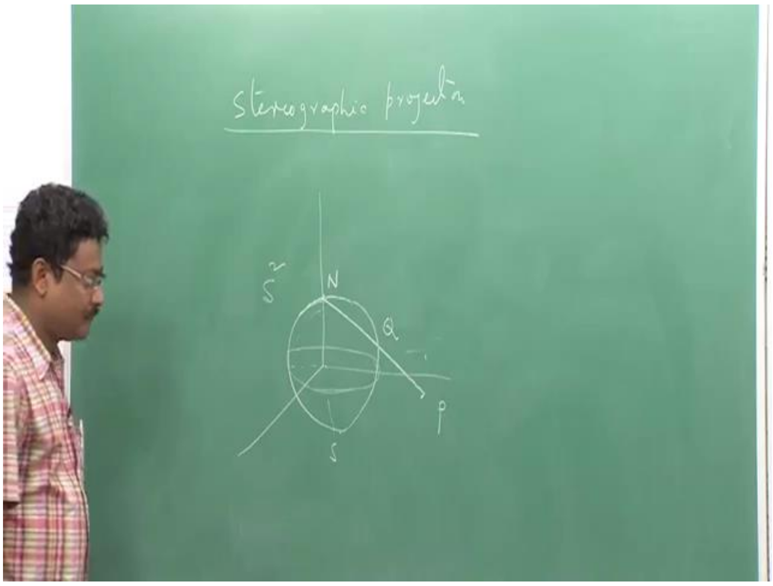

Today we will start with an example, very famous one you do it in complex analysis also, it is called the stereo graphic projection. What is, so that you do you do is, so you have the unit sphere, this is North Pole, this is the South Pole. So, this is S2, right, this surface is S2, we denote it by S2, what stereo graphic projection does, okay. We will look at it in other way round. Take any point P on the plane, YZ, XY plane. You join it through to North Pole, so that will intersect, except at the North Pole, that will intersect any other point Q. Okay.

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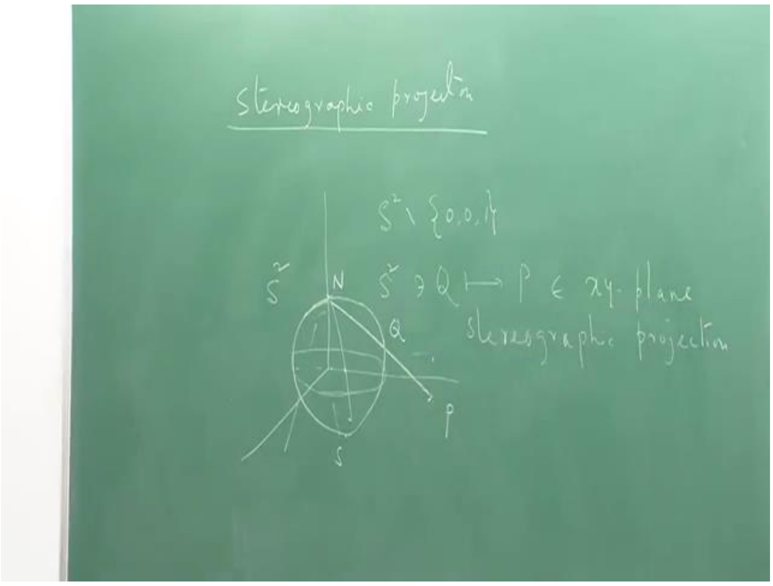

And the points, so all the points which are outside this sphere, they will be mapped to the northern hemisphere, so you take a point here, that will go here and the points inside, that will be mapped here, inside the thing. So, such a thing is called, so this map, now other way map, so any point on the sphere will be mapped to the point in the plane, except the North Pole. So, North Pole I have to remove. So, you consider myself is now, S2 - North Pole, North pole is 0, 0, 1, then this map Q going to P, this is called stereo graphic projection.

So, Q in S2 equals to P in XY plane. The use of this you do it in Atlas. When you do our cartography, if we want to put the world map into, map is a surface, globe but you want, this is a globe but you want to draw it on a paper, then that is what you do, stereo graphic projection.

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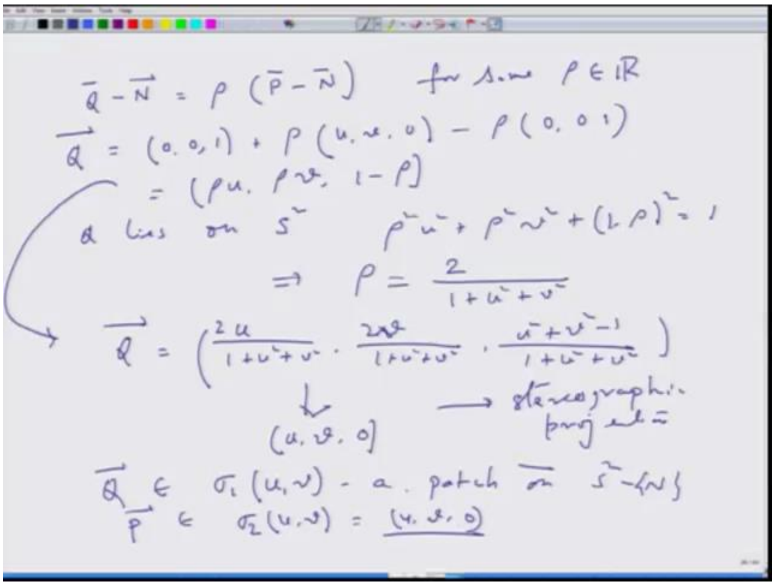

Let us do the calculations here, what does, how does this map looks like. So, what I have here? So, if I look at this point Q, if I look at this line Q - N, this line this vector and this vector, they are proportional, right, they are on the same line. So, Q - North Pole, this will be some constant times P - North Pole, correct. They are the same lines for some rho, some constant depending on Q and P.

So, what happens to Q now? Q is, so I am trying to find out the coordinates actually + rho U V0 - rho 0, 0, 1, which is actually rho U rho V 1 - rho, this is the coordinate of U in terms of rho. But Q lies on S2. So, I must have rho square U square rho square V square + 1 - rho square equal to1. So, that will give me rho equal to 2 by 1 + U square + V square. If you expand, that is what you will get. Right. This will be rho square here. So, what happens to Q now, what is the coordinate of Q? Q is U by 1 + U square + V square 2U. 2V from here, right 1 + U square + V square and then U square + V square - 1 divided by 1 + U square + V square.

This goes to U V0, so this is the stereo graphic projection. Okay. If I write down this picture coordinate wise, this will happen. So, Q belongs to some sigma 1 U V or surface patch, a patch on S2 - N and the plane P point belongs to single surface patch U V Sigma 2 which is U V0. Okay.

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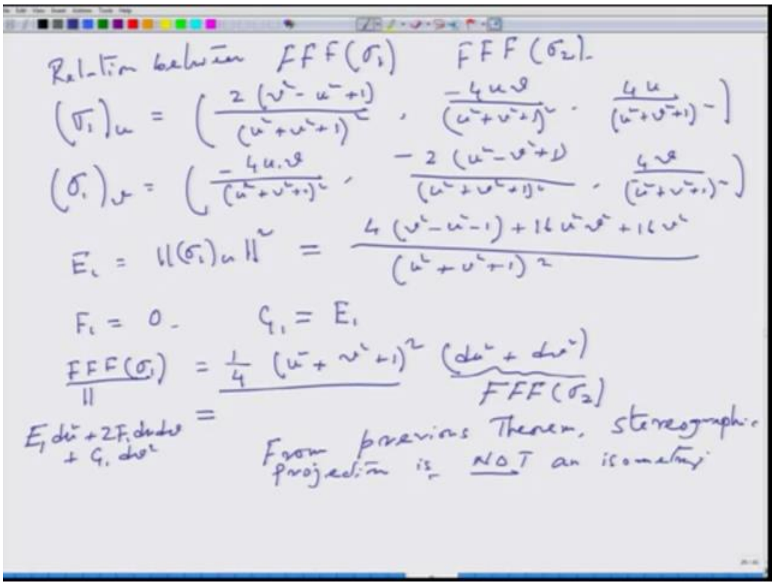

Let us see the relation between 1st fundamental form of Sigma 1 and 1st fundamental form of Sigma 2. Okay. Okay, then I have to calculate all those quantities, Sigma 1U, this will be 2 into V square - U square, you do the calculation yourself. U square + V square + 1 square - 4 U V, U square + V square + 1 whole square and then I think 4 U, what was it? Yes. 2, yes, 4 U U square + V square + 1 whole square. I have to calculate Sigma 1V also, that will have a similar expression. 4UVU square + V square + 1 whole square -2 U square - V square + 1 this time. Same denominator.

And here I will have 4 V, U square + V square + 1 whole square. So, I have to calculate E1 which is norm of Sigma 1 U square, you do it, what we have is 4 V square - U square +, sorry -1+16 U square V square +16 V square divided by U square + V square +1 whole square. You calculate F1 V0 and you calculate G1 this will be same as E1. And, so FFF of Sigma 1, this will be 1 by 4 U square + V square + 1 whole square du square dv square. So, you have to do this calculation, recall this is E1, so this fellow is E1 du square, 2 F1 du dv G1 dv square. And this fellow is what? This is precisely 1st fundamental form of plane, right.

So, what you get, FF of Sigma 1 is some multiple of 1st fundamental form of Sigma 2 and that multiple is not 1, so they are not equal. So, immediately from previous theorem + this theorem, stereo graphic projection is not an isometry. Because of isometry, then it has to be same.

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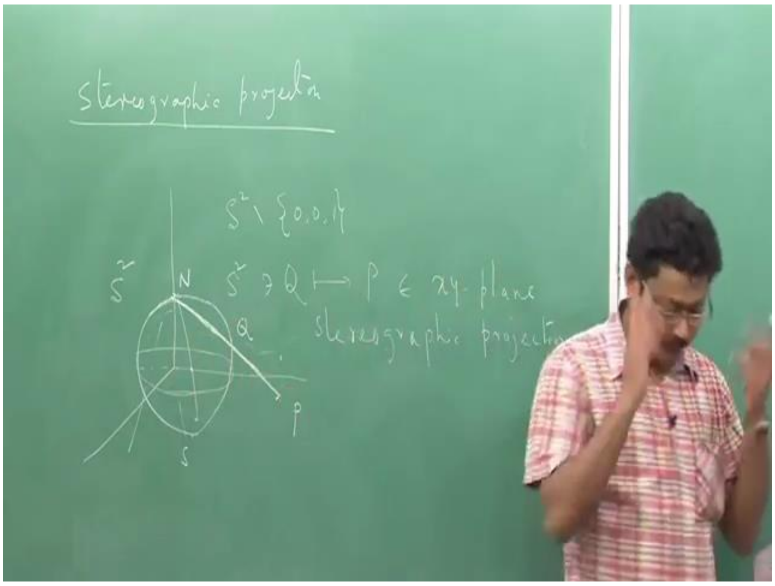

So, this is usual, because if you take a long curve on the plane, that will be wrapped on the under, inverse map of stereo graphic projection will be wrapped in the sphere. So, let us cannot be preserved. If you think off intuitively, length of curves cannot be preserved to this map.

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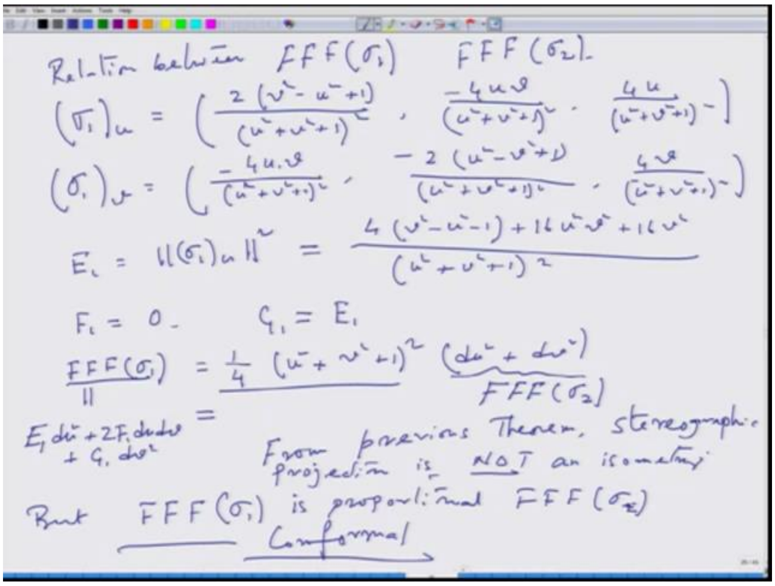

But what I have that FF1 of Sigma 1 is proportional, that is a multiple of FF of Sigma 2, 1st fundamental form of Sigma 2. So, 1st fundamental form of sphere is proportional to 1st fundamental form of plane and such a map is called conformal. This is not the definition, definition I make in the next page.

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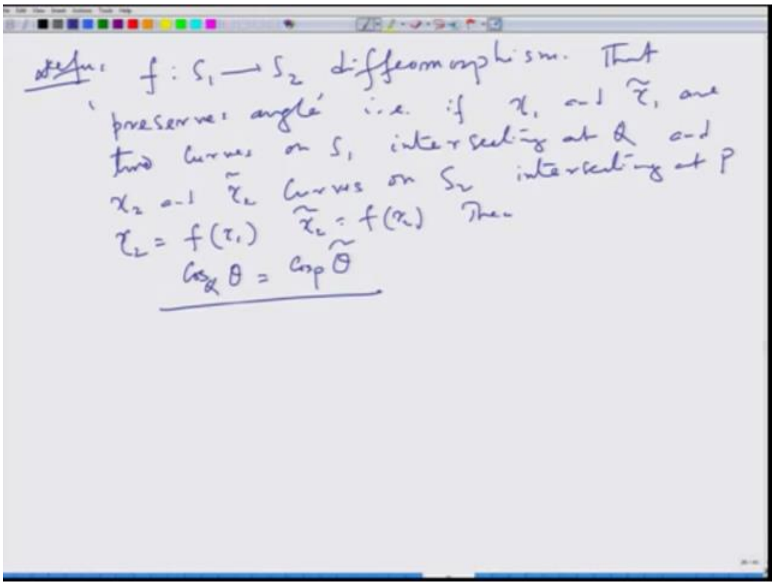

What it does that if from, suppose I have a diffeomorphism between surfaces that preserves angle, what is that, that is if gamma 1 and gamma 1 tilde are 2 curves on S1 intersecting at P and gamma 2 and gamma 2 tilde curves on S1, curves on S2 intersecting at P and of course I will have gamma 2 is F of gamma 1, gamma 2 tilde is F of gamma 2, then cos Q Theta equal to cos P Theta.

What does it mean? What is the angle between gamma 1, that is the angle between any, what is the angle between gamma 1 and gamma 2, that is the angle between their tangent vectors, right and, similarly here.

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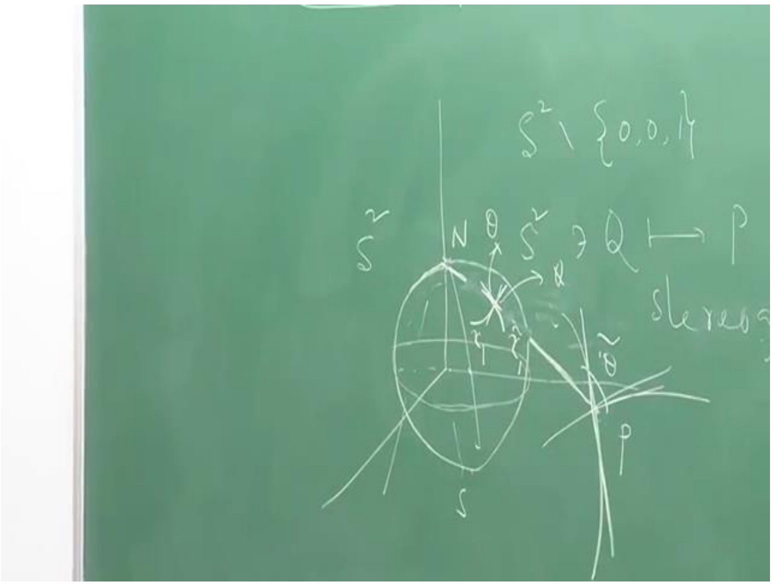

So, if I have a curve here on the surface, suppose there intersecting, so here is a curve, here is a curve gamma 1, gamma 1 tilde, here is a point Q, so angle between these 2 is angle between their tangent vectors. This will map to some other curves here and here, the length is not preserved the cos of the angles of the tangent vectors, that is preserved. Okay.

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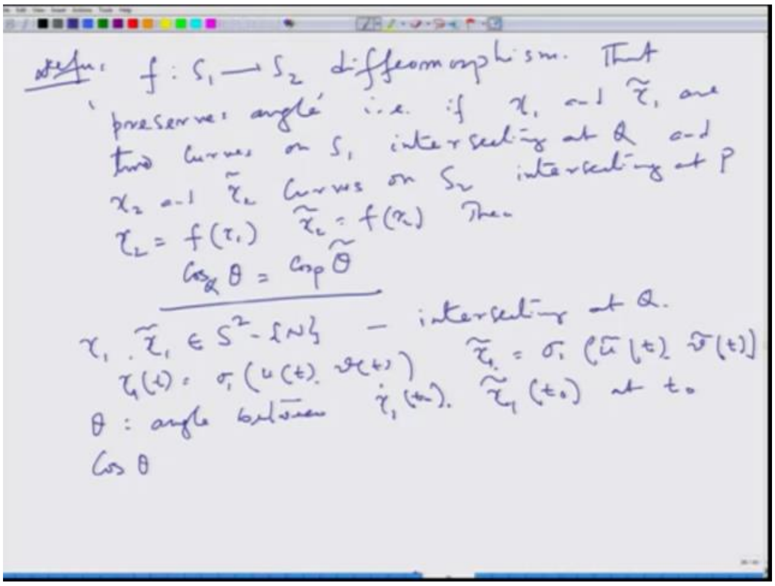

So, the angle between 2 curves are preserved, such a map is called conformal. What is happening for this one? So, let us do it for the stereo graphic projection, what is happening here. So, let us take gamma 1 and gamma 1 tilde on S2 - N intersecting at Q. What I was trying to explain on the board, okay… So, let us take gamma T, some parameterisation, Sigma 1, UP, VT and gamma 1 tilde Sigma 1 U tilde T V tilde T. Theta, the angles between gamma 1 dot T0 gamma 1 tilde dot T0, some T0, some point T0. Okay. Then what is cos theta?

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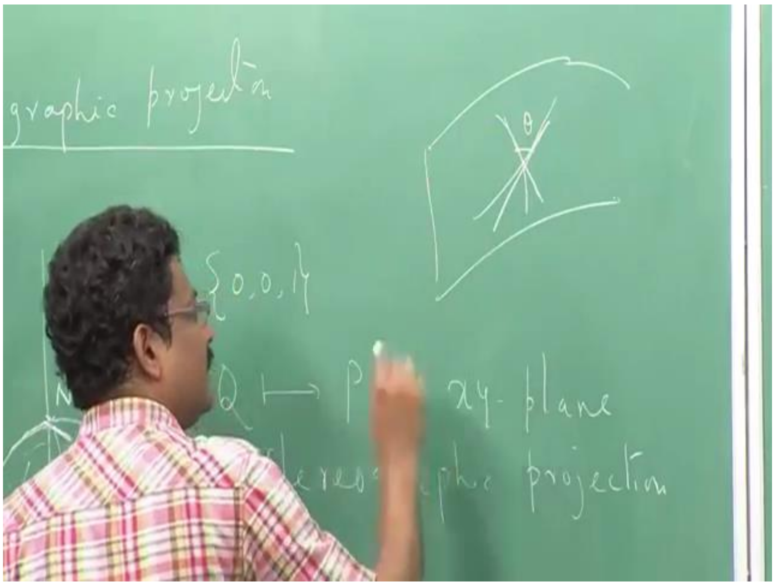

Just draw it here without the surface. If I have any surface patch, I have a curve here, I have a curve here, there intersecting, this is the tangent here, this is the tangent here, this is theta, so cos theta I can find out as how do you find cos Theta.

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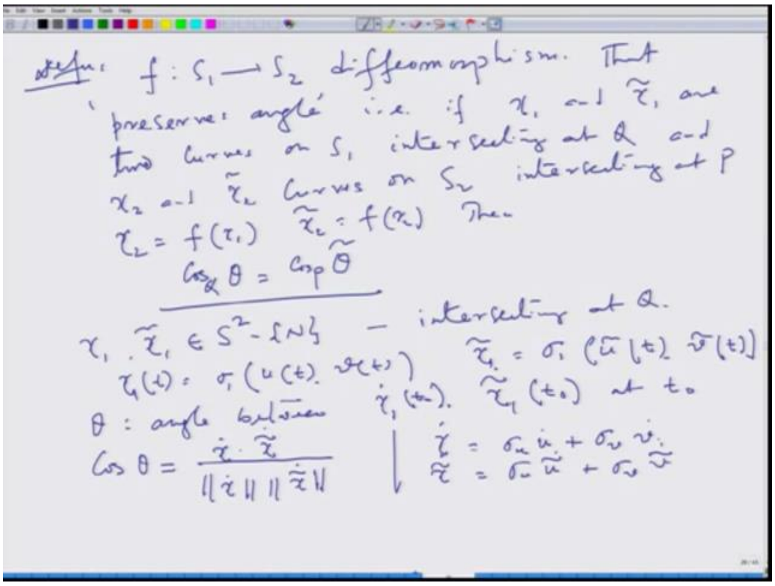

It is precisely gamma dot gamma tilde dot definition of angle, right. Gamma dot norm of gamma dot norm of gamma tilde dot. But on the other hand we also have gamma dot is Sigma U U dot + Sigma V V dot, similarly gamma tilde dot is Sigma U and U tilde dot Sigma V V tilde dot.

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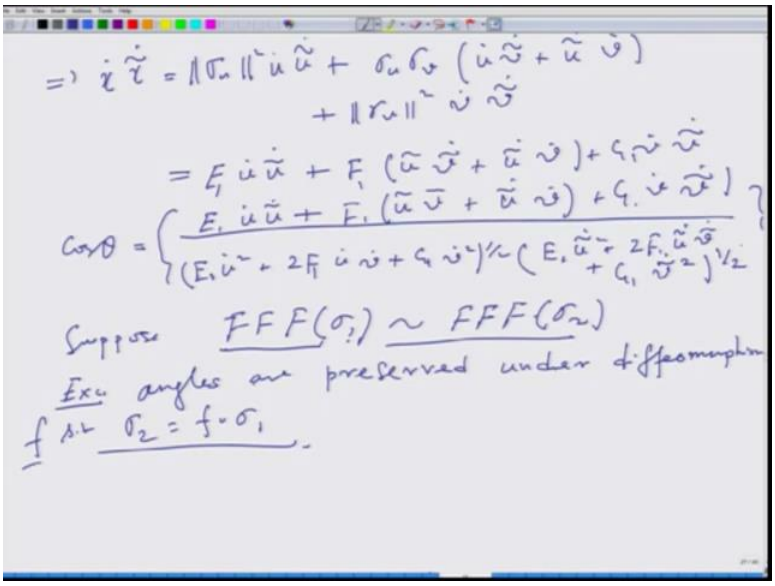

So, that will give us, let us do it, gamma dot, gamma tilde dot equal to, if we calculate this inner product, here, inner product with this and this, so you will see what will, what comes up is Sigma U square U dot U tilde dot 1st term, next term is Sigma U Sigma V and U dot cos term, V tilde dot + U tilde dot V dot + Sigma V square, last term you see… This into this. So, that will give V dot V tilde dot. But then it is what, this is E 1 U dot U tilde dot F, so the terms on 1st fundamental forms E E1 F1 and G1, I am writing here. So, what will be cos Theta? Cos Theta is this term divided by, I have to divide by norms, right. U1 U dot square 2 F U dot V dot + G V dot square half U1 V1 U tilde dot square 2 F 1U tilde dot V tilde dot + G1 G1 V tilde dot square, power half.

So, if I write down this expression cos Theta will be in terms of 1st fundamental form is this. Now, suppose since I give as an exercise. FF of Sigma 1 is proportional to FF of Sigma 2, then exercise, angles are preserved under diffeomorphism which takes Sigma 2, diffeomorphism F such that F2 is F Sigma 1. So, if this surface patch is mapped by a diffeomorphism to Sigma 2 and this happens, that, so what is exercise? If I have the map F such that Sigma 2, surface patch Sigma 1 goes to Sigma 2 under F and the 1st fundamental form of Sigma 1 and the 1st fundamental form of Sigma 2 or proportional, they are not equal. Then these angles are preserved because you will have cos Theta equal to, what you have to show that for this map also, you will have the same cos Theta, that is what you have to show.

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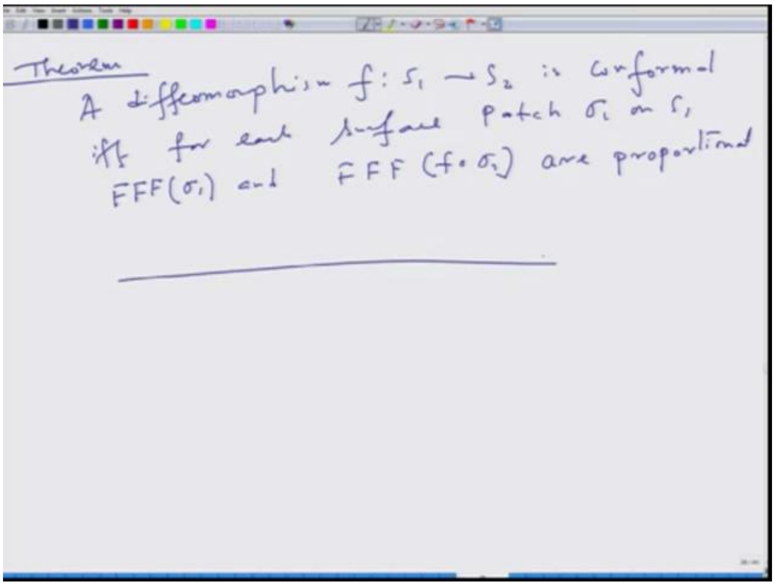

So, that is the calculation and I request you to do the calculation and finally I can then write down the theorem. A diffeomorphism F from S1 to S2 is conformal if and only if for each surface patch Sigma 1 on S1, FF, 1st fundamental forms of Sigma 1 and 1st fundamental form of as compose Sigma 1, what happened in the last example which I told you to calculate… are proportional. One way is easy, otherwise the calculation I indicated… So, that is about the conformal map, so you get conformal, you have 2 maps now, one that preserves length of the curves, it is isometry and I have these conformal maps. They will see how to do, what to do in these 2 maps on the geometry of surfaces in the next 2 lectures. Thank you.